

BLOCKS OF THE CATEGORY OF CUSPIDAL \mathfrak{sp}_{2n} -MODULES

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ABSTRACT. We show that every block of the category of cuspidal generalized weight modules with finite dimensional generalized weight spaces over the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is equivalent to the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -modules.

1. INTRODUCTION AND DESCRIPTION OF THE RESULTS

We fix the ground field to be the complex numbers. Fix $n \in \{2, 3, \dots\}$ and consider the symplectic Lie algebra $\mathfrak{sp}_{2n} =: \mathfrak{g}$ with a fixed Cartan subalgebra \mathfrak{h} and root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where Δ denotes the corresponding root system. For a \mathfrak{g} -module V and $\lambda \in \mathfrak{h}^*$ set

$$V_\lambda := \{v \in V : h \cdot v = \lambda(h)v \text{ for any } h \in \mathfrak{h}\},$$

$$V^\lambda := \{v \in V : (h - \lambda(h))^k \cdot v = 0 \text{ for any } h \in \mathfrak{h} \text{ and } k \gg 0\}.$$

A \mathfrak{g} -module V is called

- *weight* provided that $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$;
- *generalized weight* provided that $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda$;
- *cuspidal* provided that for any $\alpha \in \Delta$ the action of any nonzero element from \mathfrak{g}_α on V is bijective.

If V is a generalized weight module, then the set $\{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$ is called the *support* of V and is denoted by $\text{supp}(V)$.

Denote by $\hat{\mathcal{C}}$ the full subcategory in $\mathfrak{g}\text{-mod}$ which consist of all cuspidal generalized weight modules with finite-dimensional generalized weight spaces, and by \mathcal{C} the full subcategory of $\hat{\mathcal{C}}$ consisting of all weight modules. Understanding the categories \mathcal{C} and $\hat{\mathcal{C}}$ is a classical problem in the representation theory of Lie algebras. The first major step towards the solution of this problem was made in [Mat], where all simple objects in $\hat{\mathcal{C}}$ were classified. In [BKLM] it was shown that the category \mathcal{C} is semi-simple, hence completely understood. The aim of the present note is to describe the category $\hat{\mathcal{C}}$.

Apart from \mathfrak{sp}_{2n} , cuspidal weight modules with finite dimensional weight spaces exist only for the Lie algebra \mathfrak{sl}_n ([Fe]). In the latter case, simple objects in the corresponding category $\hat{\mathcal{C}}$ are classified in [Mat], the category \mathcal{C} is described in [GS], see also [MS], and the category $\hat{\mathcal{C}}$ is described in [MS]. Taking all these results into account, the present paper completes the study of cuspidal generalized weight modules with finite dimensional generalized weight spaces over semi-simple finite-dimensional Lie algebras.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The action of $Z(\mathfrak{g})$ on any object from $\hat{\mathcal{C}}$ is locally finite. Using this and the

standard support arguments gives the following *block decomposition* of $\hat{\mathcal{C}}$:

$$\hat{\mathcal{C}} \cong \bigoplus_{\substack{\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C} \\ \xi \in \mathfrak{h}^*/Z\Delta}} \hat{\mathcal{C}}_{\chi, \xi},$$

where $\hat{\mathcal{C}}_{\chi, \xi}$ consists of all V such that $\text{Supp}(V) \subset \xi$ and $(z - \chi(z))^k \cdot v = 0$ for all $v \in V$, $z \in Z(\mathfrak{g})$ and $k \gg 0$. Set $\mathcal{C}_{\chi, \xi} := \mathcal{C} \cap \hat{\mathcal{C}}_{\chi, \xi}$. From [Mat, Section 9] it follows that each nontrivial $\hat{\mathcal{C}}_{\chi, \xi}$ contains a unique (up to isomorphism) simple object, in particular, $\hat{\mathcal{C}}_{\chi, \xi}$ is indecomposable, hence a block. From this and [BKLM] we thus get that every nontrivial block $\mathcal{C}_{\chi, \xi}$ is equivalent to the category of finite-dimensional \mathbb{C} -modules. Our main result is the following:

Theorem 1. *Every nontrivial block $\hat{\mathcal{C}}_{\chi, \xi}$ is equivalent to the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -modules.*

To prove Theorem 1 we use and further develop the technique of extension of the module structure from a Lie subalgebra, originally developed in [MS] for the study of categories of singular and non-integral cuspidal generalized weight \mathfrak{sl}_n -modules. The proof of Theorem 1 is given in Section 4. In Section 2 we recall the standard reduction to the case of the so-called simple *completely pointed* modules (i.e. simple weight cuspidal modules for which **all** nontrivial weight spaces are one-dimensional) and a realization of such modules using differential operators. In Section 3 we define a functor from the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -modules to any block $\hat{\mathcal{C}}_{\chi, \xi}$ containing a simple completely pointed module. In Section 4 we prove that this functor is an equivalence of categories. In Section 5 we present some consequences of our main result, in particular, we recover the main result from [BKLM] stated above.

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2. COMPLETELY POINTED SIMPLE CUSPIDAL WEIGHT MODULES

A weight \mathfrak{g} -module V is called *pointed* provided that $\dim V_\lambda = 1$ for some $\lambda \in \mathfrak{h}^*$. If V is a pointed simple cuspidal weight \mathfrak{g} -module, then, obviously, all nontrivial weight spaces of V are one-dimensional, in which case one says that V is *completely pointed* (see [BKLM]). It is enough to consider blocks with completely pointed simple modules because of the following:

Lemma 2. *All nontrivial blocks of $\hat{\mathcal{C}}$ are equivalent.*

Proof. In the case of the category \mathcal{C} this is proved in [BKLM, Lemma 2]. The same argument works in the case of the category $\hat{\mathcal{C}}$ as well. \square

Let us recall the explicit realization of completely pointed simple cuspidal modules from [BL]. Denote by W_n the n -th *Weyl algebra*, that is the algebra of differential operators with polynomial coefficients in variables x_1, x_2, \dots, x_n . The algebra W_n is generated by x_i and $\frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, which satisfy the relations $[\frac{\partial}{\partial x_i}, x_j] = \delta_{i,j}$. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be the vectors of the standard basis in \mathbb{C}^n . Identify \mathbb{C}^n with \mathfrak{h}^* such that Δ becomes the following *standard root system* of type C_n :

$$\{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq n\}.$$

Then

$$\mathbf{H} = \mathbf{H}_n = \{2\varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \dots, \varepsilon_n - \varepsilon_{n-1}\}$$

is a basis of Δ . Fix a basis of \mathfrak{g} of the form

$$\mathbf{C} := \{X_{\pm\varepsilon_i \pm \varepsilon_j} : 1 \leq i < j \leq n\} \cup \{X_{\pm 2\varepsilon_i} : i = 1, 2, \dots, n\} \cup \{H_\alpha : \alpha \in \mathbf{H}\}$$

such that the following map defines an injective Lie algebra homomorphism from \mathfrak{g} to the Lie algebra associated with W_n :

$$(1) \quad \begin{aligned} X_{\varepsilon_i - \varepsilon_j} &\mapsto x_i \frac{\partial}{\partial x_j}, & 1 \leq i \neq j \leq n; \\ X_{\varepsilon_i + \varepsilon_j} &\mapsto x_i x_j, & i, j = 1, 2, \dots, n; \\ X_{-\varepsilon_i - \varepsilon_j} &\mapsto \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}, & i, j = 1, 2, \dots, n; \\ H_{\varepsilon_{i+1} - \varepsilon_i} &\mapsto x_{i+1} \frac{\partial}{\partial x_{i+1}} - x_i \frac{\partial}{\partial x_i}, & i = 1, 2, \dots, n-1; \\ H_{2\varepsilon_1} &\mapsto \frac{1}{2} \left(x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_1} x_1 \right). \end{aligned}$$

Set

$$\mathbf{B} := \{(b_1, b_2, \dots, b_n) \in \mathbb{Z}^n : b_1 + b_2 + \dots + b_n \in 2\mathbb{Z}\}.$$

For $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n$ define $N(\mathbf{a})$ to be the linear span of

$$\{\mathbf{x}^{\mathbf{b}} := x_1^{a_1+b_1} x_2^{a_2+b_2} \dots x_n^{a_n+b_n} : \mathbf{b} \in \mathbf{B}\}.$$

We first define an action of the elements from \mathbf{C} on $N(\mathbf{a})$ using the formulas from (1) as follows:

$$(2) \quad \begin{aligned} X_{\varepsilon_i - \varepsilon_j} \mathbf{x}^{\mathbf{b}} &= (a_j + b_j) \mathbf{x}^{\mathbf{b} + \varepsilon_i - \varepsilon_j}, & 1 \leq i \neq j \leq n; \\ X_{\varepsilon_i + \varepsilon_j} \mathbf{x}^{\mathbf{b}} &= \mathbf{x}^{\mathbf{b} + \varepsilon_i + \varepsilon_j}, & i, j = 1, 2, \dots, n; \\ X_{-\varepsilon_i - \varepsilon_j} \mathbf{x}^{\mathbf{b}} &= (a_i + b_i)(a_j + b_j) \mathbf{x}^{\mathbf{b} - \varepsilon_i - \varepsilon_j}, & 1 \leq i \neq j \leq n; \\ X_{-2\varepsilon_i} \mathbf{x}^{\mathbf{b}} &= (a_i + b_i)(a_i + b_i - 1) \mathbf{x}^{\mathbf{b} - 2\varepsilon_i}, & i = 1, 2, \dots, n; \\ H_{\varepsilon_{i+1} - \varepsilon_i} \mathbf{x}^{\mathbf{b}} &= (a_{i+1} + b_{i+1} - a_i - b_i) \mathbf{x}^{\mathbf{b}}, & i = 1, 2, \dots, n-1; \\ H_{2\varepsilon_1} \mathbf{x}^{\mathbf{b}} &= \frac{1}{2}(2a_1 + 2b_1 + 1) \mathbf{x}^{\mathbf{b}}. \end{aligned}$$

Theorem 3 ([BL]). (i) For every $\mathbf{a} \in \mathbb{C}^n$ formulae (2) define on $N(\mathbf{a})$ the structure of a completely pointed weight \mathfrak{g} -module.

(ii) If $a_i \notin \mathbb{Z}$ for all $i = 1, \dots, n$, then the module $N(\mathbf{a})$ is simple and cuspidal.

(iii) Every completely pointed simple cuspidal \mathfrak{g} -module is isomorphic to $N(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{C}^n$ such that $a_i \notin \mathbb{Z}$, $i = 1, \dots, n$.

3. THE FUNCTOR F

This section is similar to [MS, Subsection 3.1]. Fix $\mathbf{a} \in \mathbb{C}^n$ such that $a_i \notin \mathbb{Z}$, $i = 1, \dots, n$. Let $\hat{\mathbf{C}}_{\mathbf{a}}$ denote the block of $\hat{\mathbf{C}}$ containing $N(\mathbf{a})$. The category $\hat{\mathbf{C}}_{\mathbf{a}}$ is closed under extensions. Denote by $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -mod the category of finite dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -modules. For $V \in \mathbb{C}[[t_1, t_2, \dots, t_n]]$ -mod denote by T_i the linear operator describing the action of t_i on V . Set $\mathbf{0} = (0, 0, \dots, 0) \in \mathbf{B}$.

For $\mathbf{b} \in \mathbf{B}$ consider a copy $V^{\mathbf{b}}$ of V . Define

$$FV := \bigoplus_{\mathbf{b} \in \mathbf{B}} V^{\mathbf{b}}.$$

Define the action of elements from \mathbf{C} on the vector space FV in the following way: for $v \in V^{\mathbf{b}}$ set

$$(3) \quad \begin{aligned} X_{\varepsilon_i - \varepsilon_j} v &= (T_j + (a_j + b_j) \text{Id}_V) v && \in V^{\mathbf{b} + \varepsilon_i - \varepsilon_j}; \\ X_{\varepsilon_i + \varepsilon_j} v &= v && \in V^{\mathbf{b} + \varepsilon_i + \varepsilon_j}; \\ X_{-\varepsilon_i - \varepsilon_j} v &= (T_i + (a_i + b_i) \text{Id}_V)(T_j + (a_j + b_j) \text{Id}_V) v && \in V^{\mathbf{b} - \varepsilon_i - \varepsilon_j}; \\ X_{-2\varepsilon_i} v &= (T_i + (a_i + b_i) \text{Id}_V)(T_i + (a_i + b_i - 1) \text{Id}_V) v && \in V^{\mathbf{b} - 2\varepsilon_i}; \\ H_{\varepsilon_{i+1} - \varepsilon_i} v &= (T_{i+1} - T_i + (a_{i+1} + b_{i+1} - a_i - b_i) \text{Id}_V) v && \in V^{\mathbf{b}}; \\ H_{2\varepsilon_1} v &= \frac{1}{2}(2T_1 + (2a_1 + 2b_1 + 1) \text{Id}_V) v && \in V^{\mathbf{b}}; \end{aligned}$$

where i and j are as in the respective row of (2). For a homomorphism $f : V \rightarrow W$ of $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -modules denote by Ff the diagonally extended linear map from FV to FW , i.e. for every $\mathbf{b} \in \mathbf{B}$ and $v \in V^{\mathbf{b}}$ set

$$(4) \quad Ff(v) = f(v) \in W^{\mathbf{b}}.$$

Proposition 4. (i) Formulae (3) define on FV the structure of a \mathfrak{g} -module.

(ii) Every $V^{\mathbf{b}}$ is a generalized weight space of FV . Moreover, for $\mathbf{b} \neq \mathbf{b}'$ the weights of $V^{\mathbf{b}}$ and $V^{\mathbf{b}'}$ are different.

(iii) The module FV belongs to $\hat{\mathcal{C}}_{\mathbf{a}}$.

(iv) Formulas (3) and (4) turn F into a functor

$$F : \mathbb{C}[[t_1, t_2, \dots, t_n]]\text{-mod} \rightarrow \hat{\mathcal{C}}_{\mathbf{a}}.$$

(v) The functor F is exact, faithful and full.

Proof. Consider the \mathfrak{g} -module $N(\mathbf{a})$ for \mathbf{a} as above. Then, for every \mathbf{b} the defining relations of \mathfrak{g} (in terms of elements from \mathbf{C}), applied to $\mathbf{x}^{\mathbf{b}}$, can be written as some polynomial equations in the a_i 's. Since (2) defines a \mathfrak{g} -module for any \mathbf{a} (Theorem 3(i)), these equations hold for any \mathbf{a} , that is they are actually formal identities in the a_i 's. Write now $T_j + (a_j + b_j)\text{Id}_V = A_j + B_j$, a sum of matrices, where $A_j = T_j + a_j\text{Id}_V$ and $B_j = b_j\text{Id}_V$. Note that all A_i and B_j commute with each other and with all T_i 's. For a fixed \mathbf{b} , the defining relations for \mathfrak{g} on FV reduce to our formal identities (in the A_i 's) and hence are satisfied. This proves claim (i). Claim (ii) follows from the last two lines in (3) and the fact that all T_i 's are nilpotent (hence zero is the only eigenvalue).

As f commutes with all T_i , the map Ff commutes with the action of all elements from \mathbf{C} and hence defines a homomorphism of \mathfrak{g} -modules. By construction we also have $F(f \circ f') = Ff \circ Ff'$, which implies claim (iv).

By construction, F is exact and faithful. It sends the simple one-dimensional $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -module to $N(\mathbf{a})$ (as in this case all $T_i = 0$ and hence (3) gives (2)), which is an object of the category $\hat{\mathcal{C}}_{\mathbf{a}}$ closed under extensions. Claim (iii) follows.

To complete the proof of claim (v) we are left to show that F is full. Let $\varphi : FV \rightarrow FW$ be a \mathfrak{g} -homomorphism. Then φ commutes with the action of all elements from \mathfrak{h} . Using claim (ii), we get that φ induces, by restriction, a linear map $f : V = V^{\mathbf{0}} \rightarrow W^{\mathbf{0}} = W$. As φ commutes with all $H_{\varepsilon_{i+1} - \varepsilon_i}$, the map f commutes with all operators $T_{i+1} - T_i$. As φ commutes with $H_{2\varepsilon_1}$, the map f commutes with T_1 . It follows that f is a homomorphism of $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -modules. This yields $\varphi = Ff$ and thus the functor F is full. This completes the proof of claim (v) and of the whole proposition. \square

4. PROOF OF THEOREM 1

Because of Lemma 2 it is enough to fix one particular block and show there that F is an equivalence. Thus, we may assume that $a_i + a_j \notin \mathbb{Z}$ for all i, j (in particular, $a_i \notin \mathbb{Z}$ for all i). After Proposition 4, we are only left to show that F is dense (i.e. essentially surjective). We establish density of F by induction on n . We first prove the induction step and then the basis of the induction, which is the case $n = 2$.

Denote by λ the weight of $\mathbf{x}^{\mathbf{0}} \in N(\mathbf{a})$ (see Proposition 4(ii)). Let $M \in \hat{\mathcal{C}}_{\mathbf{a}}$. Set $V := M_{\lambda}$ and denote by M' the \mathfrak{a} -module $U(\mathbf{a})V$.

4.1. Reduction to the case $n = 2$. The main result of this subsection is the following:

Proposition 5. *If the functor F is dense for $n = 2$, then it is dense for any $n \geq 2$.*

Proof. Assume that $n > 2$ and that the functor F is dense in the case of the algebra \mathfrak{sp}_{2n-2} . We realize \mathfrak{sp}_{2n-2} as the subalgebra \mathfrak{a} of \mathfrak{g} corresponding to the subset $\mathbf{H}_{n-1} \subset \mathbf{H}$ of simple roots.

Let Y_1, Y_2, \dots, Y_n be the linear operators representing the action of the elements $H_{2\varepsilon_1}, H_{\varepsilon_2-\varepsilon_1}, H_{\varepsilon_3-\varepsilon_2}, \dots, H_{\varepsilon_n-\varepsilon_{n-1}}$ on V , respectively. Set

$$(5) \quad \begin{aligned} T_1 &:= Y_1 - \frac{1}{2}(2a_1 + 1)\text{Id}_V; \\ T_2 &:= Y_2 + T_1 - (a_2 - a_1)\text{Id}_V; \\ T_3 &:= Y_3 + T_2 - (a_3 - a_2)\text{Id}_V; \\ &\vdots \\ T_n &:= Y_n + T_{n-1} - (a_n - a_{n-1})\text{Id}_V. \end{aligned}$$

The T_i 's are obviously pairwise commuting nilpotent linear operators.

The module M' is a cuspidal generalized weight \mathfrak{a} -module with finite-dimensional weight spaces. Moreover, as all composition subquotients of M are of the form $N(\mathfrak{a})$, all composition subquotients of M' are of the form $N(\mathfrak{a})'$, the latter being a completely pointed simple cuspidal \mathfrak{a} -module. By our inductive assumption, the functor F is dense in the case of the algebra \mathfrak{a} . Hence $M' \cong N' := \bigoplus_{\mathbf{b}} V^{\mathbf{b}}$, where $\mathbf{b} \in \mathbf{B}$ is such that $b_n = 0$, and the action of \mathfrak{a} on N' is given by (3).

Lemma 6. *There is a unique (up to isomorphism) \mathfrak{g} -module $Q \in \hat{\mathcal{C}}_{\mathfrak{a}}$ such that $Q' = N'$ and which gives the linear operator T_n when computed using (5).*

Proof. The existence statement is clear, so we need only to show uniqueness. Assume that $Q \in \hat{\mathcal{C}}_{\mathfrak{a}}$ is such that $Q' = N'$ and the formulae (5), applied to Q , produce the linear operator T_n . Since $a_n \notin \mathbb{Z}$, the endomorphism $T_n + (a_n + b_n)\text{Id}_V$ is invertible for all $b_n \in \mathbb{Z}$. As the action of $X_{\varepsilon_n-\varepsilon_{n-1}}$ on Q is bijective, we can fix a weight basis in Q such that both the \mathfrak{a} -action on $Q' = N'$ and the action of $X_{\varepsilon_n-\varepsilon_{n-1}}$ on the whole Q is given by (3). As $n > 2$, the elements $X_{\pm 2\varepsilon_1}$ commute with $X_{\varepsilon_n-\varepsilon_{n-1}}$ and hence their action extends uniquely to the whole of Q using this commutativity. Similarly for all elements $X_{\pm(\varepsilon_i-\varepsilon_{i-1})}$, $i < n-1$, and for the element $X_{\varepsilon_{n-2}-\varepsilon_{n-1}}$. This leaves us with the elements $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ and $X_{\varepsilon_{n-1}-\varepsilon_n}$. Note that the simple roots $\varepsilon_{n-1}-\varepsilon_{n-2}$ and $\varepsilon_n-\varepsilon_{n-1}$ corresponding to the elements $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ and $X_{\varepsilon_n-\varepsilon_{n-1}}$ generate a root system of type A_2 (this corresponds to the algebra \mathfrak{sl}_3). Therefore the fact that the action of $X_{\varepsilon_{n-1}-\varepsilon_{n-2}}$ extends uniquely to Q is proved in [MS, Lemma 21], and the fact that the action of $X_{\varepsilon_{n-1}-\varepsilon_n}$ extends uniquely to Q is proved in [MS, Lemma 22]. This completes the proof. \square

The module FV obviously satisfies $(FV)' = N'$ and defines the linear operator T_n when computed using (5). Hence Lemma 6 implies $M \cong FV$. Since $M \in \hat{\mathcal{C}}_{\mathfrak{a}}$ was arbitrary, this shows that the functor F is dense, completing the proof. \square

4.2. Base of the induction: some \mathfrak{sl}_2 -theory as preparation. In this subsection we will recall (and slightly improve) some classical \mathfrak{sl}_2 -theory. We refer the reader to [Maz] for more details. Consider the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ with standard basis

$$\mathbf{e} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{f} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{h} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let V be a finite-dimensional vector space and A and B be two commuting linear operators on V . For $i \in \mathbb{Z}$ denote by $V^{(i)}$ a copy of V and consider the vector space $\bar{V} := \bigoplus_{i \in \mathbb{Z}} V^{(i)}$ (a direct sum of copies of V indexed by i). Define the actions of \mathbf{e} , \mathbf{f} and \mathbf{h} on \bar{V} as follows: for $v \in V^{(i)}$ set

$$(6) \quad \begin{aligned} \mathbf{e}v &:= (P - i\text{Id}_V)v && \in V^{(i+1)} \\ \mathbf{f}v &:= (Q + i\text{Id}_V)v && \in V^{(i-1)} \\ \mathbf{h}v &:= (Q - P + 2i\text{Id}_V)v && \in V^{(i)}. \end{aligned}$$

This can be depicted as follows (here right arrows represent the action of \mathbf{e} , left arrows represent the action of \mathbf{f} and loops represent the action of \mathbf{h}):

$$\begin{array}{ccccccc} \cdots & \xleftrightarrow{P+2\text{Id}_V} & V^{(-1)} & \xleftrightarrow{P+\text{Id}_V} & V^{(0)} & \xleftrightarrow{P} & V^{(1)} & \xleftrightarrow{P-\text{Id}_V} & \cdots \\ & \xleftarrow{Q-\text{Id}_V} & \circlearrowleft & \xleftarrow{Q} & \circlearrowleft & \xleftarrow{Q+\text{Id}_V} & \circlearrowleft & \xleftarrow{Q+2\text{Id}_V} & \\ & & Q-P-2\text{Id}_V & & Q-P & & Q-P+2\text{Id}_V & & \end{array}$$

- Proposition 7.** (i) Formulae (6) define on \bar{V} the structure of a generalized weight \mathfrak{sl}_2 -module with finite dimensional generalized weight spaces.
- (ii) Every cuspidal generalized weight \mathfrak{sl}_2 -module with finite dimensional generalized weight spaces is isomorphic to \bar{V} for some V with P and Q as above.
- (iii) The action of the Casimir element $\mathbf{c} := (\mathbf{h} + 1)^2 + 4\mathbf{fe}$ on \bar{V} is given by the linear operator $(P + Q + \text{Id}_V)^2$.
- (iv) Let \mathbb{C}^2 denote the natural \mathfrak{sl}_2 -module (the unique two-dimensional simple \mathfrak{sl}_2 -module). Then the linear operator $(\mathbf{c} - (P + Q + 2\text{Id}_V)^2)(\mathbf{c} - (P + Q)^2)$ annihilates the \mathfrak{sl}_2 -module $\mathbb{C}^2 \otimes \bar{V}$.
- (v) Let \mathbb{C}^3 denote the unique three-dimensional simple \mathfrak{sl}_2 -module. Then the linear operator $(\mathbf{c} - (P + Q + 3\text{Id}_V)^2)(\mathbf{c} - (P + Q + \text{Id}_V)^2)(\mathbf{c} - (P + Q - \text{Id}_V)^2)$ annihilates the \mathfrak{sl}_2 -module $\mathbb{C}^3 \otimes \bar{V}$.

Proof. The fact that \bar{V} is an \mathfrak{sl}_2 -module is checked by a direct computation. That \bar{V} is a generalized weight module follows from the fact that the action of \mathbf{h} on \bar{V} preserves (by (6)) each V^i and hence is locally finite. Since the category of generalized weight modules is closed under extensions, to prove that \bar{V} has finite dimensional generalized weight spaces it is enough to consider the case when \mathbf{h} has a unique eigenvalue on $V^{(0)}$, say λ . However, in this case \mathbf{h} has a unique eigenvalue on V^i , namely $\lambda + 2i$, which implies that $\bar{V}^\lambda = V$ is finite dimensional. Claim (i) follows. To prove Claim (iii) we observe that the action of \mathbf{c} on V^i is given by:

$$(Q - P + (2i + 1)\text{Id}_V)^2 + 4(Q + (i + 1)\text{Id}_V)(P - i\text{Id}_V) = (P + Q + \text{Id}_V)^2.$$

Claim (ii) can be found with all details in [Maz, Chapter 3].

To prove claim (iv) choose a basis $\{v_1, \dots, v_k\}$ in V , which gives rise to a basis $\{v_1^{(i)}, \dots, v_k^{(i)}, i \in \mathbb{Z}\}$ in \bar{V} . Choose the standard basis $\{e_1, e_2\}$ in \mathbb{C}^2 . Since $\mathbf{h}e_1 = e_1$, $\mathbf{h}e_2 = -e_2$ and \mathbf{h} acts by $Q - P + 2i\text{Id}_V$ on $V^{(i)}$, we obtain that \mathbf{h} acts by $Q - P + (2i + 1)\text{Id}_V$ on the vector space $W^{(i)}$ with basis

$$\{e_1 \otimes v_1^{(i)}, \dots, e_1 \otimes v_1^{(i)}, e_2 \otimes v_1^{(i+1)}, \dots, e_2 \otimes v_1^{(i+1)}\}.$$

We have $\mathbb{C}^2 \otimes \bar{V} \cong \bigoplus_{i \in \mathbb{Z}} W^{(i)}$ and one easily computes that in the above basis the actions of \mathbf{e} and \mathbf{f} on $\mathbb{C}^2 \otimes \bar{V}$ is given by the following picture:

$$\begin{array}{ccccccc} \cdots & \xleftrightarrow{\quad} & W^{(-1)} & \xleftrightarrow{\quad} & W^{(0)} & \xleftrightarrow{\quad} & W^{(1)} & \xleftrightarrow{\quad} & \cdots \\ & \xleftarrow{\quad} & \begin{pmatrix} P+\text{Id} & \text{Id} \\ 0 & P \end{pmatrix} & \xleftarrow{\quad} & \begin{pmatrix} P & \text{Id} \\ 0 & P-\text{Id} \end{pmatrix} & \xleftarrow{\quad} & \begin{pmatrix} Q+\text{Id} & 0 \\ \text{Id} & Q+2\text{Id} \end{pmatrix} & \xleftarrow{\quad} & \\ & & \begin{pmatrix} Q & 0 \\ \text{Id} & Q+\text{Id} \end{pmatrix} & & \begin{pmatrix} Q+\text{Id} & 0 \\ \text{Id} & Q+2\text{Id} \end{pmatrix} & & & & \end{array}$$

The action of \mathbf{c} on $W^{(0)}$ is now easily computed to be given by the linear operator

$$G := \begin{pmatrix} (Q - P + 2\text{Id})^2 + 4(Q + \text{Id})P & 4(Q + \text{Id}) \\ 4P & (Q - P + 2\text{Id})^2 + 4(Q + 2\text{Id})(P - \text{Id}) + 4\text{Id} \end{pmatrix}.$$

The characteristic polynomial of G is

$$\chi_G(\lambda) = (\lambda - (P + Q + 2\text{Id})^2)(\lambda - (P + Q)^2).$$

Claim (iv) now follows from the Cayley-Hamilton theorem.

We have an isomorphism of \mathfrak{sl}_2 -modules as follows: $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^3 \oplus \mathbb{C}$ (here \mathbb{C} is the trivial module), and hence claim (v) follows applying claim (iv) twice.

Alternatively, one could do a direct calculation (similar to the proof of (iii)). The proposition follows. \square

We note that the statement of Proposition 7(ii) is a special case of a more general result of Gabriel and Drozd describing blocks of the category of (generalized) weight \mathfrak{sl}_2 -modules, in particular, simple weight \mathfrak{sl}_2 -modules (see [Di, 7.8.16] and [Dr]). The statements of Proposition 7(iv) and (v) are \mathfrak{sl}_2 -refinements of a theorem of Kostant describing possible (generalized) central characters of the tensor product of a finite dimensional module with an infinite dimensional module ([Ko, Theorem 5.1]).

4.3. The case $n = 2$. Assume now that $n = 2$. We have $a_1, a_2, a_1 + a_2 \notin \mathbb{Z}$. Let \mathfrak{a} denote the Lie subalgebra of \mathfrak{g} generated by $X_{\pm(\varepsilon_2 - \varepsilon_1)}$. The algebra \mathfrak{a} is isomorphic to \mathfrak{sl}_2 .

Let $M \in \hat{\mathcal{C}}_{\mathfrak{a}}$. Denote by λ the weight of $\mathbf{x}^0 \in N(\mathfrak{a})$ and set $V := M_\lambda$. Let Y_1 and Y_2 be the linear operators representing the actions of the elements $H_{\varepsilon_2 - \varepsilon_1}$ and $C := (H_{\varepsilon_2 - \varepsilon_1} + 1)^2 + 4X_{\varepsilon_1 - \varepsilon_2}X_{\varepsilon_2 - \varepsilon_1}$ on V . The element C is a Casimir element for \mathfrak{a} , in particular, the operators Y_1 and Y_2 commute. Our first observation is the following:

Lemma 8. *The action of C on V is invertible and hence has a square root.*

Proof. From (2) we have that C acts on \mathbf{x}^0 by

$$(a_2 - a_1 + 1)^2 + 4(a_2 + 1)a_1 = (a_1 + a_2 + 1)^2.$$

Since $a_1 + a_2 \notin \mathbb{Z}$ by our assumptions, \mathbf{x}^0 is an eigenvector of C with a nonzero eigenvalue. As the module M has a composition series with subquotients isomorphic to $N(\mathfrak{a})$, the complex number $(a_1 + a_2 + 1)^2 \neq 0$ is the only eigenvalue of C on V . The claim follows. \square

Consider the \mathfrak{a} -module $M' := U(\mathfrak{a})M_\lambda$. Let Y'_2 denote any square root of Y_2 , which is a polynomial in Y_2 (it exists by Lemma 8). Then Y'_2 commutes with Y_1 . Set

$$T_1 := \frac{Y'_2 - Y_1 - \text{Id}_V}{2} - a_1 \text{Id}_V, \quad T_2 := \frac{Y'_2 + Y_1 - \text{Id}_V}{2} - a_2 \text{Id}_V.$$

Then T_1 and T_2 are two commuting nilpotent linear operators (it is easy to check that 0 is the unique eigenvalue for both T_1 and T_2), hence define on V the structure of a $\mathbb{C}[[t_1, t_2]]$ -module. The aim of this subsection is to establish an isomorphism $FV \cong M$, which would complete the proof of Theorem 1.

Set $R' := U(\mathfrak{a})(FV)_\lambda$. A direct computation (using (3)) shows that $H_{\varepsilon_2 - \varepsilon_1}$ and C act on $(FV)_\lambda = V^0$ as the linear operators Y_1 and Y_2 , respectively. As any cuspidal generalized weight \mathfrak{a} -module is uniquely determined by the actions of $H_{\varepsilon_2 - \varepsilon_1}$ and C (see [Dr] or [Maz, 3.7] for full details), it follows that $M' \cong R'$. The isomorphism $FV \cong M$ now follows from the following statement:

Proposition 9. *There is at most one (up to isomorphism) \mathfrak{g} -module $R \in \hat{\mathcal{C}}_{\mathfrak{a}}$ such that $U(\mathfrak{a})R_\lambda = R'$.*

Proof. Let $R \in \hat{\mathcal{C}}_{\mathfrak{a}}$ be such that $U(\mathfrak{a})R_\lambda = R'$. We choose a weight basis in R such that the action of \mathfrak{a} on R' and the action of $X_{2\varepsilon_1}$ on R is given by (3) (in other words these actions coincide with the corresponding actions on FV). Since $X_{\varepsilon_1 - \varepsilon_2}$ commutes with $X_{2\varepsilon_1}$, it follows that the action of $X_{\varepsilon_1 - \varepsilon_2}$ on R is also given by (3).

It is left to show that the action of $X_{\varepsilon_2 - \varepsilon_1}$ extends uniquely from R' to R and then that there is a unique way to define the action of $X_{-2\varepsilon_1}$. This will be done in the Lemmata 10 and 11 below. \square

Lemma 10. *There is a unique way to extend the action of $X_{\varepsilon_2 - \varepsilon_1}$ from R' to R .*

Proof. Let us first show that for every $k \in \{1, 2, \dots\}$ the action of $X_{\varepsilon_2 - \varepsilon_1}$ extends uniquely from $X_{2\varepsilon_1}^{k-1}R'$ to $X_{2\varepsilon_1}^k R'$ (here $X_{2\varepsilon_1}^0 R' = R'$).

Consider the following picture:

$$(7) \quad \begin{array}{ccccc} & & \overset{X}{\dashrightarrow} & & \\ & \bullet & \xleftarrow{Q} & \bullet & \\ & \uparrow 1 & & \uparrow 1 & \\ & \bullet & \xrightarrow{P+1} & \bullet & \xrightarrow{P} & \bullet \\ & \downarrow Q & & \downarrow Q+1 & & \bullet \end{array}$$

Here bullets are weight spaces with some fixed bases. The lower row is a part of $X_{2\varepsilon_1}^{k-1}R'$ where the \mathfrak{a} -action is already known by induction. The bases in the weight spaces in the lower row are chosen such that the action of \mathfrak{a} in the lower row is given by (3). The upper row is a part of $X_{2\varepsilon_1}^k R'$ where the \mathfrak{a} -action is to be determined. Arrows pointing up indicate the action of $X_{2\varepsilon_1}$. The bases of the weight spaces in the upper row are chosen such that the action of $X_{2\varepsilon_1}$ is given by the operator Id_V (as in (3)). Left arrows indicate the action of $X_{\varepsilon_1 - \varepsilon_2}$. The latter commutes with the action of $X_{2\varepsilon_1}$ and hence is given by the same linear operator in each column. Right arrows indicate the action of $X_{\varepsilon_2 - \varepsilon_1}$ (which is known for $X_{2\varepsilon_1}^{k-1}R'$ and is to be determined for $X_{2\varepsilon_1}^k R'$). The part to be determined is given by the dashed arrow. Labels P and Q represent coefficients (which are linear operators on V) appearing in the corresponding parts of formulae (3). Note that P and Q commute. The action of $X_{\varepsilon_2 - \varepsilon_1}$ on $X_{2\varepsilon_1}^k R'$ which is to be determined is given by some unknown linear operators X .

From $H_{\varepsilon_2 - \varepsilon_1} = [X_{\varepsilon_2 - \varepsilon_1}, X_{\varepsilon_1 - \varepsilon_2}]$ we compute that the action of $H_{\varepsilon_2 - \varepsilon_1}$ on the middle weight space in the lower row is given by $Q - P$. Using $[H_{\varepsilon_2 - \varepsilon_1}, X_{2\varepsilon_1}] = -2X_{2\varepsilon_1}$ we get that $H_{\varepsilon_2 - \varepsilon_1}$ acts on the right dot of the upper row via $Q - P - 2$. Using $[H_{\varepsilon_2 - \varepsilon_1}, X_{\varepsilon_1 - \varepsilon_2}] = -2X_{\varepsilon_1 - \varepsilon_2}$ we get that $H_{\varepsilon_2 - \varepsilon_1}$ acts on the left dot of the upper row via $Q - P - 4$. Hence the action of C on the upper row is given by $(Q - P - 3)^2 + 4XQ$. The action of C on the lower row is given by $(Q - P - 1)^2 + 4(P + 1)Q = (Q + P + 1)^2$.

The elements $X_{2\varepsilon_1}$, $X_{2\varepsilon_2}$ and $X_{\varepsilon_1 + \varepsilon_1}$ form a weight basis of a simple three-dimensional \mathfrak{a} -module \mathbb{C}^3 with respect to the adjoint action of \mathfrak{a} . Hence the upper row of our picture is a subquotient of the tensor product of the lower row and \mathbb{C}^3 . Therefore, from Proposition 7(v) we obtain that the linear operator

$$(C - (Q + P - 1)^2)(C - (Q + P + 1)^2)(C - (Q + P + 3)^2)$$

annihilates the upper row. A direct computation using (3) shows that the action of the operators $C - (Q + P - 1)^2$ and $C - (Q + P + 1)^2$ on the part $X_{2\varepsilon_1}^k N(\mathfrak{a})'$ of the module $N(\mathfrak{a})$ is invertible. As the \mathfrak{g} -module we are working with must have a composition series with subquotients $N(\mathfrak{a})$, it follows that the action of both $C - (Q + P - 1)^2$ and $C - (Q + P + 1)^2$ on $X_{2\varepsilon_1}^k R'$ is invertible. Hence $C - (Q + P + 3)^2$ annihilates $X_{2\varepsilon_1}^k R'$, which gives us the equation

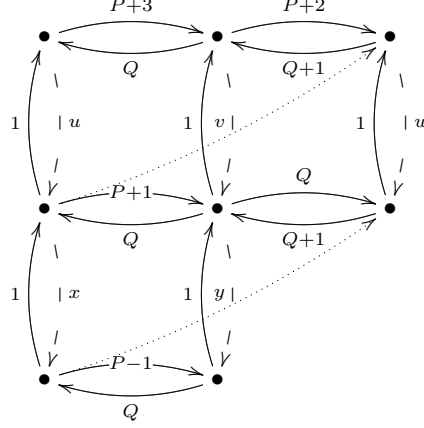
$$(Q - P - 3)^2 + 4XQ = (Q + P + 3)^2.$$

This equation has a unique solution, namely $X = Q + 3$, which gives the required extension.

Similarly one shows that for $k \in \{-1, -2, \dots\}$ the action of $X_{\varepsilon_2 - \varepsilon_1}$ extends uniquely from $X_{2\varepsilon_1}^{k+1}R'$ to $X_{2\varepsilon_1}^k R'$ (here again $X_{2\varepsilon_1}^0 R' = R'$). This completes the proof of our lemma. \square

Lemma 11. *There is a unique way to define the action of $X_{-2\varepsilon_1}$ on N .*

Proof. To determine this action of $X_{-2\varepsilon_1}$ on N we consider the following extension of the picture (7) with the same notation as in the proof of Lemma (10):



Here all right arrows, representing the action of $X_{\varepsilon_2-\varepsilon_1}$, are now determined by Lemma 10 and we have to figure out the down arrows, representing the action of $X_{-2\varepsilon_1}$. The two dotted arrows will be used later on in the proof.

Consider the \mathfrak{sl}_2 -subalgebra \mathfrak{c} of \mathfrak{g} generated by $e := X_{2\varepsilon_1}$ and $f := X_{-2\varepsilon_1}$. Set $h := [e, f]$. Denote by Z the action of h in the leftmost weight space of the middle row. Then $Z = x - u$. The element h commutes with both e and f . Therefore, by (3), the operator Z commutes with both T_1 and T_2 and hence with both P and Q .

The algebra \mathfrak{c} has the quadratic Casimir element $C_{\mathfrak{c}}$, whose action on the \mathfrak{c} -module given by the leftmost column of our picture is given by $x + f(Z)$, where f is some polynomial of degree two. From (3) it follows that the unique eigenvalue of this action is nonzero, in particular, $x + f(Z)$ is invertible. Let x' be a fixed square root $x + f(Z)$, which is a polynomial in $x + f(Z)$.

The elements $X_{\varepsilon_2-\varepsilon_1}$ and $X_{\varepsilon_2+\varepsilon_1}$ form a basis of a simple two-dimensional \mathfrak{c} -module with respect to the adjoint action. Using Proposition 7(iv) and arguments similar to those used in the proof of Lemma 10, we get that $C_{\mathfrak{c}} - (x' + 1)^2$ or $C_{\mathfrak{c}} - (x' - 1)^2$ annihilates the middle column (the sign depends on the original choice of x'). Note that the middle column equals $X_{\varepsilon_2-\varepsilon_1}$ applied to the leftmost column.

Similarly, the elements $X_{\varepsilon_1-\varepsilon_2}$ and $X_{-\varepsilon_2-\varepsilon_1}$ form a basis of a simple two-dimensional \mathfrak{c} -module with respect to the adjoint action. Applying the same arguments as in the previous paragraph we get that $C_{\mathfrak{c}} - (x')^2$ annihilates any vector of the form $X_{\varepsilon_1-\varepsilon_2}X_{\varepsilon_2-\varepsilon_1}\mathbf{v}$, where \mathbf{v} is from the leftmost column. This implies that the actions of $C_{\mathfrak{c}}$ and $X_{\varepsilon_1-\varepsilon_2}X_{\varepsilon_2-\varepsilon_1}$ and thus the actions of $C_{\mathfrak{c}}$ and C on the leftmost column commute. As the action of H commutes with the action of C , we thus obtain that x commutes with the action of C . This implies that x commutes with $T_1 + T_2$. As it obviously commutes with $T_1 - T_2$, we get that x commutes with both T_1 and T_2 and hence with both P and Q .

Similarly one shows that y, u, v and w commute with both P and Q . From the commutativity of $X_{\varepsilon_2-\varepsilon_1}$ and $X_{-2\varepsilon_1}$ we get the following conditions:

$$y(P+1) = (P-1)x, \quad v(P+3) = (P+1)u, \quad w(P+2)(P+3) = P(P+1)u.$$

Here everything commutes by the above and $P+1, P+2$ and $P+3$ are invertible (as $X_{\varepsilon_2-\varepsilon_1}$ acts bijectively). Therefore

$$y = (P-1)(P+1)^{-1}x, \quad v = (P+1)(P+3)^{-1}u, \quad w = P(P+1)(P+3)^{-1}(P+2)^{-1}u.$$

This implies that y , v and w are uniquely determined by x and u .

Since the actions of both $X_{\varepsilon_2-\varepsilon_1}$ and $X_{2\varepsilon_1}$ are completely determined, we can compute the action of $X_{2\varepsilon_2}$ and see that it is given (similarly to the action of $X_{2\varepsilon_1}$) by Id_V (this is depicted by the dotted arrows in the picture). As $X_{-2\varepsilon_2}$ and $X_{2\varepsilon_2}$ commute, we obtain that $w = x$, that is

$$(8) \quad x = P(P+1)(P+3)^{-1}(P+2)^{-1}u.$$

Therefore the only parameter left for now is u .

On the one hand, the action of the element h on the middle dot of the second row is given by $y - v = (P-1)(P+1)^{-1}x - (P+1)(P+3)^{-1}u$. On the other hand, from $[h, X_{\varepsilon_2-\varepsilon_1}] = 4X_{\varepsilon_2-\varepsilon_1}$ we have that this action equals $Z + 4 = x - u + 4$. This gives us the equation

$$(9) \quad (P-1)(P+1)^{-1}x - (P+1)(P+3)^{-1}u = x - u + 4.$$

Using (9) and (8) we get the equation

$$\frac{P(P-1)}{(P+2)(P+3)}u + \frac{P+1}{P+3}u = \frac{P(P+1)}{(P+2)(P+3)}u - u + 4.$$

This is a linear equation with nonzero coefficients and thus it has a unique solution, namely $u = (P+3)(P+2)$. Hence u is uniquely defined. The claim of the lemma follows. \square

5. CONSEQUENCES

Corollary 12. *Let $\mathbf{a} \in \mathbb{C}^n$ be such that $a_i \notin \mathbb{Z}$ and $a_i + a_j \notin \mathbb{Z}$ for all i and j . Let $M \in \hat{\mathcal{C}}$ and $\lambda \in \text{supp}(M)$. Denote by U_0 the centralizer of \mathfrak{h} in $U(\mathfrak{g})$. Then for any $A, B \in U_0$ the actions of A and B on M_λ commute.*

Proof. By Proposition 4, we may assume that $M \cong \text{FV}$. For the module FV the claim follows from the formulae (3). \square

Corollary 13. *For any simple weight cuspidal \mathfrak{g} -module L with finite dimensional weight spaces we have $\dim \text{Ext}_{\mathfrak{g}}^1(L, L) = n$.*

Proof. This follows from Theorem 1 and the observation that a similar equality is true for the unique simple $\mathbb{C}[[t_1, t_2, \dots, t_n]]$ -module. \square

We also recover the main result of [BKLM]:

Corollary 14 ([BKLM]). *The category of all weight cuspidal \mathfrak{g} -modules is semi-simple.*

Proof. By [BKLM, Lemma 2], all blocks of the category of weight cuspidal \mathfrak{g} -modules are equivalent. Hence it is enough to prove the claim for the block containing $N(\mathbf{a})$ for some $\mathbf{a} \in \mathbb{C}^n$ such that $a_i + a_j \notin \mathbb{Z}$ for all i, j . From (3) it follows that the module FV is weight if and only if all operators T_i are semi-simple, hence zero. Therefore from Theorem 1 we get that the block of the category of weight cuspidal modules is equivalent to the category of finite dimensional modules over $\mathbb{C}[[t_1, t_2, \dots, t_n]]/(t_1 - 0, t_2 - 0, \dots, t_n - 0) \cong \mathbb{C}$. The claim follows. \square

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