

CATEGORIFICATION OF WEDDERBURN'S BASIS FOR $\mathbb{C}[S_n]$

VOLODYMYR MAZORCHUK AND CATHARINA STROPPEL

ABSTRACT. M. Neunhöffer studies in [Ne] a certain basis of $\mathbb{C}[S_n]$ with the origins in [Lu] and shows that this basis is in fact Wedderburn's basis. In particular, in this basis the right regular representation of S_n decomposes into a direct sum of irreducible representations (i.e. Specht or cell modules). In the present paper we rediscover essentially the same basis with a categorical origin coming from projective-injective modules in certain subcategories of the BGG-category \mathcal{O} . An important role in our arguments is played by the dominant projective module in each of these categories. As a byproduct of the study of this dominant projective module we show that *Kostant's problem* ([Jo]) has a negative answer for some simple highest weight module over the Lie algebra \mathfrak{sl}_4 , which disproves the general belief that Kostant's problem should have a positive answer for all simple highest weight modules in type A .

1. THE MAIN RESULT

Let n be a positive integer and S_n the group of permutations of the elements from $\{1, 2, \dots, n\}$. Denote by \mathcal{S} the usual set of Coxeter generators of S_n and by $\mathcal{H} = \mathcal{H}(S_n, \mathcal{S})$ the associated (generic) Iwahori-Hecke algebra. The algebra \mathcal{H} is a free $\mathbb{Z}[v, v^{-1}]$ -module with basis $\{H_w | w \in S_n\}$ and multiplication given by

$$H_x H_y = H_{xy} \text{ if } l(x) + l(y) = l(xy), \quad \text{and } H_s^2 = H_e + (v^{-1} - v)H_s \text{ for } s \in \mathcal{S},$$

where $l : S_n \rightarrow \mathbb{Z}$ denotes the length function with respect to \mathcal{S} . Denote by $\{\underline{H}_w | w \in S_n\}$ the *Kazhdan-Lusztig basis*, and by $\{\hat{H}_w | w \in S_n\}$ the dual Kazhdan-Lusztig basis of \mathcal{H} (both in the normalization of [So]).

The group algebra $\mathbb{C}[S_n]$ of S_n is obtained by specializing \mathcal{H} , more precisely: let $\mathbb{Z}[v, v^{-1}]$ act trivially on \mathbb{C} via $v \cdot 1 = 1$, then $\mathbb{C} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathcal{H} = \mathbb{C}[S_n]$ and we have the \mathbb{C} -linear evaluation morphism of algebras as follows:

$$\text{ev} : \mathbb{C} \otimes_{\mathbb{Z}[v, v^{-1}]} \mathcal{H} \rightarrow \mathbb{C}[S_n], \quad 1 \otimes H_w \mapsto w.$$

The Robinson-Schensted correspondence (see e.g. [Sa, 3.1]) defines a bijection between elements $w \in S_n$ and pairs $(a(w), b(w))$ of standard tableaux

The first author was partially supported by STINT, the Royal Swedish Academy of Sciences, and the Swedish Research Council, the second author was partially supported by EPSRC.

with n boxes, such that $a(w)$, $b(w)$ are of the same shape. For every element $w \in S_n$ we denote by $\mathbf{R}_w = \{x \in S_n \mid a(x) = a(w)\}$ the *right cell* of S_n which contains w . Let \bar{w} denote the unique involution in \mathbf{R}_w . The element \bar{w} satisfies (and is defined by the property) $a(\bar{w}) = b(\bar{w}) = a(w)$ and is the *Duflo involution* of \mathbf{R}_w .

Our main result is the construction of a basis of $\mathbb{C}[S_n]$ compatible with its regular right S_n -module structure in the following way:

Theorem 1. *For $w \in S_n$ set $f_w = \text{ev}(\hat{H}_{\bar{w}}H_w)$. Then the following holds:*

- (a) *The elements $\{f_w \mid w \in S_n\}$ form a basis of $\mathbb{C}[S_n]$.*
- (b) *Let $x \in S_n$ and consider the linear span $\mathbf{S}(x)$ of all f_w , $w \in \mathbf{R}_x$. Then $\mathbf{S}(x)$ is invariant with respect to the right action of $\mathbb{C}[S_n]$ and isomorphic to the (irreducible) cell module associated with \mathbf{R}_x .*

In other words, there is a decomposition of the right regular representation of S_n into a direct sum of irreducible modules which is compatible with the basis $\{f_w \mid w \in S_n\}$. (In fact the theorem and its proof are valid over any field of characteristic zero.)

Theorem 1 turns out to be related to the recent paper [Ne], where a similar bases was studied. Let $\{\mathbf{R}_i : i \in I\}$ be a list of right cells in S_n containing exactly one representative from each two-sided cell. For each $i \in I$ and $(x, y) \in \mathbf{R}_i \times \mathbf{R}_i$ set $h_{(x,y)}^i = \text{ev}(\hat{H}_{x^{-1}}H_y)$. From [Ne] it follows that $\{h_{(x,y)}^i : i \in I, (x, y) \in \mathbf{R}_i \times \mathbf{R}_i\}$ has the same properties as $\{f_w \mid w \in S_n\}$; in particular, this set is a basis of $\mathbb{C}[S_n]$ and in this basis the right regular representation of $\mathbb{C}[S_n]$ decomposes into a direct sum of irreducible representations. Moreover, in [Ne] it is even proved that a normalized version of $\{h_{(x,y)}^i \mid i \in I, (x, y) \in \mathbf{R}_i \times \mathbf{R}_i\}$ is in fact Wedderburn's basis of $\mathbb{C}[S_n]$, which means that the elements of this set correspond to matrix units in some decomposition of $\mathbb{C}[S_n]$ into a direct sum of matrix algebras. The origins of the basis $\{h_{(x,y)}^i \mid i \in I, (x, y) \in \mathbf{R}_i \times \mathbf{R}_i\}$ go further back to [Lu]. There is an asymptotic version \mathcal{J} of the Hecke algebra, introduced by Lusztig in [Lu] together with a homomorphism $\Psi : \mathcal{H} \rightarrow \mathbb{Z}[v, v^{-1}] \otimes_{\mathbb{Z}} \mathcal{J}$ which becomes an isomorphism over $\mathbb{Q}(t)$. As pointed out to us by Neunhöffer, the basis $\{h_{(x,y)}^i \mid i \in I, (x, y) \in \mathbf{R}_i \times \mathbf{R}_i\}$ is exactly Lusztig's basis for \mathcal{J} pulled back via the homomorphism Ψ to \mathcal{H} . The precise connection to [Ne] is the following:

Theorem 2. $\{f_w \mid w \in S_n\} = \{h_{(x,y)}^i \mid i \in I, (x, y) \in \mathbf{R}_i \times \mathbf{R}_i\}$.

The origins of Theorem 1, as well as the proof of Theorem 2, are categorical; and this is absolutely crucial for our arguments. In particular, our approach is completely different from the combinatorial approach of [Ne]. There are alternative combinatorial approaches to the construction of a basis for $\mathbb{C}[S_n]$ and some related algebras in which the regular representation decomposed into a direct sum of irreducibles, see [RW], [Mu1], [Mu2], [Mat1], [Mat2]. Some other classical approaches (e.g. [KL], [A11], [A12]) towards

a decomposition of the regular representation of S_n into irreducible representations using an explicit basis lead only to filtrations whose successive subquotients are irreducible.

We finish this introduction with an explicit example:

Example 3. For $n = 3$ let $\mathcal{S} = \{s, t\}$ and $S_3 = \{e, s, t, st, ts, sts = tst\}$. In this case the right cells are $\{e\}$, $\{s, st\}$, $\{t, ts\}$, $\{sts\}$ and we have

| w, \bar{w} | \underline{H}_w | \hat{H}_w | $\hat{H}_{\bar{w}}\underline{H}_w$ |
|--------------|---|--|---|
| e, e | H_e | $H_e - vH_s - vH_t + v^2H_{ts} + v^2H_{st} - v^3H_{tst}$ | $H_e - vH_s - vH_t + v^2H_{ts} + v^2H_{st} - v^3H_{tst}$ |
| s, s | $H_s + vH_e$ | $H_s - vH_{st} - vH_{ts} + v^2H_{tst}$ | $H_e + v^{-1}H_s - vH_t - H_{ts}$ |
| t, t | $H_t + vH_e$ | $H_t - vH_{ts} - vH_{st} + v^2H_{sts}$ | $H_e + v^{-1}H_t - vH_s - H_{st}$ |
| st, s | $H_{st} + vH_t + vH_s + v^2H_e$ | $H_{st} - vH_{sts}$ | $v^{-1}H_{st} + H_s - H_{sts} - vH_{ts}$ |
| ts, t | $H_{ts} + vH_t + vH_s + v^2H_e$ | $H_{ts} - vH_{sts}$ | $v^{-1}H_{ts} + H_t - H_{sts} - vH_{st}$ |
| sts, sts | $H_{sts} + vH_{ts} + vH_{st} + v^2H_s + v^2H_e$ | H_{sts} | $H_e + v^{-1}H_t + v^{-1}H_s + v^{-2}H_{st} + v^{-2}H_{ts} + v^{-3}H_{sts}$ |

Hence, our basis consists of the elements

$$e - s - t + st + ts - sts, e + s - t - ts, e + t - s - st, \\ st + s - ts - sts, ts + t - st - sts, e + t + s + st + ts + sts.$$

Acknowledgment. We thank Ken Brown for very helpful suggestions, Meinhold Geck for information about [Ne], and Michael Rapoport for helpful discussions. We also thank Max Neunhöffer, Susumu Ariki and Andrew Mathas for their remarks on a preliminary version of the paper.

2. PROOF OF THEOREM 1

We prove Theorem 1 by giving an explicit categorical interpretation of all ingredients. This interpretation is based on the categorification of cell modules as established in [MS, Section 4] (the original idea of categorifying the Hecke algebra goes back to [KL] and [BG]). The main players here are certain subquotient categories of the famous BGG category \mathcal{O} (for the latter see [BGG]).

Let \mathcal{O}_0 be the principal block of \mathcal{O} for the simple complex Lie algebra \mathfrak{sl}_n with its standard triangular decomposition. The simple objects in \mathcal{O}_0 are the $L(w)$, $w \in S_n$, the simple highest weight modules with the highest weight $w(\rho) - \rho$, where ρ is the half-sum of all positive roots. Let $\Delta(w)$ and $P(w)$ denote the Verma module, resp. the indecomposable projective

module with unique simple quotient isomorphic to $L(w)$ respectively. Further, denote by θ_w the indecomposable projective endofunctor of \mathcal{O}_0 , with the property $\theta_w P(e) \cong \theta_w \Delta(e) \cong P(w)$ (see [BG]). Finally, let $[\mathcal{O}_0]$ denote the complexified Grothendieck group of \mathcal{O}_0 . For $M \in \mathcal{O}_0$ we denote by $[M]$ its image in $[\mathcal{O}_0]$.

There is a \mathbb{C} -linear isomorphism $\varphi : [\mathcal{O}_0] \rightarrow \mathbb{C}[S_n]$ with $\varphi([\Delta(w)]) = w$. The Kazhdan-Lusztig conjecture ([KL], proved in [BeBe], [BK]) implies that $\varphi([P(w)]) = \text{ev}(\underline{H}_w)$ (for an overview see e.g. [MS, Subsection 3.4]). The standard bilinear form on $\mathbb{C}[S_n]$ is categorified via the bifunctor $\text{Ext}^*(-, -)$ ([KMS1, Section 5] or [MS, Subsection 4.6]). Indecomposable projective and simple modules form dual bases with respect to this categorical form, and so

$$(2.1) \quad \varphi([L(w)]) = \text{ev}(\hat{H}_w)$$

The functors θ_w are exact and induce therefore \mathbb{C} -linear endomorphisms $[\theta_w]$ of $[\mathcal{O}_0]$. By [BG, Theorem 3.4(iv)] and [So] (for a more adjusted reformulation see [MS, Subsection 3.4]) we have, for all M in \mathcal{O}_0 ,

$$(2.2) \quad \varphi([\theta_w M]) = [\theta_w](\varphi([M])) = \varphi([M])\text{ev}(\underline{H}_w).$$

Recall the right cells mentioned above and let $\leq_{\mathbb{R}}$ be the right preorder on S_n . Fix $w \in W$ and set

$$\hat{\mathbb{R}}_w = \{x \in S_n \mid x \leq_{\mathbb{R}} y \text{ for some } y \in \mathbb{R}_w\}.$$

Associated with the right cell \mathbb{R}_w of w we have the full subcategory $\mathcal{O}_0^{\hat{\mathbb{R}}_w}$ of \mathcal{O}_0 , which consists of all modules M with all composition subquotients of the form $L(x)$ with $x \in \hat{\mathbb{R}}_w$. Let

$$Z^{\hat{\mathbb{R}}_w} : \mathcal{O}_0 \rightarrow \mathcal{O}_0^{\hat{\mathbb{R}}_w}$$

be the natural projection functor which takes the maximal quotient that lies in $\mathcal{O}_0^{\hat{\mathbb{R}}_w}$. All this is built up such that we have

$$(2.3) \quad Z^{\hat{\mathbb{R}}_w} \theta_x \cong \theta_x Z^{\hat{\mathbb{R}}_w}$$

for any $x, w \in S_n$, ([MS, Lemma 19]). For $x \in S_n$ we define $P^{\hat{\mathbb{R}}_w}(x) = Z^{\hat{\mathbb{R}}_w} P(x)$, and it follows that

$$(2.4) \quad P^{\hat{\mathbb{R}}_w}(x) \neq 0 \text{ if and only if } x \in \hat{\mathbb{R}}_w.$$

Moreover, the set $\{P^{\hat{\mathbb{R}}_w}(x) \mid x \in \hat{\mathbb{R}}_w\}$ constitutes a complete list of indecomposable projective modules in $\mathcal{O}_0^{\hat{\mathbb{R}}_w}$.

The following Proposition provides a basis of $\mathbb{C}[S_n]$ which has already most of the desired properties:

Proposition 4. *For $x \in S_n$ define $g_x = \varphi([P^{\hat{\mathbb{R}}_x}(x)]) \in \mathbb{C}[S_n]$. Then the following holds:*

(a) $\{g_x \mid x \in S_n\}$ is a basis of $\mathbb{C}[S_n]$.

(b) For every $w \in S_n$ the linear span of $\{g_x | x \in \mathbf{R}_w\}$ is invariant with respect to the right action of S_n and is isomorphic to the cell module associated with \mathbf{R}_w .

Proof. As $|\{g_x | x \in S_n\}| = |S_n| = \dim_{\mathbb{C}} \mathbb{C}[S_n]$, it is enough to show that the elements from $\{g_x | x \in S_n\}$ are linearly independent. By definition of the category $\mathcal{O}_0^{\hat{\mathbf{R}}_x}$, all the simple composition factors of $P^{\hat{\mathbf{R}}_x}(x)$ are of the form $L(z)$ where z is smaller or equal to x in the right cell order. Therefore, when expressed in the specialization $\{\text{ev}(\hat{H}_z) | z \in \mathbb{C}[S_n]\}$ of the dual Kazhdan-Lusztig basis, the element g_x is a linear combination of basis elements, corresponding to $z \in \hat{\mathbf{R}}_x$. By induction on the right order, it is then enough to show that for any $w \in S_n$ the elements from $\{g_x | x \in \mathbf{R}_w\}$ are linearly independent. By [KMS1, Theorem 1] and [MS, Theorem 18], these elements form the Kazhdan-Lusztig basis in the cell module associated with \mathbf{R}_w . The cell module is a subquotient of $\mathbb{C}[S_n]$. Hence these elements are linearly independent already in $\mathbb{C}[S_n]$. The first statement follows.

To prove the invariance it is enough to show, thanks to (2.2), that projective functors preserve the additive subcategory \mathcal{A} of $\mathcal{O}_0^{\hat{\mathbf{R}}_w}$ generated by the indecomposable projective modules $P^{\hat{\mathbf{R}}_w}(x)$, $x \in \mathbf{R}_w$. Since \mathcal{H} is generated by the \underline{H}_s , where s runs through \mathcal{S} , it is enough to show that for any $s \in \mathcal{S}$ and $x \in \mathbf{R}_w$ the module $\theta_s P^{\hat{\mathbf{R}}_w}(x)$ belongs to \mathcal{A} . Now (2.3), [MS, (4.1)] and (2.4) provide the following three isomorphisms:

$$\begin{aligned} \theta_s P^{\hat{\mathbf{R}}_w}(x) = \theta_s Z^{\hat{\mathbf{R}}_w} \theta_x \Delta(e) &\cong Z^{\hat{\mathbf{R}}_w} \theta_s \theta_x \Delta(e) \\ &\cong Z^{\hat{\mathbf{R}}_w} \oplus_{y \geq_{\mathbf{R}_w} x} \theta_y^{m_y} \Delta(e) = \oplus_{y \geq_{\mathbf{R}_w} x} (Z^{\hat{\mathbf{R}}_w} P(y))^{\oplus m_y} \\ &\cong \oplus_{y \in \mathbf{R}_w} \oplus_{i=1}^{m_y} P^{\hat{\mathbf{R}}_w}(y) \end{aligned}$$

for some non-negative integers m_y .

The claim about the invariance follows. The claim about the cell module follows from [MS, Theorem 16 and Theorem 18]. \square

Now the statement of Theorem 1 follows if we prove the following statement

Proposition 5. *We have $f_w = g_w$ for all $w \in S_n$. In particular, Theorem 1 follows from Proposition 4.*

Proof. We already know that $\varphi([L(w)]) = \text{ev}(\hat{H}_w)$ for all $w \in S_n$. Thanks to (2.2) and the definitions of f_w and g_w , the statement of the proposition follows from the following

$$\boxed{\text{Key statement: Let } w \in S_n, \text{ then } \theta_w L(\bar{w}) \cong P^{\hat{\mathbf{R}}_w}(w).}$$

In what follows we prove this statement.

Recall that $P^{\hat{\mathbf{R}}_w}(w) \cong \theta_w P^{\hat{\mathbf{R}}_w}(e)$ by (2.3). To prove the key statement we have to study of the dominant projective module $P^{\hat{\mathbf{R}}_w}(e)$ in $\mathcal{O}_0^{\hat{\mathbf{R}}_w}$ in more detail.

Lemma 6. *Let $x \in \mathbf{R}_w$ be such that $x \neq \bar{w}$. Then $[P^{\hat{\mathbf{R}}_w}(e) : L(x)] = 0$.*

Proof. Recall that the functor θ_x is both left and right adjoint to the functor $\theta_{x^{-1}}$. Hence we have

$$\begin{aligned} [P^{\hat{\mathbf{R}}_w}(e) : L(x)] &= \dim \operatorname{Hom}_{\mathcal{O}}(P^{\hat{\mathbf{R}}_w}(x), P^{\hat{\mathbf{R}}_w}(e)) \\ &= \dim \operatorname{Hom}_{\mathcal{O}}(\theta_x P^{\hat{\mathbf{R}}_w}(e), P^{\hat{\mathbf{R}}_w}(e)) \\ &= \dim \operatorname{Hom}_{\mathcal{O}}(P^{\hat{\mathbf{R}}_w}(e), \theta_{x^{-1}} P^{\hat{\mathbf{R}}_w}(e)). \end{aligned}$$

On the other hand $a(x^{-1}) = b(x)$ and $b(x^{-1}) = a(x)$ ([Sa, Theorem 3.6.6]), in particular, $a(\bar{w}) = b(\bar{w})$ for the involution \bar{w} . As $x \neq \bar{w}$, we have $x \neq x^{-1}$, and hence

$$a(x^{-1}) = b(x) \neq a(x) = a(\bar{w}).$$

Thus $x^{-1} \notin \mathbf{R}_w$. Since $a(x^{-1})$ and $a(x)$ still have the same shape, it follows that $x^{-1} \notin \hat{\mathbf{R}}_w$ ([BjBr, Exercise 10, page 198]). Therefore

$$\theta_{x^{-1}} P^{\hat{\mathbf{R}}_w}(e) = \theta_{x^{-1}} Z^{\hat{\mathbf{R}}_w} \Delta(e) \cong Z^{\hat{\mathbf{R}}_w} \theta_{x^{-1}} \Delta(e) \cong Z^{\hat{\mathbf{R}}_w} P(x^{-1}) = 0$$

and therefore $\dim \operatorname{Hom}_{\mathcal{O}}(P^{\hat{\mathbf{R}}_w}(e), \theta_{x^{-1}} P^{\hat{\mathbf{R}}_w}(e)) = 0$ as well. \square

Lemma 7. *For any $x \in \mathbf{R}_w$ and $y \in \hat{\mathbf{R}}_w \setminus \mathbf{R}_w$ we have $\theta_x L(y) = 0$. In particular, $[P^{\hat{\mathbf{R}}_w}(e) : L(\bar{w})] > 0$.*

Proof. As $P^{\hat{\mathbf{R}}_w}(y) \twoheadrightarrow L(y)$ and θ_x is exact, we have $\theta_x P^{\hat{\mathbf{R}}_w}(y) \twoheadrightarrow \theta_x L(y)$. Applying (2.3) we even have that $\theta_x L(y)$ is a homomorphic image of the module $Z^{\hat{\mathbf{R}}_w} \theta_x \theta_y \Delta(e)$.

Note that $\theta_x L(y) \in \mathcal{O}_0^{\hat{\mathbf{R}}_y}$, in particular, all simple subquotients of $\theta_x L(y)$ have the form $L(z)$, $z \in \hat{\mathbf{R}}_y$.

On the other hand, it follows from [MS, (4.1)] that $\theta_x \theta_y$ is a direct sum of functors of the form θ_z , where $z \geq_{\mathbf{R}} x$. Hence, by (2.4), all simple quotients of the module $Z^{\hat{\mathbf{R}}_w} \theta_x \theta_y \Delta(e)$ have the form $L(z)$, $z \in \mathbf{R}_w$. As $\mathbf{R}_w \cap \hat{\mathbf{R}}_y = \emptyset$, we must have $\theta_x L(y) = 0$.

We know that $P^{\hat{\mathbf{R}}_w}(\bar{w}) = \theta_{\bar{w}} P^{\hat{\mathbf{R}}_w}(e) \neq 0$. By Lemma 6 and the above, $L(\bar{w})$ is the only subquotient of $P^{\hat{\mathbf{R}}_w}(e)$ which has the chance not to be annihilated by $\theta_{\bar{w}}$. Altogether we must have $[P^{\hat{\mathbf{R}}_w}(e) : L(\bar{w})] > 0$ \square

Lemma 8. $[P^{\hat{\mathbf{R}}_w}(e) : L(\bar{w})] = 1$.

Proof. Assume for a moment that \mathbf{R}_w contains an element of the form $w'_0 w_0$, where w_0 is the longest element of S_n and w'_0 is the longest element of some parabolic (Young) subgroup W of S_n . Then the modules $P^{\hat{\mathbf{R}}_w}(x)$, $x \in \mathbf{R}_w$, are exactly the indecomposable projective-injective modules in the parabolic subcategory \mathcal{O}_0^W (in the sense of [R-C]) of \mathcal{O}_0 ([MS, Remark 14]).

Amongst the indecomposable projective-injective modules in \mathcal{O}_0^W there is, due to [IS, 3.1], a special one which is obtained as a translation of some simple projective module (out of possibly several walls). Since translation to walls maps simple modules to simples or zero, the special module, call it P , is thus obtained as a translation of some $L(x)$ for some $x \in \mathbf{R}_w$.

From [KMS1, Theorem 1] it further follows that translating P and taking appropriate direct summands, we will finally get all $P^{\hat{\mathbf{R}}_w}(x)$, $x \in \mathbf{R}_w$. This implies the existence of an indecomposable projective functor θ_y such that the module $\theta_y L(\bar{w})$ contains $P^{\hat{\mathbf{R}}_w}(\bar{w})$ as a direct summand (see [MS, 5.1]). By [MS, Theorem 18], the above restriction that the right cell should contain $w'_0 w_0$ is in fact superfluous. Moreover, from [MS, Theorem 18] it also follows that the module $P^{\hat{\mathbf{R}}_w}(\bar{w})$ is an injective object in $\mathcal{O}^{\hat{\mathbf{R}}_w}$ (and so the same holds for any $P^{\hat{\mathbf{R}}_w}(x)$, $x \in \mathbf{R}_w$).

Consider now $\theta_y P^{\hat{\mathbf{R}}_w}(e) \cong P^{\hat{\mathbf{R}}_w}(y)$. As $P^{\hat{\mathbf{R}}_w}(\bar{w})$ is both projective and injective, from Lemma 7 it follows that $P^{\hat{\mathbf{R}}_w}(\bar{w})$ must be a direct summand of $P^{\hat{\mathbf{R}}_w}(y)$. As $P^{\hat{\mathbf{R}}_w}(y)$ is indecomposable, this forces $P^{\hat{\mathbf{R}}_w}(y) \cong P^{\hat{\mathbf{R}}_w}(\bar{w})$, $y = \bar{w}$, and finally $[P^{\hat{\mathbf{R}}_w}(e) : L(\bar{w})] = 1$. \square

From Lemma 7 and Lemma 8 it follows that for any $x \in \mathbf{R}_w$ we have $\theta_x P^{\hat{\mathbf{R}}_w}(e) \cong \theta_x L(\bar{w})$. This finally proves the key statement and at the same time completes the proof of Proposition 5 and Theorem 1. \square

Remark 9. Let $w \in S_n$ be such that the right cell \mathbf{R}_w contains the element $w'_0 w_0$ for some Young subgroup W of S_n . Then $\mathcal{O}_0^{\hat{\mathbf{R}}_w}$ is the regular block of the parabolic category \mathcal{O} (in the sense of [R-C]) associated with W . The elements f_x , $x \not\leq_{\mathbf{R}} w$, form a basis of a submodule N of $\mathbb{C}[S_n]$. The quotient $\mathbb{C}[S_n]/N$ is isomorphic to the induced sign module $\mathbb{C}[S_n] \otimes_{\mathbb{C}[W]} \text{sign}$ (see [MS, 6.2.1] for details). The classes of the elements f_x , $x \leq_{\mathbf{R}} w$, thus form a basis of the induced sign module. Alternatively, the elements f_x , $x \leq_{\mathbf{R}} w$, form a basis of a submodule of $\mathbb{C}[S_n]$, which is isomorphic to the induced sign module.

3. PROOF OF THEOREM 2

Analogously to the previous section we interpret $h_{(x,y)}^i = \varphi([\theta_y L(x^{-1})])$ for each $i \in I$ and $(x, y) \in \mathbf{R}_i \times \mathbf{R}_i$. Let $i \in I$ be fixed. Because of Proposition 5 and the definition of g_w 's, to prove Theorem 2 it is enough to show that every $\theta_y L(x^{-1})$ is a projective-injective module in $\mathcal{O}_0^{\hat{\mathbf{R}}_x^{-1}}$. In the case $x = \bar{y}$ this follows from the Key statement of Section 2.

Let now $x \in \mathbf{R}_i$ be arbitrary. As x and \bar{y} belong to the same right cell, the elements x^{-1} and \bar{y} belong to the same left cell. Let \mathcal{A} and \mathcal{B} denote the additive categories of projective-injective modules in $\mathcal{O}_0^{\hat{\mathbf{R}}_{\bar{y}}}$ and $\mathcal{O}_0^{\hat{\mathbf{R}}_x^{-1}}$

respectively. In [MS, Section 5] it was shown that there exists an equivalence $F : \mathcal{A} \rightarrow \mathcal{B}$ which commutes with projective functors and satisfies $F(P^{\hat{R}_{\bar{y}}}(y)) = P^{\hat{R}_{x^{-1}}}(x^{-1})$.

Let $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ denote the full subcategories of respectively $\mathcal{O}_0^{\hat{R}_{\bar{y}}}$ and $\mathcal{O}_0^{\hat{R}_{x^{-1}}}$ which consist of all modules X having a two step presentation $M_1 \rightarrow M_0 \rightarrow X \rightarrow 0$, where $M_1, M_0 \in \mathcal{A}$ or $M_1, M_0 \in \mathcal{B}$ respectively. Then F extends, in the obvious way, to an equivalence $\overline{F} : \overline{\mathcal{A}} \rightarrow \overline{\mathcal{B}}$ which commutes with projective functors.

Let $\overline{L(\bar{y})}$ denote the quotient of $P^{\hat{R}_{\bar{y}}}(y)$ modulo the trace of all modules from \mathcal{A} in the radical of $P^{\hat{R}_{\bar{y}}}(y)$. Define $\overline{L(x^{-1})}$ analogously. Then $\overline{L(\bar{y})}$ has simple top $L(\bar{y})$ and all other subquotients of $\overline{L(\bar{y})}$ are of the form $L(z)$, where $z <_{\mathbb{R}} \bar{y}$. Analogously $\overline{L(x^{-1})}$ has simple top $L(x^{-1})$ and all other subquotients of $\overline{L(x^{-1})}$ are of the form $L(z)$, where $z <_{\mathbb{R}} x^{-1}$. From the above construction we have $\overline{F}(\overline{L(\bar{y})}) = \overline{L(x^{-1})}$. Further $\theta_y \overline{L(\bar{y})} = \theta_y L(\bar{y})$ by Lemma 7. Analogous arguments imply $\theta_y \overline{L(x^{-1})} = \theta_y L(x^{-1})$. Adding everything up we have

$$\theta_y L(x^{-1}) = \theta_y \overline{L(x^{-1})} = \theta_y \overline{F}(\overline{L(\bar{y})}) = \overline{F}(\theta_y \overline{L(\bar{y})}) = \overline{F}(\theta_y L(\bar{y})) = F(\theta_y L(\bar{y})).$$

Hence $\theta_y L(x^{-1}) = F(\theta_y L(\bar{y}))$ is a projective-injective module in $\mathcal{O}_0^{\hat{R}_{x^{-1}}}$. The claim follows.

4. WHAT HAPPENS FOR THE INTEGRAL HECKE ALGEBRA?

Unfortunately, the elements $\overline{f}_w = \hat{H}_w H_w$, $w \in S_n$, do not form a basis in \mathcal{H} for $n > 1$. Indeed, already for $n = 2$ we have $S_2 = \{e, s\}$ and it is easy to compute that $\overline{f}_e = H_e - v H_s$ and $\overline{f}_s = H_e + v^{-1} H_s$. Now, the element H_s is not a $\mathbb{Z}[v, v^{-1}]$ -linear combination of \overline{f}_e and \overline{f}_s (in order to get H_s we had to be allowed to divide by $v + v^{-1}$). The categorical reason for this failure is the fact that the algebra

$$\text{End}_{\mathcal{O}}(\oplus_{w \in S_n} P^{\hat{R}_w}(w))$$

is self-injective and not semi-simple for $n > 1$. In particular, for such n it has infinite global dimension. This means that there is a simple module, whose image in the Grothendieck group can only be presented as an infinite linear combination of images of projective modules.

After a proper reformulation, this can be rectified by imposing an appropriate condition of divisibility (for example the divisibility by $v + v^{-1}$ in the example above). This explains in particular why we complexified the Grothendieck group, i.e. extended the scalars to \mathbb{C} (or any other field of characteristic zero) from the very beginning: this ensure this divisibility.

However, the situation is not too bad: the elements \overline{f}_w generate in \mathcal{H} a submodule, which is isomorphic to the direct sum of all cell modules.

We also note that in [Ne] the author does not work with the integral Hecke algebra but rather extends the scalars to a certain field of characteristic zero. In particular, the normalized basis defined in [Ne] is not a basis for \mathcal{H} either.

5. AN APPLICATION TO KOSTANT'S PROBLEM

The core object $\Delta^{\hat{R}_w}(e)$ of our study in Section 2 has an unexpected application to the so-called Kostant's problem from [Jo]; see also [Ja, Kapitel 6].

Let \mathfrak{g} be a complex reductive finite-dimensional Lie algebra. For every \mathfrak{g} -module M we have the bimodule $\mathcal{L}(M, M)$ of all \mathbb{C} -linear endomorphisms of M on which the adjoint action of the universal enveloping algebra $U(\mathfrak{g})$ is locally finite. (That means any vector $f \in \mathcal{L}(M, M)$ lies inside a finite dimensional subspace which is stable under the adjoint action defined as $x.f(m) = x(f(m)) - f(xm)$ for $x \in \mathfrak{g}$, $m \in M$). Initiated by [Jo], *Kostant's problem* became the standard terminology for the following question concerning an arbitrary \mathfrak{g} -module M :

Is the natural injection $U(\mathfrak{g})/\text{Ann}(M) \hookrightarrow \mathcal{L}(M, M)$ surjective?

Although there are several classes of modules for which the answer is known to be positive (see [Jo], [Ma], [MS] and references therein), a complete answer to this problem seems to be far away - the problem is not even solved for simple highest weight modules. In [Jo, 9.5] an example of a simple highest weight module in type B_2 , for which the answer is negative is mentioned (for details see [MS, 11.5]). In this section we use the module $\Delta^{\hat{R}_w}(e)$ to construct another example in type A_3 , which disproves a general belief that the answer to Kostant's problem is positive for simple highest weight modules in type A (this belief was based on [Jo, 9.1] and further strengthened by [MS, Theorem 60]).

Let $n = 4$ and $r = (12)$, $s = (23)$, $t = (34)$ be the standard Coxeter generators of S_4 . Consider $w = rt = \bar{w}$. In this case we have $R_w = \{rt, rts\}$ and $\hat{R}_w = \{rt, rts, t, ts, tsr, r, rs, rst, e\}$. We consider the graded version of \mathcal{O} as worked out in [St1]. Using [St2, Appendix] one computes that the module $N = \Delta^{\hat{R}_w}(e)$ has the following graded filtration (resp. socle or radical filtration)

$$N = \begin{array}{ccc} & e & \\ & r & t \\ & & rt \end{array}$$

where we abbreviate $L(x)$ simply by x .

Lemma 10. $\text{Ann}(L(rt)) = \text{Ann}(N)$

Proof. Let Y_r and Y_t denote some non-zero elements from the negative root spaces corresponding to r and t respectively. Let further U' be the localization of $U(\mathfrak{sl}_4)$ with respect to the multiplicative set $\{Y_r^i Y_t^j \mid i, j \geq 0\}$. As $rt > r$ and $rt > t$ with respect to the Bruhat order, both Y_r and Y_t act

injectively on $L(rt)$. Hence $L(rt)$ will be the simple socle of the \mathfrak{sl}_4 -module $N' = U' \otimes_{U(\mathfrak{sl}_4)} L(rt)$. As $t > e$ it is further easy to see (for example using the results of [KM, Section 4]) that N is a submodule of N' . Hence the statement of the lemma would follow if we would prove that $\text{Ann}(L(rt)) = \text{Ann}(N')$. In fact, as $L(rt) \subset N'$, we have only to prove that $\text{Ann}(L(rt)) \subset \text{Ann}(N')$. This however, follows from the following statement:

Lemma 11. *Let \mathfrak{g} be a semi-simple finite-dimensional Lie algebra, $0 \neq x \in \mathfrak{g}$ some root vector, and M a \mathfrak{g} -module on which x acts injectively. Let U' be the localization of $U(\mathfrak{g})$ with respect to the powers of X . Then $\text{Ann}(M) \subset \text{Ann}(M')$, where $M' = U' \otimes_{U(\mathfrak{g})} M$.*

Proof. The set $X := \{x^i \mid i \geq 0\}$ is an Ore set in $\mathcal{U}(\mathfrak{g})$ with $X \cap \text{Ann}(M) = \emptyset$ by hypothesis. So $U' \text{Ann}(M) = \text{Ann}(M)U'$ is a proper ideal in U' . This means $\text{Ann}(M)M' = \text{Ann}(M)U'M = U' \text{Ann}(M)M = \{0\}$. This completes the proof. \square

\square

Lemma 12. (a) *The module $\theta_t \theta_s \theta_r N$ has the following graded filtration:*

$$\begin{array}{cccc} & & rst & \\ & rs & & rt \\ rst & tsr & trs & r \\ & & rt & \end{array}$$

(b) *The module $\theta_t \theta_s \theta_r L(rt)$ is a submodule of the module $\theta_t \theta_s \theta_r N$ and has the following graded filtration:*

$$\begin{array}{ccc} & & rt \\ tsr & trs & r \\ & & rt \end{array}$$

Proof. This is verified by direct computations. \square

Theorem 13. *Kostant's problem has a negative answer for $L(rt)$.*

Proof. As N is a quotient of the dominant Verma module, Kostant's problem has a positive solution for N by [Ja, 6.9]. Hence $\mathcal{L}(N, N) = U(\mathfrak{sl}_4)/\text{Ann}(N)$. By Lemma 10, we have $\text{Ann}(N) = \text{Ann}(L(rt))$ and hence we also have $U(\mathfrak{sl}_4)/\text{Ann}(N) = U(\mathfrak{sl}_4)/\text{Ann}(L(rt))$. From Lemma 12(a) we obtain that $\dim \text{Hom}_{\mathcal{O}}(N, \theta_t \theta_s \theta_r N) = 0$ (as for the top $L(e)$ of N we have $[\theta_t \theta_s \theta_r N : L(e)] = 0$), while $\dim \text{Hom}_{\mathcal{O}}(L(rt), \theta_t \theta_s \theta_r L(rt)) \neq 0$ by Lemma 12(b) (as $L(rt)$ obviously occurs in the socle of $\theta_t \theta_s \theta_r L(rt)$). This implies $\mathcal{L}(N, N) \neq \mathcal{L}(L(rt), L(rt))$, which, in turn, yields $\mathcal{L}(L(rt), L(rt)) \neq U(\mathfrak{sl}_4)/\text{Ann}(L(rt))$. The claim follows. \square

REFERENCES

- [All1] E. Allen, The decomposition of a bigraded left regular representation of the diagonal action of S_n . J. Combin. Theory Ser. A **71** (1995), no. 1, 97–111.

- [Al2] E. Allen, A conjecture of Procesi and a new basis for the decomposition of the graded left regular representation of S_n . *Adv. Math.* **100** (1993), no. 2, 262–292.
- [BeBe] A. Beilinson, J. Bernstein, Localisation de g -modules. *C. R. Acad. Sci. Paris Sr. I Math.* **292** (1981), no. 1, 15–18.
- [BG] J. Bernstein, S. Gelfand, Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. *Compositio Math.* **41** (1980), no. 2, 245–285.
- [BGG] J. Bernstein, I. Gelfand, S. Gelfand, A certain category of \mathfrak{g} -modules. *Funkcional. Anal. i Priložen.* **10** (1976), no. 2, 1–8.
- [BjBr] A. Björner, F. Brenti, *Combinatorics of Coxeter groups*. Graduate Texts in Mathematics, **231**. Springer, 2005.
- [BK] J.-L. Brylinski, M. Kashiwara, Kazhdan-Lusztig conjecture and holonomic systems. *Invent. Math.* **64** (1981), no. 3, 387–410.
- [IS] R. Irving, B. Shelton, Loewy series and simple projective modules in the category \mathcal{O}_S . *Pacific J. Math.* **132** (1988), no. 2, 319–342.
- [Ja] J. C. Jantzen, *Einhüllende Algebren halbeinfacher Lie-Algebren*. *Ergebnisse der Mathematik und ihrer Grenzgebiete 3*. Springer-Verlag, Berlin, 1983.
- [Jo] A. Joseph, Kostant's problem, Goldie rank and the Gelfand-Kirillov conjecture. *Invent. Math.* **56** (1980), no. 3, 191–213.
- [KL] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras. *Invent. Math.* **53** (1979), no. 2, 165–184.
- [KM] O. Khomenko, V. Mazorchuk, On Arkhipov's and Enright's functors. *Math. Z.* **249** (2005), no. 2, 357–386.
- [KMS1] M. Khovanov, V. Mazorchuk, C. Stroppel, A categorification of integral Specht modules, to appear in *Proc. AMS*, preprint math.RT/0607630.
- [Lu] G. Lusztig, Leading coefficients of character values of Hecke algebras. *The Arcata Conference on Representations of Finite Groups* (Arcata, Calif., 1986), 235–262, *Proc. Sympos. Pure Math.*, **47**, Part 2, Amer. Math. Soc., Providence, RI, 1987.
- [Mat1] A. Mathas, Matrix units and generic degrees for the Ariki-Koike algebras. *J. Algebra* **281** (2004), no. 2, 695–730.
- [Mat2] A. Mathas, Seminormal forms and Gram determinants for cellular algebras, preprint math/0604108, to appear in *J. Reine Angew. Math.*
- [Ma] V. Mazorchuk, A twisted approach to Kostant's problem. *Glasg. Math. J.* **47** (2005), no. 3, 549–561.
- [MS] V. Mazorchuk, C. Stroppel, Categorification of (induced) cell modules and the rough structure of generalized Verma modules, arXiv:math/0702811
- [Mu1] G. Murphy, On the representation theory of the symmetric groups and associated Hecke algebras. *J. Algebra* **152** (1992), no. 2, 492–513.
- [Mu2] G. Murphy, The representations of Hecke algebras of type A_n . *J. Algebra* **173** (1995), no. 1, 97–121.
- [Ne] M. Neunhöffer, Kazhdan-Lusztig basis, Wedderburn decomposition, and Lusztig's homomorphism for Iwahori-Hecke algebras. *J. Algebra* **303** (2006), no. 1, 430–446.
- [RW] A. Ram, H. Wenzl, Matrix units for centralizer algebras. *J. Algebra* **145** (1992), no. 2, 378–395.
- [R-C] A. Rocha-Caridi, Splitting criteria for \mathfrak{g} -modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite-dimensional, irreducible \mathfrak{g} -module. *Trans. Amer. Math. Soc.* **262** (1980), no. 2, 335–366.
- [Sa] B. Sagan, *The symmetric group. Representations, combinatorial algorithms, and symmetric functions*. Second edition. Graduate Texts in Mathematics, **203**. Springer-Verlag, New York, 2001.

- [So] W. Soergel, Kazhdan-Lusztig polynomials and a combinatorics for tilting modules. Represent. Theory **1** (1997), 83–114.
- [St1] C. Stroppel, Category \mathcal{O} : gradings and translation functors. J. Algebra **268** (2003), no. 1, 301–326.
- [St2] C. Stroppel, Category \mathcal{O} : quivers and endomorphism rings of projectives. Represent. Theory **7** (2003), 322–345.

V. M.: DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY (SWEDEN).
E-mail address: `mazor@math.uu.se`

C. S.: DEPARTMENT OF MATHEMATICS, GLASGOW UNIVERSITY (UNITED KINGDOM).
E-mail address: `c.stroppel@maths.gla.ac.uk`