Affine varieties

1 Algebraic sets

K : field $\quad K^n$: affine n-dim space.

Any $f(x) \in K[x_1, \dots, x_n]$ gives a function

$$\tilde{f} : K^n \longrightarrow K, \quad p = (a_1, \dots, a_n) \mapsto f(p) = f(a_1, \dots, a_n)$$

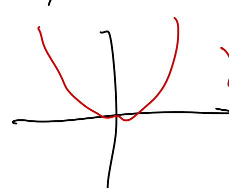
Write $A = K[x_1, \dots, x_n]$

Def A subset $X \subseteq K^n$ is called algebraic if $\exists\, U \subseteq A$ st $X = V(U) := \{p \in K^n \mid f(p) = 0 \ \forall f \in U\}$.

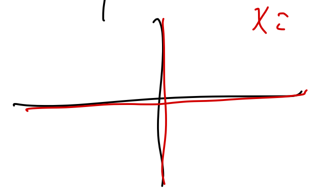
Ex 1) $U = \emptyset \Rightarrow V(U) = K^n$

2) $U = \{f\}$, $n = 1 \Rightarrow V(f)$ any finite subset of K

$n = 2$ $\quad f(x,y) = x^2 - y \quad V(U)$



$f(x,y) = xy \quad V(U)$



Let $J = (U) \subseteq A$

$f(p) = 0$, $g(p) = 0 \Rightarrow (f+g)(p) = 0$, $f(p)h(p) = 0 \quad \forall h \in A$

so $V(U) = V(J) = V(\{f_1, \dots, f_r\})$

where $J = (f_1, \dots, f_r)$ (Exist by Hilbert's basis Thm)

<u>Prop / Def</u> 1) $V(0) = K^n$, $V(A) = \emptyset$

2) $I \subseteq J \Rightarrow V(I) \supseteq V(J)$

3) $V(I \cap J) = V(I) \cup V(J)$

4) $V(\sum_i I_i) = \bigcap_i V(J_i)$

Hence $\{ K^n \setminus V(J) \mid J \subseteq A \text{ ideal} \}$ is a topology on K^n called Zariski topology.

<u>proof</u> 1), 2), 4) clear

2) gives $V(I \cap J) \supseteq V(I) \cup V(J)$

If $p \notin V(I) \cup V(J)$. Then $\exists f \in I$, $g \in J$ st

$f(p) \neq 0$, $g(p) \neq 0$ $\Rightarrow f(p)g(p) \neq 0$ but $fg \in I \cap J$

$\Rightarrow p \notin V(I \cap J)$ $\square$

<u>Def</u> Let $X \subset K^n$. Then

$$I(X) := \{ f \in A \mid f(p) = 0 \ \forall p \in X \}.$$

is an ideal in A

<u>Prop</u> Let $X, Y \in K^n$ are algebraic, $J \subseteq A$ an ideal.

1) $X \subseteq Y \Rightarrow I(X) \supseteq I(Y)$

2) $X = V(I(X))$

3) $J \subseteq I(V(J))$.

Prop Let $X, Y \subseteq K^n$ are algebraic, $J \subseteq A$ an ideal.

1) $X \subseteq Y \Rightarrow I(X) \supseteq I(Y)$

2) $X = V(I(X))$

3) $J \subseteq I(V(J))$.

proof 1) $X \subseteq V(I(X))$, $J \subseteq I(V(J))$. clear

2) Write $X = V(J)$ for some $J \subseteq I(X)$. by 3)

so $X = V(J) \supseteq V(I(X))$. $\square$

Ex 1) $K = \mathbb{R}$ $A = K[x]$, $J = (x^2+1)$

$\Rightarrow V(J) = \emptyset \Rightarrow I(V(J)) = A \neq J$.

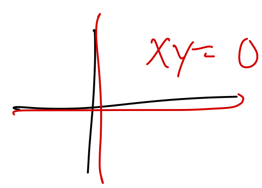
2) $K = \mathbb{C}$ $A = \mathbb{C}[x]$, $J = (x^2+1)$

$\Rightarrow V(J) = \{\pm i\}$, $I(V(J)) = (x^2+1) = J$

3) $K = \mathbb{C}$ $A = \mathbb{C}[x]$, $J = (x^2)$, $\Rightarrow V(J) = \{0\}$

$I(V(J)) = (x) \supsetneq (x^2)$

4) $K = \mathbb{R}$ $A = K[x,y]$, $I = (xy)$

$V(I) =$ 

$= V((x)) \cup V((y))$

Def An algebraic set $X \subseteq K^n$ is irreducible

if $X = X_1 \cup X_2$ X_i alg $\Rightarrow X = X_1$ or $X = X_2$

Prop Let $X \subseteq K^n$ alg. set

$$X \text{ irr} \iff I(X) \text{ prime}$$

proof Suppose $X = X_1 \cup X_2$ $\quad X_i \subsetneq X$ then $I(X_i) \supsetneq I(X)$

Let $f_i \in I(X_i) \setminus I(X)$

$\Rightarrow$ $f_1 f_2 \in I(X_1) \cap I(X_2)$ but

$$V(I(X_1) \cap I(X_2)) = V(I(X_1)) \cup V(I(X_2))$$
$$= X_1 \cup X_2 = X$$

$\Rightarrow$ $f_1 f_2 \in I(X)$ so $I(X)$ is not prime.

Suppose $I(X)$ not prime and $f_1, f_2 \notin I(X)$ st $f_1 f_2 \in I(X)$

Let $I_i := (I(X), f_i)$ and $V(I_i) = X_i$ $\quad i = 1, 2,$

Then $X_i \subsetneq X$ but $X \subseteq X_1 \cup X_2$ as

$\forall p \in X$ $\quad f_1 f_2(p) = 0 \Rightarrow f_1(p) = 0$ or $f_2(p) = 0$ $\quad \square$

(2) Polynomial functions

Let $V \subseteq K^n$ alg. set. Any $f(x) \in A$ gives

$$\tilde{f} : V \to K, \quad p \longmapsto f(p),$$

set $K[V] = \{ g : V \to K \mid g = \tilde{f} \text{ for some } f \in A \}$.

i.e $\phi : A \longrightarrow K[V], \quad f \longmapsto \tilde{f}$

surjective algebra morphism.

Moreover $\ker \phi = I(V)$

Hence $A / I(V) \simeq K[V]$.

Note V irreducible $\Leftrightarrow I(V)$ prime $\Leftrightarrow K[V]$ domain.

In this case we call V an affine variety

$K[V]$: coordinate ring of V

$K(V) := Q(K[V])$: function field of V

We say $\alpha \in K(V)$ is regular at $p \in V$ if

$\alpha = \frac{f}{g}$ where $g(p) \neq 0$.

$$K[V]_p = \{ \alpha \in K(V) \mid \alpha \text{ regular at } p \}$$
$$= S^{-1} K[V] \text{ where } S = \{ g \in K[V] \mid g(p) \neq 0 \}$$
$$= K[V] \setminus I(\{p\})$$

Note If $p=(a_1,\dots,a_n)\in K^n$ then

$m_p := I(\{p\}) = (x_1-a_1, x_2-a_2, \dots, x_n-a_n)$

is maximal $A/m_p \simeq K$ field

Q 1) Are there other maximal ideals in A?

2) When $I(V(J)) = J$?

In general 1) is no $\mathbb{R}[x]/(x^2+1) \simeq \mathbb{C}$ field

Def Let R: commutative ring $I\subseteq R$ ideal

1) The radical of I is $\sqrt{I} := \{a\in R \mid \exists n\geq 1\ a^n\in I\}$

2) I is called a radical ideal if $I=\sqrt{I}$.

Prop $\sqrt{I}$ is an ideal.

proof Let $a,b\in\sqrt{I}$, $r\in R$ $\exists m,n$ $a^m, b^n\in I$

$(ra)^m = r^m a^m \in I \Rightarrow ra\in I$.

Let $N=m+n-1$ $(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} \underset{I}{a^k} \text{ or } \underset{I}{b^{N-k}} \in I$ $\square$.

Rmk $\sqrt{I}\supseteq I$ $\sqrt{\sqrt{I}} = \sqrt{I}$.

Ex 1) If R: UFD $a=\prod_i a_i^{n_i}$ a_i irreducible $i\neq j \Rightarrow (a_i)\neq(a_j)$

$\sqrt{(a)} = (\prod_i a_i)$

2) If I prime then $\sqrt{I}=I$ as $a\in\sqrt{I} \Rightarrow \exists n\ a^n\in I$

$\Rightarrow a\in I$

<u>Thm</u> (Hilbert's Nullstellensatz)

Let K: algebraically closed field and $A = K[X_1, \dots, X_n]$

a) The maximal ideals of A are precisely

$$m_p = (X_1 - a_1, \dots, X_n - a_n) \qquad p = (a_1, \dots, a_n) \in K^n.$$

b) Let $J \subsetneq A$ ideal. Then $V(J) \neq \emptyset$

c) Let $J \subsetneq A$ ideal $\quad I(V(J)) = \sqrt{J}$.

<u>Rmk</u> b) $\Rightarrow$ K: algebraically closed

<u>Corollary</u> If K alg. closed then V and I induce inverse bijections

$$\begin{tikzcd}
\{\text{radical ideals}\} \arrow[r, leftrightarrow] & \{\text{alg. sets}\} \\
\{\text{prime ideals}\} \arrow[u, phantom, "\cup" rotate=90] \arrow[r, leftrightarrow] & \{\text{irr. alg. sets}\} \arrow[u, phantom, "\cup" rotate=90] \\
\{\text{max ideals}\} \arrow[u, phantom, "\cup" rotate=90] \arrow[r, leftrightarrow] & \{\text{singletons}\} \arrow[u, phantom, "\cup" rotate=90]
\end{tikzcd}$$

proof

Lemma (Zariski) Let E be a f.g. K-alg

If E is a field, then $\dim_K E < \infty$.

proof see prop 7.9. in Introduction to Commutative Algebra

by Atiyah-Macdonald

a) Let $m \in K[x_1, \dots, x_n]$ maximal ideal. Then

$F := K[x_1, \dots, x_n]/m$; field and f.g. K-alg

so $\dim_K F < \infty$ $\Rightarrow$ $F = K$ as K alg. closed

Hence $\forall i$ $x_i \equiv a_i$ mod m for some $a_i \in K$

$\Rightarrow$ $x_i - a_i \in m$ so $m_p := (x_1 - a_1, \dots, x_n - a_n) \subseteq m$

But m_p is maximal so $m_p = m$.

a) $\Rightarrow$ b) Let $J \subsetneq A$. Then there is $m \supseteq J$

m ; maximal. By a) $m = m_p$ for some $p \in K^n$

Let $f \in J$ then $f \in m_p$ so $f(p) = 0$

Hence $p \in V(J)$ and $V(J) \neq \emptyset$

b) $\Rightarrow$ c) Let $J \subseteq A$ and

If $f \in \sqrt{J} \Rightarrow \exists n \; f^n \in J \subseteq I(V(J))$

$\Rightarrow (f(p))^n = 0 \quad \forall p \in V(J) \Rightarrow f(p) = 0 \quad \forall p \in V(J)$

$\Rightarrow f \in I(V(J))$ so $\sqrt{J} \subseteq I(V(J))$

Now assume $f \in I(V(J))$. We show $f^N \in J$ for some N.

WLOG $f \neq 0$

set $J_1 = (J, fY - 1) \subseteq K[X_1, \dots, X_n, Y]$

$q = (a_1, \dots, a_n, b) \in V(J_1)$

$\Rightarrow p = (a_1, \dots, a_n) \in V(J)$ and $f(p)\, b = 1$

$\Rightarrow f(p) \neq 0 \Rightarrow p \notin V(J)$ $\lightning$

So $V(J_1) = \emptyset$. Now by b) $J_1 = K[X_1, \dots, X_n, Y] \ni 1$

$\Rightarrow 1 = \sum_{i=1}^{m} g_i f_i + g_0 (fY - 1)$ for some $f_i \in J$.

Now let $N = \max_i \deg_Y g_i$ so that

$f^N g_i = G_i(X_1, \dots, X_n, fY)$ for some polynomial G_i

$f^N \overset{(*)}{=} \sum_{i=1}^{m} G_i(X_1, \dots, X_n, fY) f_i + G_0(X_1, \dots, X_n, fY)(fY - 1)$

Define $\phi : K[X_1, \dots, X_n, Y] \longrightarrow K(X_1, \dots, X_n)$

$X_i \longmapsto X_i$

$Y \longmapsto \frac{1}{f}$

evaluate ϕ on $(*)$

Then

$$f^N \overset{(*)}{=} \sum_{i=1}^{m} \underbrace{G_i(x_1,\dots,x_n,1)}_{\in A} \underbrace{f_i}_{\in J} + 0$$

in $K(x_1,\dots,x_n)$

in $K[x_1,\dots,x_n]$

$\Rightarrow \quad f^N \in J \quad$ and $\quad f \in \sqrt{J}$

So $\quad I(V(J)) \subseteq \sqrt{J} \quad$ so $\quad I(V(J)) = \sqrt{J}$. $\square$