Problem set 2

Algebra for PhD students

Solutions should contain detailed arguments for all statements made. The total amount of points for this set is 100. Hand in before April 29.

Problem 1. (10 credits) Let K be a field, A be a K-algebra and M a left A-module. Show that M is simple if and only if $M \simeq A/I$ where I is a maximal left ideal of A.

Problem 2. (10 credits) Let R be a commutative ring with identity and $S \subseteq R$ a multiplicative set. Recall that $S^{-1}R = (S \times R)/\sim$, where $(s, a) \sim (t, b)$ if and only if there is $u \in S$ such that $uta = usb$. Moreover, multiplication and addition are defined on $S^{-1}R$ by

$$[(s, a)][(t, b)] = [(st, ab)]$$

and

$$[(s, a)] + [(t, b)] = [(st, ta + sb)].$$

Show that

(a) $\sim$ is an equivalence relation on $S \times R$,

(b) the operations above are well-defined and endow $S^{-1}R$ with a ring structure,

(c) for all $s, t \in S$, the element $[(s, t)] \in S^{-1}R$ is invertible with inverse $[(t, s)]$.

Problem 3. (10 credits) Let K be a field. Recall that the polynomial ring in finitely many variables $K[x_1, \ldots, x_n]$ is a Noetherian unique factorization domain. Show that the polynomial ring in infinitely many variables $K[x_1, x_2, \ldots,]$ is a unique factorization domain, but not Noetherian.

Problem 4. (10 credits) Let K be a field and set $J = (x_1^2 + x_2^2 - 1, x_2 - 1) \subseteq K[x_1, x_2]$. Compute $V(J)$ and $I(V(J))$.

Problem 5. (20 credits) Let R be a commutative Noetherian ring. Show that

(a) R/I is Noetherian for any ideal I,

(b) $S^{-1}R$ is Noetherian for any regular multiplicative set S.

Problem 6. (20 credits) Let R be a commutative ring with identity and $I, J \subseteq R$ ideals. Recall that $IJ \subseteq R$ is the ideal

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i \mid n \geq 0,\ a_i \in I,\ b_i \in J \right\}.$$

Show that

(a) If $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$,

(b) $\sqrt{IJ} = \sqrt{\sqrt{I}\sqrt{J}}$

(c) $\sqrt{I^n} = \sqrt{I}$

Problem 7. (20 credits) Let A be the subring of $M_{2\times 2}(\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$. For simplicity we write

$$A = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R} \end{pmatrix}.$$

Next consider the A-modules

$$\begin{pmatrix} \mathbb{C} \\ 0 \end{pmatrix} := \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \mid a \in \mathbb{C} \right\}, \quad \begin{pmatrix} \mathbb{C} \\ \mathbb{R} \end{pmatrix} := \left\{ \begin{pmatrix} b \\ c \end{pmatrix} \mid b \in \mathbb{C},\ c \in \mathbb{R} \right\},$$

$$\begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix} := \left\{ \begin{pmatrix} b \\ c \end{pmatrix} \mid b, c \in \mathbb{C} \right\}, \quad \begin{pmatrix} 0 \\ \mathbb{R} \end{pmatrix} := \begin{pmatrix} \mathbb{C} \\ \mathbb{R} \end{pmatrix} \Big/ \begin{pmatrix} \mathbb{C} \\ 0 \end{pmatrix},$$

with action given by matrix multiplication. Show that every left A-module can be written as a direct sum of some number of copies of the above modules.