Koszul duality for extension algebras of standard modules

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Abstract

We define and investigate a class of Koszul quasi-hereditary algebras for which there is a natural equivalence between the bounded derived category of graded modules and the bounded derived category of graded modules over (a proper version of) the extension algebra of standard modules. Examples of such algebras include, in particular, the multiplicity free blocks of the BGG category $\mathcal{O}$, and some quasi-hereditary algebras with Cartan decomposition in the sense of König.

1 Introduction

For a finite-dimensional Koszul algebra, A, of finite global dimension there is a nutural equivalence between the bounded derived category $\mathcal{D}^b(A-\mathrm{gmod})$ of graded A-modules and the bounded derived category of graded modules over the Yoneda extension algebra $E(A)$ of A, see [BGS]. This equivalence is produced by the so-called Koszul duality functor. If A is quasi-hereditary and satisfies some natular assumptions on the resolutions of standard and costandard modules (see [ADL]), then the algebra $E(A)$ is also quasi-hereditary and the Koszul duality functor behaves well with respect to this structure. Some time ago S. Ovsienko in a private communication expressed a hope that for (some) graded Koszul quasi-hereditary algebras it might be possible that $\mathcal{D}^b(A-\mathrm{gmod})$ is equivalent to the bounded derived category of graded modules for the extension algebra $\mathrm{Ext}_A^*(\Delta,\Delta)$ of the direct sum Δ of all standard modules for A. The reason for this hope is the fact that every quasi-hereditary algebra has two natural families of homologically orthogonal modules, namely standard and costandard modules. Both these families generate $\mathcal{D}^b(A-\mathrm{gmod})$ as a triangulated category. The idea of Ovisenko was to organize the equivalence between the derived categories such that the standard A-modules become projective objects and the corresponding costandard A-modules become simple objects. In particular, it should follow automatically that $\mathrm{Ext}_A^*(\Delta,\Delta)$ is Koszul, and its Koszul dual should be isomorphic to the extension algebra $\mathrm{Ext}_A^*(\nabla,\nabla)$ of the costandard module ∇ for A.

In the present paper we define and investigate a big family of graded quasi-hereditary algebras for which Ovsienko's idea works. However, the passage from A to $\mathrm{Ext}_A^*(\Delta,\Delta)$ is not painless. There is of course a trivial case, when A is directed. In this case we have either $A \cong \mathrm{Ext}_A^*(\Delta,\Delta)$ or $E(A) \cong \mathrm{Ext}_A^*(\Delta,\Delta)$. In all other cases one quickly comes to the problem that the "natural" gradation induced on $\mathrm{Ext}_A^*(\Delta, \Delta)$ from $\mathcal{D}^b(A-\mathrm{gmod})$ is a $\mathbb{Z}^2$-gradation and not a $\mathbb{Z}$-gradation. In fact, we were not able to find any "natural" copy of the category of graded $\mathrm{Ext}_A^*(\Delta, \Delta)$-modules inside $\mathcal{D}^b(A-\mathrm{gmod})$. However, under two special conditions, (I) and (II), see Subsection 2.5, which we impose on the algebras we consider in this paper, we single out inside $\mathcal{D}^b(A-\mathrm{gmod})$ a subcategory of graded modules over certain $\mathbb{Z}$-graded category (not algebra!), $\mathcal{B}$, whose bounded derived category is naturally equivalent to $\mathcal{D}^b(A-\mathrm{gmod})$. Additionally, the category $\mathcal{B}$ carries a natural free action of $\mathbb{Z}$. The quotient modulo this action happens to be exactly $\mathrm{Ext}_A^*(\Delta, \Delta)$ with the induced $\mathbb{Z}^2$-gradation. As a consequence, we have to extend our setup and consider modules over categories rather than those over algebras. This forces us to reformulate and extend many classical notions and results (like Koszul algebras, quasi-hereidtary algebras, Rickard-Morita Theorem etc.) in our more general setup.

The paper is organized as follows: in Section 2 we collect all necessary preliminaries about the categories and algebras we consider. In Section 3 we get some preliminary information about the quasi-hereditary categories satisfying (I) and (II). In Section 4 we formulate and prove our main result. We finish the paper with a discussion on several applications of our result in Section 5.

2 Some generalities

2.1 Modules over categories and Rickard–Morita Theorem

Let $\Bbbk$ be an algebraically closed field and $D = \mathrm{Hom}_{\Bbbk}(\_, \Bbbk)$ denote the usual duality. Since we need not only modules over algebras, but also those over categories, we include the main definitions concerning them. All categories under consideration will be *linear categories* over some $\Bbbk$. It means that all sets of morphisms $\mathscr{A}(x, y)$ in such a category, $\mathscr{A}$, are $\Bbbk$-vector spaces and the multiplication is $\Bbbk$-bilinear. Moreover, we suppose that these categories are *small*, i.e. their classes of objects are sets. All functors are supposed to be $\Bbbk$-linear. An *$\mathscr{A}$-module* is, by definition, a functor $M : \mathscr{A} \to \mathscr{V}ec$ (the category of $\Bbbk$-vector spaces). For an element, $m \in M(x)$, and a morphism, $\alpha : x \to y$, we write αm instead of $M(\alpha)m$, etc. We denote by $\mathscr{A}$-Mod the category of all $\mathscr{A}$-modules. A *representable* module is one isomoprhic to $\mathscr{A}^x = \mathscr{A}(x, \_)$ for some object x. Such functors are projective objects in the category $\mathscr{A}$-Mod and every projective object in this category is a direct summand of a direct sum (maybe infinite) of representable functors. Just in the same way, the functors $\mathscr{A}_x = \mathscr{A}(\_, x)$ are projective objects in the category of $\mathscr{A}^{\mathrm{op}}$-modules, where $\mathscr{A}^{\mathrm{op}}$ denotes the *opposite category.* A *set of generators* of an $\mathscr{A}$-module, M, is a subset, $S \subseteq \bigcup_{x \in \mathrm{ob}\, \mathscr{A}} M(x)$, such that any element $m \in M$ can be expressed as $\sum_{u \in S} \alpha_u u$, where all $\alpha_u \in \mathrm{mor}\, \mathscr{A}$ and only finitely many of these morphisms are nonzero. Especially, $\{ 1_x \}$ is a set of generators of $\mathscr{A}^x$, as well as of $\mathscr{A}_x$.

Recall that, if a category, $\mathscr{C}$, has infinite direct sums, an object, C, is called *compact* if the functor $\mathscr{C}^C$ preserves arbitrary direct sums. For instance, finitely generated modules are compact objects of $\mathscr{A}$-Mod. Suppose now that $\mathscr{A}$ is *basic*, i.e. different objects of $\mathscr{A}$ are nonisomorphic and there are no nontrivial idempotents in all algebras $\mathscr{A}(x,x)$, $x \in \operatorname{ob}\mathscr{A}$. We denote by $\mathscr{A}$-mod the category of *finite dimensional* $\mathscr{A}$-modules., that is those modules M for which all spaces $M(x)$ are finite dimensional and $M(x) = 0$ for all but a finite number of objects x. Equivalently, $\bigoplus_{x\in\operatorname{ob}\mathscr{A}} M(x)$ is finite dimensional. If all modules $\mathscr{A}^x$ and $\mathscr{A}_x$ are finite dimensional, we call $\mathscr{A}$ a *bounded category.*

We denote by $\mathcal{D}\mathscr{A}$ the *derived category* of the category $\mathscr{A}$-Mod; by $\mathcal{D}^+\mathscr{A}$, $\mathcal{D}^-\mathscr{A}$ and $\mathcal{D}^b\mathscr{A}$, respectively, its full subcategories consiting of *right bounded, left bounded* and (two-sided) bounded complexes. The shift in $\mathcal{D}\mathscr{A}$ will be denoted by $C^\bullet \mapsto C^\bullet[1]$; actually $C^n[1] = C^{n+1}$. By $\mathcal{D}^{\mathrm{per}}\mathscr{A}$ we denote the full subcategory of $\mathcal{D}\mathscr{A}$ consisting of *perfect complexes*, i.e. those isomorphic (in $\mathcal{D}\mathscr{A}$) to bounded complexes of finitely generated projective modules. It is known that perfect complexes are just the compact objects of $\mathcal{D}\mathscr{A}$. The category $\mathcal{D}^{\mathrm{per}}\mathscr{A}$ can be identified with the *bounded homotopy category* $\mathcal{H}^b(\mathscr{A}\text{-proj})$, i.e. the factorcategory of the category of finite complexes of finitely generated projective $\mathscr{A}$-modules modulo homotopy. The projective modules (or further their canonical images) generate $\mathcal{D}^{\mathrm{per}}\mathscr{A}$ as a triangulated category. We recall the following theorem by Rickard [Ric]. Though in [Ric], as well as in other papers, such as [Ke], only the case, when $\mathscr{A}$ and $\mathscr{B}$ are $\mathbb{k}$-algebras (or, the same, categories with one object) was considered, the proofs remain almost intact in this more general case, so we omit them.

Theorem 2.1 (Rickard–Morita Theorem). *Let $\mathscr{A}$ and $\mathscr{B}$ be two small $\mathbb{k}$-linear categories. Then the follwoing conditions are equivalent:*

1. *There is a triangle equivalence $\mathcal{D}\mathscr{A} \xrightarrow{\sim} \mathcal{D}\mathscr{B}$.*
2. *There is a triangle equivalence $\mathcal{D}^*\mathscr{A} \xrightarrow{\sim} \mathcal{D}^*\mathscr{B}$, where $*$ can be replaced by any of the symbols $+$, $-$, b or ${}^{\mathrm{per}}$.*
3. *There is a full subcategory $\mathscr{X} \subset \mathcal{D}^{\mathrm{per}}\mathscr{A}$ such that*
   (a) *$\mathscr{X} \simeq \mathscr{B}^{\mathrm{op}}$;*
   (b) *$\operatorname{Hom}_{\mathcal{D}\mathscr{A}}(X, X'[k]) = 0$ for any $X, X' \in \mathscr{X}$ and any $k \neq 0$;*
   (c) *$\mathscr{X}$ generates $\mathcal{D}^{\mathrm{per}}\mathscr{A}$ as a triangulated category.*

Moreover, in this case, any equivalence $T : \mathscr{X} \xrightarrow{\sim} \mathscr{B}^{\mathrm{op}}$ can be extended to a triangle equivalence $F : \mathcal{D}\mathscr{A} \xrightarrow{\sim} \mathcal{D}\mathscr{B}$ such that $FX = \mathscr{B}^{TX}$ for every $X \in \mathscr{X}$. In particular, if $\mathscr{B} = \mathscr{X}^{\mathrm{op}}$, $FX = \mathscr{X}_X = \operatorname{Hom}_{\mathcal{D}\mathscr{A}}(\_, X)$.

In fact, given an equivalence, $\Phi : \mathcal{D}\mathscr{B} \xrightarrow{\sim} \mathcal{D}\mathscr{A}$, one can set $\mathscr{X} = \{\, \Phi\mathscr{B}^x \mid x \in \operatorname{ob}\mathscr{B}\}$. Note that, since $\mathscr{B}^{TX}$ is a finitely generated projective $\mathscr{B}$-module, then one also has $\operatorname{RHom}_{\mathscr{A}}(X, \_) \simeq \operatorname{RHom}_{\mathscr{B}}(\mathscr{B}^{TX}, F\_)$. Thus, for every complex $C^\bullet$ of $\mathscr{A}$-modules we have $\operatorname{RHom}_{\mathscr{A}}(X, \mathscr{C}^\bullet) \simeq FC^\bullet(TX)$ in $\mathcal{D}\mathscr{B}$.

The set of objects of a full subcategory $\mathscr{X} \subseteq \mathcal{D}\mathscr{A}$ satisfying conditions 3b and 3c will be called a *tilting subset* in $\mathcal{D}\mathscr{A}$.

2.2 Graded categories, graded modules and group actions

Let $\mathbf{G}$ be a semigroup. A $\mathbf{G}$-*grading* of a category, $\mathscr{A}$, consists of decompositions $\mathscr{A}(x,y) = \bigoplus_{\sigma\in\mathbf{G}} \mathscr{A}(x,y)_\sigma$ given for any objects $x, y \in \mathscr{A}$, such that, for every $x, y, z \in \operatorname{ob}\mathscr{A}$ and for every $\sigma, \tau \in \mathbf{G}$, $\mathscr{A}(y,z)_\tau \mathscr{A}(x,y)_\sigma \subseteq \mathscr{A}(x,z)_{\sigma\tau}$. A category $\mathscr{A}$ with a fixed $\mathbf{G}$-grading is called a $\mathbf{G}$-*graded category.* The morphisms $\alpha \in \mathscr{A}(x,y)_\sigma$ are called *homogeneous of degree* σ, and we shall write $\deg x = \sigma$. If $\mathscr{A}$ is a $\mathbf{G}$-graded category, a $\mathbf{G}$-*graded module* (or simply a *graded module*) over $\mathscr{A}$ is an $\mathscr{A}$-module M with fixed decompositions $M(x) = \bigoplus_{\sigma\in\mathbf{G}} M(x)_\sigma$, given for all objects $x \in \mathscr{A}$, such that $\mathscr{A}(x,y)_\tau M(x)_\sigma \subseteq M(y)_{\sigma\tau}$ for any x, y, σ, τ. We denote by $\mathscr{A}$-GMod the category of graded $\mathscr{A}$-modules and by $\mathscr{A}$-gmod the category of finite dimensional graded $\mathscr{A}$-modules. Again we call elements $u \in M(x)_\sigma$ *homogeneous elements* of degree σ and write $\deg u = \sigma$. For any graded $\mathscr{A}$-module M and an element $\tau \in \mathbf{G}$, we define the *shifted graded* module $M\langle\tau\rangle$, which coincide with M as $\mathscr{A}$-module, but the grading is given by the rule: $M\langle\tau\rangle_\sigma = M_{\tau\sigma}$. Obviously, the shift $M \mapsto M\langle\tau\rangle$ is an autoequivalence of the category $\mathscr{A}$-GMod.

We shall usually consider the case, when $\mathbf{G}$ is a group (mainly $\mathbb{Z}$ or $\mathbb{Z}^2$). Such group gradings are closely related to the *group actions.* We say that a group $\mathbf{G}$ *acts on a category,* $\mathscr{A}$, if a map $T : \mathbf{G} \to \operatorname{Func}(\mathscr{A},\mathscr{A})$ is given such that $T1 = \operatorname{Id}$, where 1 is the unit of $\mathbf{G}$, and $T(\tau\sigma) = T(\tau)T(\sigma)$ for all $\sigma, \tau \in \mathbf{G}$. We do not consider here more general actions with systems of factors, when in the last formula the equality is replaced by an isomorphism of functors. We shall write σx instead of $T(\sigma)x$ both for objects and for morphisms from $\mathscr{A}$. Given such an action, we can define the *quotient category* $\mathscr{A}/\mathbf{G}$ as follows:

- The objects of $\mathscr{A}/\mathbf{G}$ are the orbits of $\mathbf{G}$ on $\operatorname{ob}\mathscr{A}$.
- $(\mathscr{A}/\mathbf{G})(\mathbf{G}x,\mathbf{G}y)$ is defined as the factorspace of $\bigoplus_{\substack{x'\in\mathbf{G}x\\ y'\in\mathbf{G}y}} \mathscr{A}(x,y)$ modulo the subspace generated by all differences $\alpha - \sigma\alpha$ $(\sigma \in \mathbf{G})$.
- The product of morphisms is defined in obvious way using representatives (one can easily check that their choice does not imply the result).

The action is called *free* if $\sigma x \neq x$ for every object $x \in \mathscr{A}$ and any $\sigma \neq 1$ from $\mathbf{G}$. In this case it is easy to see that

$$(\mathscr{A}/\mathbf{G})(\mathbf{G}x,\mathbf{G}y) \simeq \bigoplus_{y'\in\mathbf{G}y} \mathscr{A}(x,y') \simeq \bigoplus_{x'\in\mathbf{G}x} \mathscr{A}(x',y).$$

This allows us to define a $\mathbf{G}$-grading of $\mathscr{A}/\mathbf{G}$. Namely, we fix a representative $\widehat{\mathbf{x}}$ in every orbit $\mathbf{x}$ and consider morphisms $\widehat{\mathbf{x}} \to \sigma\widehat{\mathbf{y}}$ as homogeneous morphisms $\mathbf{x} \to \mathbf{y}$ of degree σ. One can verify that, whenever the action is free, the quotient category $\mathscr{A}/\mathbf{G}$ is equivalent to the skew group category $\mathscr{A} * \mathbf{G}$ as defined, for instance, in [RR].

Moreover, if the action is free, there is a good correspondence between $\mathscr{A}$-modules and graded $\mathscr{A}/\mathbf{G}$-modules. Given an $\mathscr{A}$-module, M, we define the graded $\mathscr{A}/\mathbf{G}$-module GM putting $GM(\mathbf{x})_\sigma = M(\sigma\widehat{\mathbf{x}})$ and, for $u \in GM(\mathbf{x})_\sigma$ and $\alpha \in (\mathscr{A}/\mathbf{G})_\tau(\mathbf{x},\mathbf{y})$, defining their product as $(\sigma\alpha)u$. It gives a functor, $G : \mathscr{A}\text{-Mod} \to \mathscr{A}/\mathbf{G}\text{-GMod}$. On the contarary, given a graded $\mathscr{A}/\mathbf{G}$-module N, we define the $\mathscr{A}$-module $G'N$ putting $G'N(x) = N(\mathbf{G}x)_\sigma$, where $x = \sigma\widehat{\mathbf{G}x}$. One immediately checks that G and G' define an equivelance $\mathscr{A}\text{-Mod} \xrightarrow{\sim} \mathscr{A}/\mathbf{G}\text{-GMod}$ (cf. [RR]). Moreover, the restrictions of these functors to the categories $\mathscr{A}$-mod and $\mathscr{A}/\mathbf{G}$-gmod induce an equivalence of these categories as well.

If the category $\mathscr{A}$ has already been $\mathbf{H}$-graded with a grading semigroup $\mathbf{H}$ and the action of $\mathbf{G}$ preserves this grading, the factorcategory $\mathscr{A}/\mathbf{G}$ becomes $\mathbf{H} \times \mathbf{G}$ graded, and the functors G, G' above induce an equivalence of the categories of $\mathbf{H}$-graded $\mathscr{A}$-modules and of $\mathbf{H} \times \mathbf{G}$-graded $\mathscr{A}/\mathbf{G}$-modules.

Actually, any group grading can be obtained as the result of a free group action. Namely, given a $\mathbf{G}$-graded category $\mathscr{A}$, define a new category $\widetilde{\mathscr{A}}$ with a $\mathbf{G}$-action as follows:

- The *objects* of $\widetilde{\mathscr{A}}$ are pairs (x, σ) with $x \in \operatorname{ob}\mathscr{A}$, $\sigma \in \mathbf{G}$.
- The morphisms $(x, \sigma) \to (y, \tau)$ is a pair, (α, σ), where α is a homogeneous morphism $x \to y$ of degree $\sigma^{-1}\tau$.
- The product $(\beta, \tau)(\alpha, \sigma)$ is defined as $(\beta\alpha, \sigma)$.
- $\tau(x, \sigma) = (x, \sigma\tau)$, where x is an object or a morphism from $\mathscr{A}$, $\sigma, \tau \in \mathbf{G}$.

Obviously, this action is free and $\mathscr{A}$ can be identified with $\widetilde{\mathscr{A}}/\mathbf{G}$ as a graded category. Just in the same way, given any graded $\mathscr{A}$-module M, we turn it into an $\widetilde{\mathscr{A}}$-module, denoted by $\widetilde{M}$, setting

- $\widetilde{M}(x, \sigma) = M(x)_\sigma$.
- $(\alpha, \sigma)m = \alpha m$ if $m \in \widetilde{M}(x, \sigma)$, $(\alpha, \sigma) \in \widetilde{\mathscr{A}}((x, \sigma), (y, \tau))$.

This correspondence gives the same equivalence $\widetilde{\mathscr{A}}\text{-Mod} \xrightarrow{\sim} \mathscr{A}\text{-GMod}$ as above.

This allows us to extend all results about module categories to the categories of graded modules. Especially, we can apply the Rickard–Morita Theorem to the category $\mathscr{A}$-GMod (note that the category $\mathscr{B}$ from this theorem remains ungraded). We denote by $\mathcal{D}_{gr}\mathscr{A}$ the derived category of $\mathscr{A}$-GMod. The grading shift $M \mapsto M\langle\sigma\rangle$ naturally extends to the category $\mathcal{D}_{gr}\mathscr{A}$ and commutes with the triangle shift $M \mapsto M[1]$.

There is an important class of gradings, defined as follows.

Definition 2.2. *Let $\mathscr{A}$ be a $\mathbf{G}$-graded category. We say that it is* naturally graded *if the category $\widetilde{\mathscr{A}}$ defined above contains a full subcategory $\widetilde{\mathscr{A}}^0 \simeq \mathscr{A}$ such that $\widetilde{\mathscr{A}} = \bigsqcup_{\sigma\in\mathbf{G}} \sigma(\widetilde{\mathscr{A}}^0)$, i.e.*

- $\operatorname{ob}\widetilde{\mathscr{A}} = \bigsqcup_{\sigma\in\mathbf{G}} \sigma(\operatorname{ob}\widetilde{\mathscr{A}}^0)$ *(a disjoint union).*
- $\widetilde{\mathscr{A}}(x, \sigma y) = 0$ *if $x, y \in \widetilde{\mathscr{A}}^0$ and $\sigma \neq 1$.*

Actually, it means that one can prescribe a degree $\deg x \in \mathbf{G}$ to every object $x \in \mathscr{A}$ so that $\mathscr{A}(x, y) = \mathscr{A}(x, y)_{\sigma^{-1}\tau}$ whenever $\deg x = \sigma$, $\deg y = \tau$. In this case also $\mathscr{A}\text{-GMod} \simeq \bigsqcup_{\sigma\in\mathbf{G}} \sigma(\widetilde{\mathscr{A}}^0)$-Mod and every component of this coproduct is equivalent to $\mathscr{A}$-Mod.

2.3 Yoneda categories

For any triangulated category $\mathscr{C}$ and any set $\mathfrak{X} \subseteq \operatorname{ob}\mathscr{C}$ we define the *Yoneda category* $\mathscr{E} = \mathscr{E}(\mathfrak{X})$, which is a $\mathbb{Z}$-graded category, as follows:

- $\operatorname{ob}\mathscr{E}(\mathfrak{X}) = \mathfrak{X}$.
- $\mathscr{E}(X,Y)_n = \operatorname{Hom}_{\mathscr{C}}(X, Y[n])$.
- The product $\beta\alpha$, where $\alpha : X \to Y[n]$, $\beta : Y \to Z[m]$, is defined as $\beta[n]\alpha : X \to Z[n+m]$.

Note that if $\mathscr{C} = \mathcal{D}\mathscr{A}$ and $\mathfrak{X} \subseteq \mathscr{A}$-Mod, then $\operatorname{Hom}_{\mathscr{C}}(X, Y[n]) = \operatorname{Ext}^n_{\mathscr{A}}(X,Y)$ and the product $\beta\alpha$ defined above coincides with the Yoneda product $\operatorname{Ext}^m_{\mathscr{A}}(Y,Z) \times \operatorname{Ext}^n_{\mathscr{A}}(X,Y) \to \operatorname{Ext}^{n+m}_{\mathscr{A}}(X,Z)$.

If $\mathscr{A}$ is a $\mathbf{G}$-graded category and $\mathscr{C} = \mathcal{D}_{gr}\mathscr{A}$, we also define the *graded Yoneda category* $\mathscr{E}_{gr}(\mathfrak{X})$, which is a $(\mathbb{Z}\times\mathbf{G})$-graded category, setting $\mathscr{E}_{gr}(X,Y)_{(n,\sigma)} = \operatorname{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(X, Y\langle\sigma\rangle[n])$, which coincide with $\operatorname{Ext}^n_{\mathscr{A}\text{-GMod}}(X, Y\langle\sigma\rangle)$ if X and Y are graded $\mathscr{A}$-modules. The product of the elements $\alpha : X \to Y\langle\sigma\rangle[n]$ and $\beta : Y \to Z\langle\tau\rangle[k]$ is then defined as $\beta\langle\sigma\rangle[n]\alpha : X \to Z\langle\tau\sigma\rangle[n+k]$. For example, let $\mathfrak{P} = \{\, \mathscr{A}^x \mid x \in \operatorname{ob}\mathscr{A} \}$, then

$$\mathscr{E}_{gr}(\mathscr{A}^x, \mathscr{A}^y)_{(n,\sigma)} = \begin{cases} 0 & \text{if } n \neq 0, \\ \mathscr{A}(y,x)_\sigma & \text{if } n = 0. \end{cases}$$

Thus $\mathscr{E}_{gr}(\mathfrak{P}) \simeq \mathscr{A}^{\mathrm{op}}$ as graded categories.

2.4 Koszul categories

In this subsection we consider $\mathbb{Z}$-graded categories $\mathscr{A}$. Moreover, we suppose that $\mathscr{A}$ is basic and *positively graded*, i.e. $\mathscr{A}(x,y)_n = 0$ if either $n < 0$ or $n = 0$ and $x \neq y$, while $\mathscr{A}(x,x)_0 = \Bbbk$. In particular, the objects of $\mathscr{A}$ are pairwise nonisomorphic and their endomorphism algebras contain no nontrivial idempotents. Then the modules $S(x)\langle 0\rangle = \operatorname{top}\mathscr{A}^x = \mathscr{A}(x,\_)_0$ and their shifts $S(x)\langle m\rangle$ are the only simple graded $\mathscr{A}$-modules. If we consider them as $\mathscr{A}$-modules *without grading*, we write $S(x)$ for them. Let $\mathfrak{S} = \{\, S(x) \,\}$ and $\mathfrak{S}_{gr} = \{\, S(x)\langle m\rangle \,\}$. We call the Yoneda category $\mathscr{E}(\mathfrak{S})$ and the graded Yoneda category $\mathscr{E}_{gr}(\mathfrak{S}_{gr})$ respectively the *Yoneda category* and the *graded Yoneda category of the positively graded category* $\mathscr{A}$ and denote them respectively by $\mathscr{E}(\mathscr{A})$ and by $\mathscr{E}_{gr}(\mathscr{A})$.

Let $\mathscr{A}_+$ be the ideal of $\mathscr{A}$ consisting of morphisms of *positive degree*, i.e. $\mathscr{A}_+(x,y) = \sum_{n>0}\mathscr{A}(x,y)_n$, and $V = V_{\mathscr{A}} = \mathscr{A}_+/\mathscr{A}_+{}^2$. Then V is an $\mathscr{A}$-bimodule. Set $V^x = V(x,\_)$, which is a semisimple gradable $\mathscr{A}$-module, hence it splits into a direct sum of copies of $S(y)$ for $y \in \operatorname{ob}\mathscr{A}$. We denote by $\nu(x,y)$ the multiplicity of $S(y)$ in $V(x)$ and define the *species* (or the *Gabriel quiver*) of $\mathscr{A}$ as the graph $\Gamma(\mathscr{A})$ such that its set of vertices is $\operatorname{ob}\mathscr{A}$ and there are $\nu(x,y)$ arrows from a vertex x to a vertex y. Equivalently,

$$\nu(x,y) = \dim_{\Bbbk}\operatorname{Ext}^1_{\mathscr{A}}(S(x),S(y)) = \sum_{m=1}^{\infty}\dim_{\Bbbk}\operatorname{Ext}^1_{\mathscr{A}\text{-GMod}}(S(x)\langle 0\rangle, S(y)\langle -m\rangle).$$

Note that $\mathscr{A}_1$ embeds into $V_{\mathscr{A}}$; hence, $\nu(x,y) \geq \dim_{\Bbbk} \mathscr{A}(x,y)_1$. If $V(x,y) = \mathscr{A}(x,y)_1$ for all x, y, we say that $\mathscr{A}$ *is generated in degree* 1.

Evidently, the Yoneda category $\mathscr{E}(\mathscr{A})$ is always positively graded. Therefore, the coefficients $\nu(S(x), S(y))$ defining its species are not smaller than $\dim_{\Bbbk} \mathscr{E}(S(x), S(y))_1 = \dim_{\Bbbk} \operatorname{Ext}^1_{\mathscr{A}}(S(x), S(y))$. Thus the species of $\mathscr{A}$ naturally embed into those of $\mathscr{E}(\mathscr{A})$.

Proposition 2.3. *Suppose that* $\dim_{\Bbbk} V_{\mathscr{A}}(x,y) < \infty$ *for all* $x, y \in \operatorname{ob}\mathscr{A}$. *The following properties are equivalent:*

(i) The Yoneda category $\mathscr{E}(\mathscr{A})$ *is generated in degree* 1.

(ii) For each object $x \in \operatorname{ob}\mathscr{A}$ *there is a projective resolution* $\mathcal{P}^\bullet(x)$ *of* $S(x)$ *such that, for every integer* n, $\mathcal{P}^{-n}(x)$ *is a direct sum of modules* $\mathscr{A}^y\langle -n\rangle$, *or, the same, is generated in degree* $-n$ *(such resolution will be called* linear*).*

(iii) For each object $x \in \operatorname{ob}\mathscr{A}$ *there is an injective resolution* $\mathcal{I}^\bullet(x)$ *of* $S(x)$ *such that, for every integer* n, $\mathcal{I}^n(x)$ *is a direct sum of modules* $D\mathscr{A}_y\langle n\rangle$.

(iv) For all x, y, l, m, *and* n *the inequality* $\operatorname{Ext}^n_{\mathscr{A}\text{-GMod}}(S(x)\langle l\rangle, S(y)\langle m\rangle) \neq 0$ *implies* $n = l - m$.

(v) $\Gamma(\mathscr{E}(\mathscr{A})) = \Gamma(\mathscr{A})$.

(vi) $\mathscr{E}(\mathscr{E}(\mathscr{A})) \simeq \mathscr{A}$.

Proof. The equivalence of the properties (i)–(v) is straightforward and well known (cf. [ADL]), at least if $\mathscr{A}$ contains finitely many objects (i.e. arises from a graded $\Bbbk$-algebra). In the general case the arguments are the same. The equivalence of (v) and (vi) follows immediately from the fact that $\Gamma(\mathscr{A})$ embeds into $\Gamma(\mathscr{E}(\mathscr{A}))$ and the last one embeds into $\Gamma(\mathscr{E}(\mathscr{E}(\mathscr{A})))$. It must also be well known, but we have not found any refernce for it. □

A category, $\mathscr{A}$, satisfying one of the equivalent conditions of Proposition 2.3 (and hence all of them), will be called *Koszul category*, and the category $\mathscr{E}(\mathscr{A})$ will be called the *Koszul dual* of $\mathscr{A}$ (the word "dual" is justified by the property (vi)). The equivalence of (ii) and (iii) implies that $\mathscr{A}$ is Koszul if and only if so is $\mathscr{A}^{\mathrm{op}}$.

Let $\mathscr{A}$ be a Koszul category and $S(x,l) = S(x)\langle l\rangle[-l]$, where $x \in \operatorname{ob}\mathscr{A}$, $l \in \mathbb{Z}$. The property (iv) shows that the set $\{\, S(x,l) \,\}$ is a tilting subset in $\mathcal{D}_{gr}\mathscr{A}$. Hence Rickard–Morita Theorem can be applied to the full subcategory $\mathscr{S}$ consiting of these objects. The group $\mathbb{Z}$ acts on $\mathscr{S}$: $T_n S(x,l) = S(x, l+n)$, and the set $\{\, S(x,0) \,\}$ can be chosen as a set of representatives of the orbits of $\mathbb{Z}$ on $\operatorname{ob}\mathscr{S}$. Moreover,

$$\operatorname{Ext}^n_{\mathscr{A}}(S(x), S(y)) \simeq \bigoplus_{l\in\mathbb{Z}} \operatorname{Ext}^n_{\mathscr{A}\text{-GMod}}(S(x)\langle 0\rangle, S(y)\langle l\rangle) = \operatorname{Ext}^n_{\mathscr{A}\text{-GMod}}(S(x,0), S(y,-n)).$$

This implies the following result (mostly also well known).

Theorem 2.4 (Koszul duality). *If* $\mathscr{A}$ *is a Koszul category, then*

1. $\mathcal{D}_{gr}\mathscr{A} \simeq \mathcal{D}\mathscr{S}^{\mathrm{op}}$.

2. $\mathscr{S}/\mathbb{Z} \simeq \mathscr{E}(\mathscr{A})$ *as* $\mathbb{Z}$*-graded categories.*

3. $\mathcal{D}_{gr}\mathscr{A} \simeq \mathcal{D}_{gr}\mathscr{E}(\mathscr{A})^{\mathrm{op}}$.

2.5 Quasihereditary categories

Let now $\mathscr{A}$ be a bounded category and let a fucntion, $\mathrm{ht} : \mathrm{ob}\,\mathscr{A} \to \mathbb{N} \cup \{0\}$, be given. For every object x define the *standard module* $\Delta(x)$ as the factor of $\mathscr{A}^x$ modulo the trace of all $\mathscr{A}^y$ with $\mathrm{ht}\,y > \mathrm{ht}\,x$, and the *costandard module* $\nabla(x)$ as $D\Delta^{\mathrm{op}}(x)$, where $\Delta^{\mathrm{op}}(x)$ denotes the standard module for $\mathscr{A}^{\mathrm{op}}$. Set $\mathbf{\Delta} = \{\, \Delta(x) \,|\, x \in \mathrm{ob}\,\mathscr{A}\}$ and $\mathbf{\nabla} = \{\, \nabla(x) \,|\, x \in \mathrm{ob}\,\mathscr{A}\}$. For a set, $\mathfrak{X}$, of $\mathscr{A}$-modules, we denote by $\mathcal{F}(\mathfrak{X})$, the full subcategory of $\mathscr{A}$-mod consiting of the modules have a filtration with subfactors from $\mathfrak{X}$ (an $\mathfrak{X}$*-filtration*). We call the category $\mathscr{A}$ *quasi-hereditary* (with respect to the function ht) if $\mathscr{A}^x \in \mathcal{F}(\mathbf{\Delta})$ or, equivalently, if $\mathcal{I}^x \in \mathcal{F}(\mathbf{\nabla})$, where $\mathcal{I}^x = D\mathscr{A}_x$. Obviously, in this case both $\mathbf{\Delta}$ and $\mathbf{\nabla}$ form a set of generators for $\mathcal{D}^{\mathrm{per}}\mathscr{A}$. The notion of a quasi-hereditary category is a natural generalization to this setup of the notion of a quasi-hereditary algebra, [DR1]. One should not confuse it with the notion of a highest weight category from [CPS]. A highest weight category is the category of modules over a quesi-hereditary algebra (or category).

Assume now that $\mathscr{A}$ is a quasi-hereditary category. The arguments of [Rin] can be easily extended to show that for each $x \in \mathrm{ob}\,\mathscr{A}$ there exists a unique (up to isomorphism) indecomposable module $T(x) \in \mathcal{F}(\mathbf{\Delta}) \cap \mathcal{F}(\mathbf{\nabla})$, called *tilting* module, whose arbitrary standard filtration starts with $\Delta(x)$.

Assume further that $\mathscr{A}$ is positively graded. Following [MO, Section 5] one shows that in this case all simple, projective, standard, injective, costandard, and tilting modules admit graded lifts. For indecomposable modules such lift is unique up to isomorphism and a shift of grading. The grading on $\mathscr{A}$ gives natural graded lifts for projective, standard and simple modules such that we have natural projections $\mathscr{A}^x \twoheadrightarrow \Delta(x)\langle 0\rangle \twoheadrightarrow S(x)\langle 0\rangle$ in $\mathscr{A}-$grmod. Let $x \in \mathrm{ob}\,\mathscr{A}$ We fix the grading on $\mathcal{I}^x$ and on $\nabla(x)$ such that the natural inclusions $S(x)\langle 0\rangle \hookrightarrow \nabla(x)\langle 0\rangle \hookrightarrow \mathcal{I}^x\langle 0\rangle$ are in $\mathscr{A}-$grmod. Finally we fix a grading on $T(x)$ such that the natural inclusion $\Delta(x) \hookrightarrow T(x)$ is in $\mathscr{A}-$grmod and remark that it follows that the natural projection $T(x) \twoheadrightarrow \nabla(x)$ is in $\mathscr{A}-$grmod.

We have to remark that the lifts above are not coordinated with the isomorphism classes of modules. For example it might happen that some indecomposable A module is projective, injective and tilting at the same time. If it is not simple, this module will have different graded lifts when considered as projective module (having the top in degree 0), as injective module (having the socle in degree 0), and as tilting module (having the top in a negative degree and the socle in a positive degree).

Set $T = \oplus_{x\in\mathrm{ob}\,\mathscr{A}}T(x)$. Since all $T(x)$ are gradable, the Ringel dual $\mathscr{R} = \mathrm{End}_{\mathscr{A}-\mathrm{mod}}(T)$ of $\mathscr{A}$ automatically inherits the structure of a graded category. In the present paper we will always consider $\mathscr{R}$ as a graded category with respect to this inherited grading.

Now we are ready to formulate the principal assumption for the algebras we consider. From now on we assume that

(I) for all $l \geq 0$ and for all $x \in \mathrm{ob}\,\mathscr{A}$ the minimal graded tilting coresolution $\mathcal{T}^\bullet(\Delta(x))$ of $\Delta(x)\langle 0\rangle$ satisfies $\mathcal{T}^k(\Delta(x)) \in \mathrm{add}\left(\oplus_{y:\mathrm{ht}(y)=\mathrm{ht}(x)-k} T(y)\langle k\rangle\right)$ for all $k \geq 0$;

(II) for all $l \geq 0$ and for all $x \in \mathrm{ob}\,\mathscr{A}$ the minimal graded tilting resolution $\mathcal{T}^\bullet(\nabla(x))$ of $\nabla(x)\langle 0\rangle$ satisfies $\mathcal{T}^k(\nabla(x)) \in \mathrm{add}\left(\oplus_{y:\mathrm{ht}(y)=\mathrm{ht}(x)+k} T(y)\langle k\rangle\right)$ for all $k \leq 0$.

3 Basic properties of graded quasi-hereditary categories satisfying (I) and (II)

During this section we always assume that $\mathscr{A}$ is a bounded graded quasi-hereditary category and that both (I) and (II) are satisfied.

Proposition 3.1. *Let $x \in \mathrm{ob}\,\mathscr{A}$.*

(i) All subquotients of any standard filtration of $T(x)\langle 0\rangle$ have the form $\Delta(y)\langle k\rangle$, where $k \geq 0$ and $\mathrm{ht}(y) = \mathrm{ht}(x) - k$; moreover, $k = 0$ is possible only if $x = y$.

(ii) All subquotients of any costandard filtration of $T(y)\langle 0\rangle$ have the form $\nabla(y)\langle k\rangle$, where $k \leq 0$ and $\mathrm{ht}(y) = \mathrm{ht}(x) + k$; moreover, $k = 0$ is possible only if $x = y$.

Proof. We prove (i) using $\mathcal{T}^\bullet(\Delta(x))$ and (I), and the arguments for (ii) are similar (using $\mathcal{T}^\bullet(\nabla(x))$ and (II)). Proceed by induction in $\mathrm{ht}(x)$. If $\mathrm{ht}(x) = 0$, then $T(x)\langle 0\rangle$ is a standard module and the statement is obvious. Now assume that the statement is proved for all y with $\mathrm{ht}(y) = l - 1$, and let $\mathrm{ht}(x) = l$. Denote by C the cokernel of the graded inclusion $\Delta(x)\langle 0\rangle \hookrightarrow T(x)\langle 0\rangle$. By (I), C embedds into a direct sum of several $T(y)\langle 1\rangle$ with $\mathrm{ht}(y) = l - 1$, such that the cokernel of this embedding has a standard filtration. From the inductive assumption it follows that every subquotient of every standard filtration of such $T(y)\langle 1\rangle$ has the form $\Delta(z)\langle k+1\rangle$, where $k \geq 0$ and $\mathrm{ht}(z) = \mathrm{ht}(y) - k$. Since $\mathrm{ht}(y) = \mathrm{ht}(x) - 1$, the statement follows. □

Corollary 3.2. *Then the grading on $\mathscr{R}$, induced from the category $\mathscr{A}$-gmod, is positive. In particular, $\mathscr{R}_0 = \mathrm{End}_{\mathscr{A}\text{-gmod}}(T\langle 0\rangle)$ is semi-simple.*

Proof. Since T has both a standard and a costandard filtration, from [DR2, Section 1] it follows that every endomorphism of T is a linear combination of endomorphisms, each of which corresponds to a map from a subquotient of a standard filtration of T to a subquotient of a costandard filtration of T. By Propsotion 3.1 all subquotients in all standard filtrations of $T\langle 0\rangle$ live in non-positive degrees and all subquotients in all costandard filtrations of $T\langle 0\rangle$ live in non-negative degrees. This implies that the grading on $\mathscr{R}$, induced from $\mathscr{A}$-gmod, is non-negative. Moreover, from Propsotion 3.1 it also follows that the only non-zero graded map from $T(x)\langle 0\rangle$ to $T\langle 0\rangle$ is an inclusion into the $T(x)\langle 0\rangle$-direct summand of $T\langle 0\rangle$. This implies that the zero component of the grading is semi-simple and hence that the grading is in fact positive. □

Set $\Delta = \oplus_{x\in\mathrm{ob}\,\mathscr{A}}\Delta(x)$ and $\nabla = \oplus_{x\in\mathrm{ob}\,\mathscr{A}}\nabla(x)$.

Corollary 3.3. *(i) The canonical inclusion* $\Delta\langle 0\rangle \hookrightarrow T\langle 0\rangle$ *induces the following isomorphism:* $\mathrm{Hom}_{\mathscr{A}-\mathrm{mod}}(\Delta\langle 0\rangle, \Delta\langle 0\rangle) \cong \mathrm{Hom}_{\mathscr{A}-\mathrm{mod}}(\Delta\langle 0\rangle, T\langle 0\rangle)$.

(ii) The canonical projection $T\langle 0\rangle \twoheadrightarrow \nabla\langle 0\rangle$ *induces the following isomorphism:* $\mathrm{Hom}_{\mathscr{A}-\mathrm{mod}}(\nabla\langle 0\rangle, \nabla\langle 0\rangle) \cong \mathrm{Hom}_{\mathscr{A}-\mathrm{mod}}(T\langle 0\rangle, \nabla\langle 0\rangle)$.

Proof. Again we will prove (i) and (ii) is proved by similar arguments. It is certainly enough to show that the inclusion $\Delta(y)\langle 0\rangle \hookrightarrow T(y)\langle 0\rangle$ induces an isomorphism,

$$\mathrm{Hom}_{\mathscr{A}\text{-gmod}}(\Delta(x)\langle k\rangle, \Delta(y)\langle 0\rangle) \cong \mathrm{Hom}_{\mathscr{A}\text{-gmod}}(\Delta(x)\langle k\rangle, T(y)\langle 0\rangle),$$

for all $x, y \in \mathrm{ob}\,\mathscr{A}$ and all $k \in \mathbb{Z}$. Let $f \in \mathrm{Hom}_{\mathscr{A}\text{-gmod}}(\Delta\langle k\rangle, T\langle 0\rangle)$. If $f \neq 0$, Proposition 3.1 implies $\mathrm{ht}(x) = \mathrm{ht}(y) - k$. Further, f induces a homomorphism, $\overline{f} : \Delta\langle k\rangle \to \mathcal{T}^1(\Delta(x))$. By (I), we have that $\mathcal{T}^1(\Delta(x))$ is a direct sum of modules of the form $T(z)\langle 1\rangle$, where $\mathrm{ht}(z) = \mathrm{ht}(y) - 1$. If $\overline{f} \neq 0$, then, using the same arguments as above, we obtain $\mathrm{ht}(x) = \mathrm{ht}(z) - (k + 1)$. Combining all obtained equalities we get $\mathrm{ht}(y) - k = \mathrm{ht}(y) - 1 - k - 1$, that is $0 = 2$, a contradiction. This implies that the image of f is contained in $\Delta(y)\langle 0\rangle$ and the statement follows. □

Proposition 3.4. *Let* $x \in \mathrm{ob}\,\mathscr{A}$.

(i) All subquotients of any standard filtration of $\mathscr{A}^x\langle 0\rangle$ *have the form* $\Delta(y)\langle k\rangle$*, where* $k \leq 0$ *and* $\mathrm{ht}(y) = \mathrm{ht}(x) - k$*; moreover,* $k = 0$ *is possible only if* $x = y$.

(ii) All subquotients of any costandard filtration of $\mathcal{I}^x\langle 0\rangle$ *have the form* $\nabla(y)\langle k\rangle$*, where* $k \geq 0$ *and* $\mathrm{ht}(y) = \mathrm{ht}(x) + k$*; moreover,* $k = 0$ *is possible only if* $x = y$.

Proof. Again we will prove (i) and (ii) is proved by similar arguments.

Lemma 3.5. *Let* $\Delta(s)\langle k\rangle \hookrightarrow M \twoheadrightarrow \Delta(t)\langle 0\rangle$*,* $s, t \in \mathrm{ob}\,\mathscr{A}$*,* $k \in \mathbb{Z}$ *be a short exact sequence. Then* $k \leq 1$*. Moreover, if* M *has simple top then* $k = 1$ *and* $\mathrm{ht}(s) = \mathrm{ht}(t) + 1$.

Proof. $k \leq 1$ is an immediate consequence of Corollary 3.2 and the fact that $\mathcal{T}^1(\Delta(t))$ is a direct some of some copies of modules of the form $T(x)\langle 1\rangle$.

Assume that M has simple top. Then the preimage of the top of $\Delta(t)$ is not a top of M any more. In particular, there should exist $m < 0$ such that $M_m \neq 0$. The only $k \leq 1$ in $\Delta(s)\langle k\rangle$ for which such m exists is obviously 1. By (I), all $T(x)\langle 1\rangle$ occurring in $\mathcal{T}^1(\Delta(t))$ satisfy $\mathrm{ht}(x) = \mathrm{ht}(t) + 1$ and the statement follows. □

Since $\mathscr{A}$ is quasi-hereditary, we can construct $\mathscr{A}^x$ via a sequence of universal extensions starting from $\Delta(x)\langle 0\rangle$. Note that $\mathscr{A}^x$ has simple top. Hence, from Lemma 3.5 it follows that on step number k we can extend the module, constructed on the previous step, with some $\Delta(y)\langle -k\rangle$, where $\mathrm{ht}(y) = \mathrm{ht}(x) - k$. The statement of the proposition follows. □

Corollary 3.6. *For every* $x \in \mathrm{ob}\,\mathscr{A}$ *every direct summand of the minimal projective cover of the kernel of the canonical projection* $\mathscr{A}^x \twoheadrightarrow \Delta(x)\langle 0\rangle$ *has the form* $P(y)\langle -1\rangle$*, where* $\mathrm{ht}(y) = \mathrm{ht}(x) + 1$.

Proof. Follows from the statement and the proof of Proposition 3.4. □

Corollary 3.7. *For all $x, y \in \mathrm{ob}\,\mathscr{A}$ the inequality $\mathrm{Ext}^1_{\mathscr{A}}(S(x)\langle 0\rangle, S(y)\langle k\rangle) \neq 0$ implies $k = -1$ and $|\mathrm{ht}(x) - \mathrm{ht}(y)| = 1$.*

Proof. Since $\mathscr{A}$ is quasi-hereditary, $\mathrm{Ext}^1_{\mathscr{A}}(S(x)\langle 0\rangle, S(y)\langle k\rangle) \neq 0$, in particular, implies $\mathrm{ht}(x) \neq \mathrm{ht}(y)$. Let us first assume that $\mathrm{ht}(x) < \mathrm{ht}(y)$. Then $\mathrm{Ext}_{\mathscr{A}}(S(x)\langle 0\rangle, S(y)\langle k\rangle) \neq 0$ implies that $S(y)\langle k\rangle$ occures in the top of the kernel K of the canonical projection $\mathscr{A}^x \twoheadrightarrow \Delta(x)\langle 0\rangle$ since all composition subquotients of $\Delta(x)\langle 0\rangle$ have the form $S(z)\langle m\rangle$ with $\mathrm{ht}(z) < \mathrm{ht}(x)$. From Corollary 3.6 it follows that the top of K consists of modules of the form $S(z)\langle -1\rangle$ with $\mathrm{ht}(z) = \mathrm{ht}(x) + 1$. This proves the necessary statement.

In the case $\mathrm{ht}(x) > \mathrm{ht}(y)$ one uses the dual arguments with injective resolutions. □

Theorem 3.8. *Let $x \in \mathrm{ob}\,\mathscr{A}$ and $\mathcal{P}^\bullet(\Delta(x))$ and $\mathcal{I}^\bullet(\nabla(x))$ denote the (graded) projective resolution of $\Delta(x)\langle 0\rangle$ and the (graded) injective coresolution of $\nabla(x)\langle 0\rangle$ respectively. Then*

(i) for every $k \leq 0$ every indecoposable direct summand of $\mathcal{P}^k(\Delta(x))$ has the form $\mathscr{A}^y\langle k\rangle$ for some y such that $\mathrm{ht}(y) = \mathrm{ht}(x) - k$;

(ii) for every $k \geq 0$ every indecoposable direct summand of $\mathcal{I}^k(\nabla(x))$ has the form $\mathcal{I}^y\langle k\rangle$ for some y such that $\mathrm{ht}(y) = \mathrm{ht}(x) + k$.

Proof. We prove (i) and (ii) is proved by dual arguments. Since $\mathscr{A}$ is positively graded we have that every indecoposable direct summand of $\mathcal{P}^k(\Delta(x))$ has the form $\mathscr{A}^y\langle l\rangle$ for some $l \leq k$. Moreover, from Proposition 3.4 it follws that $\mathrm{ht}(y) = \mathrm{ht}(x) - l$. Hence it would be enough to show that $l = k$.

Assume that some indecoposable direct summand of $\mathcal{P}^k(\Delta(i))$ has the form $\mathscr{A}^y\langle l\rangle$ for some $l < k$. Since the kernel of any morphism in $\mathcal{P}^\bullet(\Delta(i))$ has a standard filtration, the presence of $\mathscr{A}^y\langle l\rangle$ gives rise to a non-zero element of $\mathrm{Ext}^{-k}_{\mathscr{A}-\mathrm{grmod}}(\Delta(x)\langle 0\rangle, \Delta(y)\langle l\rangle)$ with $\mathrm{ht}(y) = \mathrm{ht}(x) - l$.

On the other hand, consider the tilting coresolution $\mathcal{T}^\bullet(\Delta(y))$ of $\Delta(y)\langle 0\rangle$. Because of (I), every indecomposable direct summand of $\mathcal{T}^{-k}(\Delta(y))$ has the form $T(z)\langle k\rangle$ for some z such that $\mathrm{ht}(z) = \mathrm{ht}(y) + k$. Using the same arguments as in the proof of Corollary 3.2, from Proposition 3.1 and [DR2, Section 1] one gets that $\mathrm{Hom}_{\mathscr{A}-\mathrm{grmod}}(\Delta(x)\langle m\rangle, T(z)\langle k\rangle) \neq 0$ implies $m \leq -k$. In particular, $\mathrm{Ext}^{-k}_{\mathscr{A}-\mathrm{grmod}}(\Delta(x)\langle -l\rangle, \Delta(y)) = 0$ since $-l > -k$, a contradiction. □

Corollary 3.9. *A positively graded quasi-hereidaty category, $\mathscr{A}$, satisfying both* (I) *and* (II) *is standard Koszul in the sense of [ADL], in particular, $\mathscr{A}$ is Koszul.*

Proof. Follows from [ADL, Theorem 1]. □

Corollary 3.10. *A positively graded quasi-hereidaty category, $\mathscr{A}$, satisfies both* (I) *and* (II) *if and only if its Ringel dual $\mathscr{R}$ satisfies both* (I) *and* (II).

Proof. The proof is basicly similar to that of [MO, Theorem 6]. We have two Ringel duality functors $\mathrm{Hom}_{\mathscr{A}}(T, \_)$ and $D \circ \mathrm{Hom}_{\mathscr{A}}(\_, T)$, which both admit natural graded lifts. The graded lift of the first functor sends the linear tilting resolution of costandard $\mathscr{A}$-modules to the linear projective resolutions of standard $\mathscr{R}$-modules, and the linear injective resolutions of costandard $\mathscr{A}$-modules to the linear tilting coresolutions of standard $\mathscr{R}$-modules. The graded lift of the second functor sends the linear tilting coresolution of standard $\mathscr{A}$-modules to the linear injective coresolutions of costandard $\mathscr{R}$-modules, and the linear projective resolutions of standard $\mathscr{A}$-modules to the linear tilting resolutions of costandard $\mathscr{R}$-modules. Furthermore, the graded injective $\mathscr{A}$-module $\mathcal{I} = \oplus_{x \in \mathrm{ob}\,\mathscr{A}} \mathcal{I}^x$ is identified with the graded characteristic tilting module over $\mathscr{R}$ in a natural way. Hence the grading on the Ringel dual to $\mathscr{R}$, which is in fact isomorphic to $\mathscr{A}$ as an ungraded algebra, coincides with the original grading on $\mathscr{A}$. The statement follows. □

We remark that it is obvious that a positively graded quasi-hereidaty algebra, $\mathscr{A}$, satisfies both (I) and (II) if and only if $\mathscr{A}^{\mathrm{op}}$ (with the induced grading) satisfies both (I) and (II).

Proposition 3.11. *(i) For every $x \in \mathrm{ob}\,\mathscr{A}$ the module $\Delta(x)\langle 0\rangle$ is directed in the following way: for all $l > 0$ we have $[\Delta(x)\langle 0\rangle_l : S(y)\langle -l\rangle] \neq 0$ implies $\mathrm{ht}(y) = \mathrm{ht}(x) - l$.*

(ii) $\dim_{\Bbbk} \mathrm{Hom}_{\mathscr{A}\text{-gmod}}(\Delta(y)\langle -l\rangle, \Delta(x)) = [\Delta(x)_l : S(y)\langle -l\rangle]$ *for all* $y \in \Lambda$.

Proof. To prove the first statement let us first show that $[\Delta(x)\langle 0\rangle_l : S(y)\langle -l\rangle] \neq 0$ implies $\mathrm{ht}(y) \leq \mathrm{ht}(x) - l$. Indeed. Let l be maximal such this statement fails for $\Delta(x)_l$ that is $[\Delta(x)\langle 0\rangle_l : S(y)\langle -l\rangle] \neq 0$ for some y such that $\mathrm{ht}(y) > \mathrm{ht}(x) - l$. Using Corollary 3.7 we obtain that $\mathrm{Ext}^1_{\mathscr{A}\text{-gmod}}(S(y)\langle -l\rangle, \Delta(x)\langle 0\rangle_{l+1}) = 0$ that is $S(y)\langle -l\rangle$ is inthe socle of $\Delta(x)\langle 0\rangle$. This implies the existence of a non-zero homomorphism from $\Delta(y)\langle -l\rangle$ to $\Delta(x)$, which (via the arguments of Corollary 3.2) constradicts Proposition 3.1.

Now let us show that $[\Delta(x)\langle 0\rangle_l : S(y)\langle -l\rangle] \neq 0$ implies $\mathrm{ht}(y) \geq \mathrm{ht}(x) - l$. From the definition of $\Delta(x)$ it follows that $\Delta(x)\langle 0\rangle$ is obtained by a sequence of universal extensions, which starts from $S(x)\langle 0\rangle$, and where we are allowed to extend with modules $S(z)\langle m\rangle$ for $\mathrm{ht}(z) \leq \mathrm{ht}(x)$. Applying recursively Corollary 3.7 we see that all simple subquotients, which can be obtained after at most l steps must have the form $S(z)\langle m\rangle$, where $-l \leq m \leq 0$ and $\mathrm{ht}(x) - l \leq \mathrm{ht}(z) \leq \mathrm{ht}(x)$. This gives the necessary inequality.

To prove the second statement we observe that $\dim_{\Bbbk} \mathrm{Hom}_{\mathscr{A}\text{-gmod}}(\mathscr{A}^y\langle -l\rangle, \Delta(x)\langle 0\rangle) = [\Delta(x)\langle 0\rangle_l : S(y)\langle -l\rangle]$ for all $y \in \Lambda$. Because of (i) the image of any homomorphism $f \in \mathrm{Hom}_{\mathscr{A}\text{-gmod}}(\mathscr{A}^y\langle -l\rangle, \Delta(x)\langle 0\rangle)$ does not contain simple subquotients $S(xz)\langle t\rangle$ with $\mathrm{ht}(z) \geq \mathrm{ht}(y)$. Hence f factors through $\Delta(y)\langle 0\rangle$ and the statement follows. □

4 Main theorem

Throughout this section we suppose that $\mathscr{A}$ is a bounded graded category, which is quasi-hereditary with respect to a function $\mathrm{ht} : \mathrm{ob}\,\mathscr{A} \to \mathbb{N} \cup \{0\}$ and satisfies conditions (I) and

(II). We will use the following notation:

$$\begin{aligned}
\kappa(x,l) &= \lfloor (\operatorname{ht} x - l)/2 \rfloor; \\
\delta(x,l) &= \begin{cases} 0, & \text{if } \operatorname{ht} x \equiv l \pmod 2, \\ 1 & \text{otherwise;} \end{cases} \\
\Delta(x,l) &= \Delta(x)\langle l\rangle[\kappa(x,l)]; \\
\nabla(x,l) &= \nabla(x)\langle l\rangle[\kappa(x,l)]; \\
T(x,l) &= T(x)\langle l\rangle[\kappa(x,l)]; \\
\mathscr{B} &= \mathscr{B}(\mathscr{A}) = \{\, \Delta(x,l) \mid x \in \operatorname{ob}\mathscr{A},\ l \in \mathbb{Z} \,\}; \\
\mathscr{B}' &= \mathscr{B}'(\mathscr{A}) = \{\, \nabla(x,l) \mid x \in \operatorname{ob}\mathscr{A},\ l \in \mathbb{Z} \,\}.
\end{aligned}$$

We use the same symbols $\mathscr{B}$ and $\mathscr{B}'$ for the full subcategories of $\mathcal{D}_{gr}\mathscr{A}$ with the sets of objects $\mathscr{B}$ and $\mathscr{B}'$. We also denote by $\mathcal{K}$ the ideal of $\mathscr{B}$ consisting of all morphisms $\Delta(x,l) \to \Delta(y,m)$ with $\kappa(x,l) \neq \kappa(y,m)$ and $\mathscr{B}^{\text{ver}} = \mathscr{B}/\mathcal{K}$.

Theorem 4.1 (Main Theorem). *In the described situation the following hold:*

(i) $\mathscr{B}$ *and* $\mathscr{B}'$ *are Koszul categories.*

(ii) The Koszul dual of $\mathscr{B}$ *is equivalent to* $\mathscr{B}'$ *and vice versa.*

(iii) $\mathcal{D}_{gr}\mathscr{A} \simeq \mathcal{D}(\mathscr{B}\text{-Mod})^{\text{op}}$*. Moreover, an equivalence* $F : \mathcal{D}_{gr}\mathscr{A} \xrightarrow{\sim} \mathcal{D}(\mathscr{B}\text{-Mod})^{\text{op}}$ *can be chosen such that*

- *(a)* $F\Delta(x,l) \simeq \mathscr{B}_{\Delta(x,l)}$*;*
- *(b)* $F\nabla(x,l) \simeq \operatorname{top}\mathscr{B}_{\Delta(x,l)}$*;*
- *(c)* $FT(x,l) \simeq \mathscr{B}^{\text{ver}}_{\Delta(x,l)}$*;*

(iv) The category $\mathscr{B}$ *can be naturally positively* $\mathbb{Z}$*-graded with* $\deg\Delta(x,l) = \kappa(x,l)$*, i.e.* $\mathscr{B}(\Delta(x,l),\Delta(y,m)) = \mathscr{B}(\Delta(x,l),\Delta(y,m))_{\kappa(y,m)-\kappa(x,l)}$*.*

(v) The group $\mathbb{Z}$ *acts on* $\mathscr{B}$ *in the following way:* $T_n\Delta(x,l) = \Delta(x,l+n)$*, and* $\mathscr{B}/\mathbb{Z} \simeq \mathscr{E}(\boldsymbol{\Delta})$ *as* $\mathbb{Z}^2$*-graded categories.*

Certainly, the analogues of statements (iii)-(v) for the categories $\mathscr{B}'$ and $\mathscr{E}(\boldsymbol{\nabla})$ are also valid. We leave the reader to formulate them; the proofs follow by duality. Note also that, in particular, this theorem shows that the category $\mathcal{D}_{gr}\mathscr{A}$ splits as $\mathcal{D}_{gr}\mathscr{A}_{\text{even}} \bigsqcup \mathcal{D}_{gr}\mathscr{A}_{\text{odd}}$, where the first component is generated by $\{\, \Delta(x,l) \,\}$ with $\operatorname{ht} x - l$ even and the second one by $\{\, \Delta(x,l) \,\}$ with $\operatorname{ht} x - l$ odd.

The proof of this theorem includes several propositions, which will be stated separately. Most of them consist of some statements about the category $\mathscr{B}$ (especially, the modules $\Delta(x,l)$) and analogous statements about the category $\mathscr{B}'$ (especially, the modules $\nabla(x,l)$). We shall always prove the statements about $\mathscr{B}$; those about $\mathscr{B}'$ follow by duality (or can be proved quite in the same way).

Proposition 4.2. *1. If* $\mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x')\langle l'\rangle[k'], \Delta(x)\langle l\rangle[k]) \neq 0$*, then* $\mathrm{ht}(x') - 2k' - l' = \mathrm{ht}(x) - 2k - l$*.*

2. If $\mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\nabla(x')\langle l'\rangle[k'], \nabla(x)\langle l\rangle[k]) \neq 0$*, then* $\mathrm{ht}(x') - 2k' - l' = \mathrm{ht}(x) - 2k - l$*.*

Proof. Certainly, we may suppose that $k' = l' = 0$. If $\mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x'), \Delta(x)\langle l\rangle[k]) \neq 0$, also $\mathrm{Hom}_{\mathscr{A}}(\Delta(x'), \mathcal{T}^k(x)\langle l\rangle) \neq 0$, i.e. $\mathrm{Hom}_{\mathscr{A}}(\Delta(x'), \Delta(y)\langle l+k\rangle) \neq 0$ for some y with $\mathrm{ht}(y) = \mathrm{ht}(x) - k$. Proposition 3.11(i) implies that $\mathrm{ht}(x') = \mathrm{ht}(y) - (k+l) = \mathrm{ht}(x) - 2k - l$. $\square$

Since $\mathscr{A}$ is quasihereditary, the sets of objects $\mathscr{B}$ and $\mathscr{B}'$ generate $\mathcal{D}_{gr}\mathscr{A}^{\mathrm{per}}$ as triangulated category. We denote by $\mathcal{D}_{\mathrm{even}}$ and $\mathcal{D}_{\mathrm{odd}}$ the triangulated subcategories of $\mathcal{D}_{gr}\mathscr{A}$ generated by $\{\, \Delta(x,l) \mid \delta(x,l) = 0\}$ and $\{\, \Delta(x,l) \mid \delta(x,l) = 1\}$ respectively.

Corollary 4.3. $\mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\mathscr{X}, \mathscr{X}') = 0$ *if* $\mathscr{X} \in \mathcal{D}_{\mathrm{even}}$, $\mathscr{X}' \in \mathcal{D}_{\mathrm{odd}}$ *or vice versa. Thus* $\mathcal{D}_{gr}\mathscr{A} = \mathcal{D}_{\mathrm{even}} \bigsqcup \mathcal{D}_{\mathrm{odd}}$.

Corollary 4.4. *The categories* $\mathcal{D}_{\mathrm{even}}$ *and* $\mathcal{D}_{\mathrm{odd}}$ *are generated (as triangular categories), respectively, by* $\{\, \nabla(x,l) \mid \delta(x,l) = 0\}$ *and* $\{\, \nabla(x,l) \mid \delta(x,l) = 1\}$*.*

Proof. This follows from the fact that $\mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,l), \nabla(y,m)[k]) = 0$ if $k \neq 0$ or $(x,l) \neq (y,m)$. $\square$

Corollary 4.5. *The sets* $\mathscr{B}$ *and* $\mathscr{B}'$ *are tilting subsets of* $\mathcal{D}_{gr}\mathscr{A}$*.*

Proof. Indeed, it follows from Proposition 4.2 that $\mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,l), \Delta(y,m)[n]) = 0$ if $n \neq 0$. $\square$

Therefore, Rickard–Morita Theorem can be applied to these sets, which implies statements (iii) and (iiia) of the Main Theorem as well as their analogues for $\mathscr{B}'$. Moreover, if the functor F satisfies (iiia), then

$$\mathrm{Hom}_{\mathcal{D}\mathscr{B}}(F\Delta(x,l), F\nabla(y,m)[n]) \simeq \\ \simeq \mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,l), \nabla(y,m)[n]) = \begin{cases} \Bbbk, & \text{if } (x,l) = (y,m) \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

These values coincide with $\mathrm{Hom}_{\mathcal{D}\mathscr{B}}(\mathscr{B}_{\Delta(x,l)}, \mathscr{B}_{\Delta(x,l)}[n])$, which gives the statement (iiib). It also implies that the Yoneda category of $\mathscr{B}$ is equivalent to $\mathscr{B}'$ and vice versa. According to Proposition 2.3, it gives statements (i) and (ii).

Recall that $\mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x)\langle l\rangle[k], T(y,)\langle m\rangle[n]) = 0$ if $k \neq n$. Together with Corollary 3.3 it gives

$$\mathrm{Hom}_{\mathcal{D}\mathscr{B}}(F\Delta(x,l), FT(y,m)[n]) \simeq \mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,l), T(y,m)[n]) = \\ = \begin{cases} \mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,l), \Delta(y,m)), & \text{if } n \neq 0 \text{ or } \kappa(x,l) \neq \kappa(y,m), \\ 0 & \text{otherwise.} \end{cases}$$

These values coincide with $\mathrm{Hom}_{\mathcal{D}\mathscr{B}}(\mathscr{B}_{\Delta(x,l)}, \mathscr{B}^{\mathrm{ver}}_{\Delta(y,m)}[n])$, which implies the statement (iiic).

Now we define a $\mathbb{Z}$-grading in $\mathscr{B}$ setting $\deg f = \kappa(y,m) - \kappa(x,l)$ for every morphism $f : \Delta(x,l) \to \Delta(y,m)$, and consider the corresponding covering $\widetilde{\mathscr{B}}$ (cf. Subsection 2.2). The objects $\Delta(x,l)\langle\kappa(x,l)\rangle \in \widetilde{\mathscr{B}}$ form in $\widetilde{\mathscr{B}}$ a full subcategory $\mathscr{B}^0 \simeq \mathscr{B}$. Moreover, by definition, $\widetilde{\mathscr{B}}(\Delta(x,l)\langle\kappa(x,l)\rangle, \Delta(y,m)\langle n+\kappa(y,m)\rangle)$ consists of the pairs $(f,0)$, where $f : \Delta(x,l) to \Delta(y,l)$ is of degree $n + \kappa(y,m) - \kappa(x,l)$. But every morphism between these modules is of degree $\kappa(y,m) - \kappa(x,l)$. Hence, if $n \neq 0$, there are no nonzero morphisms in $\widetilde{\mathscr{B}}(\Delta(x,l)\langle\kappa(x,l)\rangle, \Delta(y,m)\langle n+\kappa(y,m)\rangle)$. It means that $\mathscr{B}$ is naturally graded thus statement *4* holds.

Proposition 4.2 also implies that

$$\mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,l), \Delta(y,m)) \simeq \mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,l+1), \Delta(y,m+1)).$$

So the functors $T_n : \Delta(x,l) \mapsto \Delta(x,l+1)$ define a free action of $\mathbb{Z}$ on $\mathscr{B}$ compatable with the grading introduced above. Then the factor $\mathscr{B}/\mathbb{Z}$ is well defined as $\mathbb{Z}^2$-graded category. The orbits of $\mathbb{Z}$ on the objects of $\mathscr{B}$ are the sets $\{\,\Delta(x,l) \mid l \in \mathbb{Z}\}$ (with fixed x). So we can identify them with the modules $\Delta(x)$ and choose $\Delta(x,0)$ as representative of such an orbit. Then

$$(\mathscr{B}/Z)(\Delta(x),\Delta(y))_{(n,m)} = \mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,0),\Delta(y,m))_n =$$
$$= \mathrm{Hom}_{\mathcal{D}_{gr}\mathscr{A}}(\Delta(x,0),\Delta(y,m))_n = \begin{cases} \mathrm{Ext}^n(\Delta(x)\langle 0\rangle, \Delta(y)\langle m\rangle), & \text{if } n = \kappa(y,m) - \kappa(x,0), \\ \quad 0 & \text{otherwise,} \end{cases}$$

which coincides with $\mathscr{E}(\Delta(x),\Delta(y))_{(n,m)}$. Thus we have proved statement (v) and accomplished the proof of Main Theorem.

5 Applications of the main result

5.1 Multiplicity free blocks of the BGG category $\mathcal{O}$

Let $\mathfrak{g}$ be a semi-simple finite-dimensional complex Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $\lambda \in \mathfrak{h}^*$ be an integral dominant weight. Let A_λ be the basic associative algebra, whose module category is equivalent to the block $\mathcal{O}_\lambda$ of the BGG-category $\mathcal{O}$, which corresponds to λ, see [BGG, So]. Let Δ denote the direct sum of all Verma modules in $\mathcal{O}_\lambda$. Let further S denote the set of simple roots associated with W_λ and $\mathcal{O}_S$ denote the corresponding S-parabolic subcategory of $\mathcal{O}_0$ (see [RC, BGS]). Let $\tilde{\Delta}$ denote the direct sum of all generalized Verma modules in $\mathcal{O}_S$. Finally, let us denote by B_λ the basic associative algbera, associated with $\mathcal{O}_S$. In [So, BGS] it was shown that the algebras A_λ and B_λ are Koszul and even Koszul dual to each other. A quasi-hereditary algebra (or the corresponding highest weight category) is said to be *multiplicity free* if all indecomposable standard modules are multiplicity free.

Theorem 5.1. *Assume that $\mathcal{O}_\lambda$ is multiplicity free. Then the following holds:*

(i) $\mathcal{O}_S$ is multiplicity free.

(ii) The algebra $\mathrm{Ext}^*_{\mathcal{O}_S}\left(\tilde{\Delta},\tilde{\Delta}\right)$ is Koszul and even Koszul self-dual.

(iii) The algebra $\mathrm{Ext}^*_{\mathcal{O}_\lambda}(\Delta,\Delta)$ is Koszul and even Koszul self-dual.

Proof. The primitive idempotents of A_λ are indexed by the highest weights of Verma modules in $\mathcal{O}_\lambda$, which are $w\cdot\lambda$, where w is a representative of a cosets W/W_λ, where W is the Weyl group of $\mathfrak{g}$ and W_λ is the stabilizer of λ in W. For the antidominant $\mu = w_0\cdot\lambda$ (here w_0 is the longest element of W) we set $\mathrm{ht}(\mu)=0$ and for all other $\nu = w\cdot\lambda$ we define $\mathrm{ht}(\nu)$ and the smallest k such that there exist simple reflections $s_1,\dots,s_k$ in W such that $\nu = s_k\dots s_1\cdot\mu$.

The primitive idempotents of B_λ are indexed by the highest weights of generalized Verma modules in $\mathcal{O}_S$, which are $w\cdot 0$, where w is the shortest representative of a cosets $W_\lambda\backslash W$. Let w_0^λ be the longest element of W_λ. For the weight $\mu = w_0^\lambda w_0\cdot\lambda$ we set $\mathrm{ht}(\mu)=0$ and for all other $\nu = w\cdot\lambda$ as above we define $\mathrm{ht}(\nu)$ and the smallest k such that there exist simple reflections $s_1,\dots,s_k$ in W such that $\nu = s_k\dots s_1\cdot\mu$.

By [MO, Sections 6,7] and [MO, Appendix], the B_λ-module $\tilde{\Delta}$ admits a linear tilting coresolution, which, under Koszul duality, becomes the A_λ-module Δ by [ADL]. Moreover, the A_λ-module Δ admits a linear tilting coresolution, which, under Koszul duality, becomes the B_λ-module $\tilde{\Delta}$ for B_λ.

Assume now that A_λ is multiplicity free. Then the condition (I) for B_λ follows from the known structure of usual Verma modules (see for example [Di, Section 7]). Using the usual duality $\star$ on B_λ (and on A_λ) we also obtain (II). Now Theorem 4.1 implies that $\mathrm{Ext}^*_{\mathcal{O}_S}\left(\tilde{\Delta},\tilde{\Delta}\right)$ is Koszul with Koszul dual $\mathrm{Ext}^*_{\mathcal{O}_S}\left(\tilde{\nabla},\tilde{\nabla}\right)$, where $\tilde{\nabla}$ is the direct sum of all costandard modules in $\mathcal{O}_S$. Applying $\star$ incudes an isomorphism of the two last algebras, which proves (ii).

Further, from the above proof of (ii) and Proposition 3.11 it follows that $\tilde{\Delta}$ is directed in the sence of Proposition 3.11. Now [Di, Section 7] implies that B_λ is multiplicity free, which gives (i).

Finally, let us prove (ii). Again it is enough to prove (I) for A_λ. If (I) is not satisfied, going to the Koszul dual B_λ we obtain a "wrong" occurence of a simple in some standard B_λ-module $\tilde{\Delta}(\nu)$. This implies the original Verma module $\Delta(\nu)$, which surjects onto $\tilde{\Delta}(\nu)$ must have higher multiplicities. Using the Kazhdan-Lusztig Theorem and induction in $\mathrm{ht}(\nu)$, we can further assume that the "wrong" occurence of a simple in $\tilde{\Delta}(\nu)\langle 0\rangle$ is in degree 1. This, in turn, would mean that for some standard A_λ-module the condition (I) fails already on the first step. However, in the multiplicity-free case all standard A_λ-modules are directed in the sence of Proposition 3.11 by [Di, Section 7]. Further from the Kazhdan-Lusztig Theorem it follows that on the first step of the construction of the tilting module $T(\nu)$ we extend $\Delta(\nu)$ with $\Delta(\xi)$ for all ξ such that $S(\xi)\langle -1\rangle$ is a subquotient of $\Delta(\nu)\langle 0\rangle$. The directedness of the standard modules and the already mentioned fact that all standard A_λ-modules have linear tilting coresolutions now imply that the first step of the tilting coresolution of every standard A_λ-module is always correct. A contradiction. This completes the proof of (ii) and of the whole theorem. □

For more information on multiplicity free blocks of $\mathcal{O}$ and $\mathcal{O}_S$ (in particular for classification in the case of maximal stabilizer) we refer the reader to [BC].

Corollary 5.2. *If $\mathcal{O}_0$ is multiplicity-free (which is the case if and only if* $\mathrm{rank}(\mathfrak{g}) \leq 2$*) then* $\mathrm{Ext}^*_{\mathcal{O}_0}(\Delta, \Delta)$ *is Koszul and even Koszul self-dual.*

Proof. By [So] we have $A_\lambda \cong B_\lambda$ in this case and the statement follows from Theorem 5.1. □

We would like to emphiasize that the algebras $\mathrm{Ext}^*_{\mathcal{O}_S}\left(\tilde{\Delta}, \tilde{\Delta}\right)$ and $\mathrm{Ext}^*_{\mathcal{O}_\lambda}(\Delta, \Delta)$ in Theorem 5.1 are not Koszul dual to each other in general, though the algebras A_λ and B_λ are.

5.2 Some Koszul quasi-hereditary algebras with Cartan decomposition

Let A be a basic quasi-hereditary algebra over $\Bbbk$ with duality and a fixed Cartan decomposition $A = B \otimes_S B^{\mathrm{op}}$, where B is a strong exact Borel subalgebra of A, see [Ko]. Let Λ be the indexing set of simple A- (and hence also of simple B-) modules.

Proposition 5.3. *Assume in the above situation that*

(1) B is Koszul;

(2) there is a function, $\mathrm{ht} : \Lambda \to \{0\} \cup \mathbb{N}$*, such that the l-th term of the minimal injective resolution of the simple B-module $L(x)$, $x \in \Lambda$, containes only indecomposable injective modules $I(y)$ such that* $\mathrm{ht}(y) = \mathrm{ht}(x) - l$*;*

(3) $A \otimes_B \_$ sends indecomposable injective B-modules to indecomposable tilting A-modules.

Then A satisfies (I) *and* (II)*. In particular, for the direct sum Δ of all standard A-modules we have that* $\mathrm{Ext}^*_A(\Delta, \Delta)$ *is Koszul and even Koszul self-dual.*

Proof. Since B is an exact Borel subalgebra of A, the functor $A \otimes_B \_$ sends simple B-modules to standard A-modules and is exact. This implies that the linear injective coresolution of any simple B-module is send by $A \otimes_B \_$ to a linear tilting coresolution of the corresponding standard A-module. This shows that A satisfies (I) and (II) follows by duality. Now Theorem 4.1 implies that $\mathrm{Ext}^*_A(\Delta, \Delta)$ is Koszul with Koszul dual $\mathrm{Ext}^*_A(\nabla, \nabla)$, where ∇ is a direct sum of all costandard A-modules. Koszul self-duality of $\mathrm{Ext}^*_A(\nabla, \nabla)$ follows by applying the duality for A. □

We note that the condition (2) is satisfied for example for incidence algebras, associated with a regular cell decomposition of the sphere $\mathbb{S}^n$, where $\mathrm{ht}(x)$ denotes the dimension of the cell x, see [KM].

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