Problem set 1

Algebra for PhD students

Solutions should contain detailed arguments for all statements made. The total amount of points for this set is 100. Hand in before March 27.

Problem 1. (10 credits) Let $(\mathbb{Q}_{>0}, \cdot)$ be the group of positive rational numbers with multiplication and $(\mathbb{Z}[x], +)$ be the group of polynomials in one variable x with integer coefficients. Show that $(\mathbb{Q}_{>0}, \cdot)$ is isomorphic to $(\mathbb{Z}[x], +)$.

Problem 2. (10 credits) Let G be a group. Consider a collection of formal generators x_g indexed by $g \in G$. Prove that G is isomorphic to $\langle x_g \mid x_g x_h = x_{gh},\ g, h \in G\rangle$.

Problem 3. (10 credits) Let $\mathcal{M}$ be the category of monoids. Show that for each $N \in \mathcal{M}$ the assingment $M \mapsto M \times N$ can be extended to a functor $F_N : \mathcal{M} \to \mathcal{M}$ (i.e. define F_N on morphisms) in such a way that each monoid morphism $\phi : N \to N'$ defines a natural transformation $F_N \to F_{N'}$.

Problem 4. (10 credits) Let G be the cyclic group of order n, i.e., $G = \langle g \mid g^n = 1\rangle$. Describe all irreducible complex representations of $G \times G$.

Problem 5. (20 credits) A semiring is an algebraic structure $(S, +, \cdot)$ such that $(S, +)$ is a commutative monoid with identity element 0 and $(S, \cdot)$ is a monoid with identity element 1 such that

- $a(b + c) = ab + ac$
- $(a + b)c = ac + bc$
- $0 \cdot a = 0 = a \cdot 0$

(a) Show that the set $M_{n\times n}(S)$ of $(n \times n)$-matrices with elements in some semiring S is a semiring with the usual operations of matrix multiplication and addition.

(b) Let $B = \{0, 1\}$ with operations defined by $0 + 0 = 0$, $1 + 0 = 0 + 1 = 1 + 1 = 1$, $1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$ and $1 \cdot 1 = 1$. Show that B is a semiring. This is called the Boolean semiring with two elements.

(c) Show that the monoid $(M_{n\times n}(B), \cdot)$ is isomorphic to the monoid B_n of binary relations on the set $\{1, \ldots, n\}$.

Problem 6. (20 credits)

(a) Show that the only ring endomorphism of $\mathbb{Q}$ is the identity.

(b) Show that the only ring endomorphism of $\mathbb{R}$ is the identity.

Problem 7. (20 credits) Describe all morphisms of $\mathbb{C}$-algebras from $M_{2\times 2}(\mathbb{C})$ to $M_{3\times 3}(\mathbb{C})$.