A NEW APPROACH TO KOSTANT'S PROBLEM

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Abstract. For every involution $\mathbf{w}$ of the symmetric group S_n we establish, in terms ofa special canonical quotient of the dominant Verma module associated with $\mathbf{w}$, an effective criterion, which allows us to verify whether the universal enveloping algebra $U(\mathfrak{sl}_n)$ surjects onto the space of all ad-finite linear transformations of the simple highest weight module $L(\mathbf{w})$. An easy sufficient condition derived from this criterion admits a straightforward computational check for example using a computer. All this is applied to get some old and many new results, which answer the classical question of Kostant in special cases, in particular we give a complete answer for simple highest weight modules in the regular block of $\mathfrak{sl}_n$, $n \leq 5$.

1. Introduction

Let $\mathfrak{g}$ be a complex semi-simple finite-dimensional Lie algebra with a fixed triangular decomposition, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and $U(\mathfrak{g})$ be its universal enveloping algebra. Then for every two $\mathfrak{g}$-modules M and N the space $\mathrm{Hom}_{\mathbb{C}}(M, N)$ may be viewed as a $U(\mathfrak{g})$-bimodule in the natural way, and, furthermore, also as a $\mathfrak{g}$-module under the adjoint action of $\mathfrak{g}$. The bimodule $\mathrm{Hom}_{\mathbb{C}}(M, N)$ has a sub-bimodule, usually denoted by $\mathcal{L}(M, N)$, which consists of all elements, on which the adjoint action of $U(\mathfrak{g})$ is locally finite (see for example [Ja, Kapitel 6]). Since $U(\mathfrak{g})$ itself consists of locally finite elements under the adjoint action, it naturally maps to $\mathcal{L}(M, M)$ for every $\mathfrak{g}$-module M, and the kernel of this map is obviously the annihilator $\mathrm{Ann}(M)$ of M in $U(\mathfrak{g})$. The classical problem of Kostant (see for example [Jo]) is formulated in the following way:

For which $\mathfrak{g}$-modules M is the natural injection

$$U(\mathfrak{g})/\mathrm{Ann}(M) \hookrightarrow \mathcal{L}(M, M)$$

surjective?

The (positive) answer to Kostant's problem is an important tool, in particular, in the study of generalized Verma modules, see [MiSo, KM1, MS1]. Unfortunately, the complete answer to this problem is not even known for simple highest weight modules. The answer is known to be positive for Verma modules (see [Jo, Corollary 6.4]) and for certain classes of simple highest weight modules (see [GJ2, Theorem 4.4]

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and [Ma1, Theorem 1]). For simple highest weight modules in type A the answer is even known to be an invariant of a left cell, see [MS1, Theorem 60]. However, already in [Jo, 9.5] it was shown that for some simple highest weight modules in type B the answer is negative. In spite of the general belief that the answer is positive for simple highest weight modules in type A, it was recently shown in [MS2, Theorem 13] that for the simple highest weight $\mathfrak{sl}_4$-module $L(rt)$, where r and t are two commuting simple reflections, the answer is negative.

The present paper is strongly inspired by the latter counter-example and is an attempt to analyze and generalize it. As in type A the answer to Kostant's problem is an invariant of a left cell, and since every left cell of the symmetric group S_n contains a unique involution, it is enough to solve Kostant's problem for all modules of the form $L(\mathbf{w})$, where $\mathbf{w} \in S_n$ is an involution. The counter-example in [MS2, Theorem 13] was constructed relating the module $L(\mathbf{w})$ to a special quotient of the dominant Verma module, which in the following will be denoted by $D^{\hat{\mathrm{R}}}$. This module is a canonical object of the category $\mathcal{O}_0^{\hat{\mathrm{R}}}$, which was used in [MS1] to categorify Kazhdan-Lusztig cell modules. The module $L(\mathbf{w})$ is the simple socle of $D^{\hat{\mathrm{R}}}$ and thus both $L(\mathbf{w})$ and $D^{\hat{\mathrm{R}}}$ are submodules of the indecomposable injective module $P^{\hat{\mathrm{R}}}(\mathbf{w})$ in $\mathcal{O}_0^{\hat{\mathrm{R}}}$, which also turns out to be projective.

The main result of the present paper relates the solution of Kostant's problem for $L(\mathbf{w})$ to the structure of $D^{\hat{\mathrm{R}}}$ as follows:

Theorem 1. *Kostant's problem has a positive answer for $L(\mathbf{w})$ if and only if every simple submodule of the cokernel of the canonical inclusion $D^{\hat{\mathrm{R}}} \subset P^{\hat{\mathrm{R}}}(\mathbf{w})$ has the form $L(x)$, where x is some element from the right cell of $\mathbf{w}$.*

We will show that Theorem 1 can be used to answer Kostant's problem in many cases, in particular, to obtain many new results and reprove some old results. The most interesting application of this theorem seems to be that it implies a sufficient condition for a *negative* answer to Kostant's problem, which is purely computational and can be realized as a relatively short and efficient program on a computer.

In Section 2 we collected all necessary preliminaries. The main results (in particular Theorem 1) are formulated in detail and proved in Section 3. In Section 4 we collected many applications, both theoretical and computational.

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2. Notation and preliminaries

From now on we assume that $\mathfrak{g} = \mathfrak{sl}_n$ and the triangular decomposition is just the usual decomposition into the upper triangular, diagonal and lower triangular matrices. The symmetric group S_n is the Weyl group W for $\mathfrak{g}$ and hence S_n acts on $\mathfrak{h}^*$ in the usual way $w\lambda$, and via the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is half the sum of all positive (with respect to the above triangular decomposition) roots of the algebra $\mathfrak{g}$.

Let $\mathcal{O}$ denote the BGG category $\mathcal{O}$, [BGG], associated with the triangular decomposition above. For $w \in W$ we let $\Delta(w)$ denote the Verma module with highest weight $w\cdot 0$, $L(w)$ denote the simple head of $\Delta(w)$, and $P(w)$ denote the indecomposable projective cover of $L(w)$. The *principal block* $\mathcal{O}_0$ of $\mathcal{O}$ is the full subcategory of $\mathcal{O}$, which contains all $L(w)$, $w \in S_n$, and is closed under isomorphisms and extensions. The category $\mathcal{O}_0$ is a direct summand of $\mathcal{O}$.

For $w \in W$ we denote by θ_w the indecomposable *projective functor* on $\mathcal{O}_0$, associated with w. This functor is a unique (up to isomorphism) indecomposable direct summand of all possible functors, which have the form $V \otimes_{\mathbb{C}} {}_- : \mathcal{O} \to \mathcal{O}$, where V is a finite-dimensional $\mathfrak{g}$-module, which satisfies $\theta_w \Delta(e) = P(w)$, see [BG, Section 3].

Denote by $\leq_{\mathtt{L}}$ and $\leq_{\mathtt{R}}$ the left and the right (pre)orders on W respectively, see [BB, Section 3]. For a fixed right cell $\mathbf{R}$ set

$$\hat{\mathbf{R}} = \{x \in W \,:\, x \leq_{\mathtt{R}} w \text{ for some } w \in \mathbf{R}\}$$

and denote by $\mathcal{O}_0^{\hat{\mathbf{R}}}$ the full subcategory of $\mathcal{O}_0$, which contains all $L(w)$, $w \in \hat{\mathbf{R}}$, and is closed under isomorphisms and extensions. The natural inclusion functor $\mathrm{i}^0_{\hat{\mathbf{R}}} : \mathcal{O}_0^{\hat{\mathbf{R}}} \to \mathcal{O}_0$ is obviously exact and hence has both the left adjoint $\mathrm{Z}_0^{\hat{\mathbf{R}}} : \mathcal{O}_0 \to \mathcal{O}_0^{\hat{\mathbf{R}}}$ and the right adjoint $\hat{\mathrm{Z}}_0^{\hat{\mathbf{R}}} : \mathcal{O}_0 \to \mathcal{O}_0^{\hat{\mathbf{R}}}$, see [MS1, 5.1]. The functor $\mathrm{Z}_0^{\hat{\mathbf{R}}}$ is just the functor of taking the maximal possible quotient, which lies in $\mathcal{O}_0^{\hat{\mathbf{R}}}$; and the functor $\hat{\mathrm{Z}}_0^{\hat{\mathbf{R}}}$ is just the functor of taking the maximal possible submodule, which lies in $\mathcal{O}_0^{\hat{\mathbf{R}}}$. All projective functors on $\mathcal{O}_0$ preserve $\mathcal{O}_0^{\hat{\mathbf{R}}}$, and both $\mathrm{Z}_0^{\hat{\mathbf{R}}}$ and $\hat{\mathrm{Z}}_0^{\hat{\mathbf{R}}}$ commute with θ_w for all $w \in W$, see [MS1, Lemma 19].

For $w \in \hat{\mathbf{R}}$ set $P^{\hat{\mathbf{R}}}(w) = \mathrm{Z}_0^{\hat{\mathbf{R}}} P(w)$ and $\Delta^{\hat{\mathbf{R}}}(w) = \mathrm{Z}_0^{\hat{\mathbf{R}}}\Delta(w)$. Then the modules $P^{\hat{\mathbf{R}}}(w)$, $w \in \hat{\mathbf{R}}$, are exactly the indecomposable projective modules in $\mathcal{O}_0^{\hat{\mathbf{R}}}$. The module $P^{\hat{\mathbf{R}}}(w)$ is injective if and only if $w \in \mathbf{R}$, see [MS1, Section 5]. Let $\mathbf{w} \in \mathbf{R}$ be a unique involution in $\mathbf{R}$. Then $P^{\hat{\mathbf{R}}}(w) = \theta_w L(\mathbf{w})$ for any $w \in \mathbf{R}$, see [MS2, Key statement]. By [MS2, Lemma 8] we have the equality $\dim \mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(e), P^{\hat{\mathbf{R}}}(\mathbf{w})) = 1$. Denote by $D^{\hat{\mathbf{R}}}$ the image of the unique (up to a scalar) non-zero homomorphism from $P^{\hat{\mathbf{R}}}(e)$ to $P^{\hat{\mathbf{R}}}(\mathbf{w})$.

Conjecture 2. $D^{\hat{\mathbf{R}}} = P^{\hat{\mathbf{R}}}(e)$.

Define the following full subcategories in $\mathcal{O}_0^{\hat{\mathbf{R}}}$:

$$\mathcal{C}_1 = \{M \in \mathcal{O}_0^{\hat{\mathbf{R}}} : [M : L(x)] > 0 \text{ implies } x <_{\mathtt{R}} \mathbf{w}\},$$
$$\mathcal{C}_2 = \{M \in \mathcal{O}_0^{\hat{\mathbf{R}}} : \mathrm{Hom}_{\mathfrak{g}}(L(x), M) \neq 0 \text{ implies } x \in \mathbf{R}\},$$
$$\mathcal{C}_3 = \{M \in \mathcal{O}_0^{\hat{\mathbf{R}}} : \mathrm{Hom}_{\mathfrak{g}}(M, L(x)) \neq 0 \text{ implies } x \in \mathbf{R}\}$$

From the definition we immediately have $\mathrm{Hom}_{\mathfrak{g}}(M,N) = 0$ for all $X \in \mathcal{C}_1$ and $Y \in \mathcal{C}_2$; and for all $X \in \mathcal{C}_3$ and $Y \in \mathcal{C}_1$.

Lemma 3. *For every $w \in W$ and $i = 1,2,3$ the functor θ_w preserves the category $\mathcal{C}_i$.*

Proof. That θ_w preserves $\mathcal{C}_1$ follows from the definitions and the fact that θ_w preserves $\mathcal{O}_0^{\hat{\mathbf{R}}}$. That θ_w preserves $\mathcal{C}_2$ follows from the fact that the injective envelope of any $X \in \mathcal{C}_2$ is projective and the fact that θ_w is exact and preserves projective-injective modules in $\mathcal{O}_0^{\hat{\mathbf{R}}}$. The proof of the fact that θ_w preserves $\mathcal{C}_3$ is dual. $\square$

Let $\mathcal{P} = \oplus_{w \in \mathbf{R}} P^{\hat{\mathbf{R}}}(w)$. For every $M \in \mathcal{O}_0^{\hat{\mathbf{R}}}$ let I_M be some injective envelope of M and set

$$M_1 = \bigcap_{\substack{f \in \mathrm{Hom}_{\mathfrak{g}}(I_M, \mathcal{P}) \\ f(M)=0}} \mathrm{Ker}(f), \qquad M_1' = \bigcap_{f \in \mathrm{Hom}_{\mathfrak{g}}(M_1, \mathcal{P})} \mathrm{Ker}(f),$$

and $M_2 = M_1/M_1'$. The correspondence $M \mapsto M_2$ is functorial and M_2 is called the *partial approximation* of M with respect to the injective module $\mathcal{P}$, see [KM2, 2.4]. We denote by $\mathrm{A} : \mathcal{O}_0^{\hat{\mathbf{R}}} \to \mathcal{O}_0^{\hat{\mathbf{R}}}$ the corresponding functor of partial approximation. From the definition we have the natural transformation nat from the identity functor ID to A, which is just the quotient map from M to $M/(M \cap M_1')$. The functor A is left exact, see [KM2, 2.4].

3. The main results

3.1. **A criterion for testing Kostant's problem.** According to [MS1, Theorem 60], the answer to Kostant's problem for $L(w)$, $w \in W$, is an invariant of a left cell. Since every left cell has a unique involution, it is thus enough to study Kostant's problem for involutions in W. The main result of the paper is the following statement:

Theorem 4. *Let $\mathbf{w} \in W$ be an involution and $\mathbf{R}$ be the right cell of W, containing $\mathbf{w}$. Then the following conditions are equivalent:*

(a) Kostant's problem has a positive solution for $L(\mathbf{w})$.
(b) Every simple module, occurring in the socle of the cokernel Coker *of the natural inclusion $D^{\hat{\mathbf{R}}} \hookrightarrow P^{\hat{\mathbf{R}}}(\mathbf{w})$, has the form $L(x)$, where $x \in \mathbf{R}$ (i.e.* Coker *belongs to $\mathcal{C}_2$).*

The idea of the proof is to compare Kostant's problem for modules $L(\mathbf{w})$ and $D^{\hat{\mathbf{R}}}$. The former is exactly the module for which we would like to solve Kostant's problem, while the latter is, by definition, a quotient of $\Delta(e)$, and hence Kostant's problem for it has a positive solution by [Ja, 6.9(10)]. The relation between these two modules is again given by definition: $L(\mathbf{w})$ is the simple socle of $D^{\hat{\mathbf{R}}}$. So, to compare $\mathcal{L}(L(\mathbf{w}), L(\mathbf{w}))$ and $\mathcal{L}(D^{\hat{\mathbf{R}}}, D^{\hat{\mathbf{R}}})$ one might first try to show that these two modules have the same annihilators, and then try to show that

$$\mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w L(\mathbf{w})) = \mathrm{Hom}_{\mathfrak{g}}(D^{\hat{\mathbf{R}}}, \theta_w D^{\hat{\mathbf{R}}}) \tag{1}$$

for all $w \in W$. This would be enough to conclude that $\mathcal{L}(L(\mathbf{w}), L(\mathbf{w})) = \mathcal{L}(D^{\hat{\mathbf{R}}}, D^{\hat{\mathbf{R}}})$ by [Ja, 6.8(3)], thus solving positively Kostant's problem for $L(\mathbf{w})$. The best way to prove (1) would be to construct a functor, which commutes with all θ_w, and sends $L(\mathbf{w})$ to $D^{\hat{\mathbf{R}}}$. It turns out that the functor A defined above does this job. So now let's do the work.

Lemma 5. *For all $w \in W$ there is an isomorphism of functors as follows: $\mathrm{A}\theta_w \cong \theta_w\mathrm{A}$.*

Proof. As A is left exact and θ_w is exact, both $\mathrm{A}\theta_w$ and $\theta_w\mathrm{A}$ are left exact.

Let $I \in \mathcal{O}_0^{\hat{\mathbf{R}}}$ be injective. Consider the short exact sequence

$$0 \to K \to I \xrightarrow{\mathrm{nat}_I} \mathrm{A}I \to 0, \tag{2}$$

where K is just the kernel of nat_I. Since the socle of $\mathcal{P}$ coincides with $\oplus_{w\in\mathbf{R}} L(w)$, from the definition of A we have that $K \in \mathcal{C}_1$, while $\mathrm{A}I \in \mathcal{C}_2$.

Applying θ_w to (2) and using Lemma 3 we obtain that $\theta_w K \in \mathcal{C}_1$ and $\theta_w \mathrm{A}I \in \mathcal{C}_2$. In particular, $\theta_w K$ is the maximal submodule of $\theta_w I$, which belongs to $\mathcal{C}_1$. Furthermore, the morphism $\theta_w(\mathrm{nat}_I)$ is surjective.

At the same time, the module $\theta_w I$ is injective as θ_w is right adjoint to the exact functor $\theta_{w^{-1}}$. From the definition of A we have that the morphism $\mathrm{nat}_{\theta_w I}$ is surjective and that its kernel coincides with the maximal submodule of $\theta_w I$, which belongs to $\mathcal{C}_1$. In other words, the kernels of $\mathrm{nat}_{\theta_w I}$ and $\theta_w(\mathrm{nat}_I)$ coincide.

Now the statement of the lemma follows from [KM2, Lemma 1], applied to the situation $\mathrm{F} = \mathrm{A}\theta_w$, $\mathrm{G} = \theta_w\mathrm{A}$ and $\mathrm{H} = \theta_w$. □

Set $\overline{D}^{\hat{\mathbf{R}}} = \mathrm{A}L(\mathbf{w})$.

Lemma 6. *(i) $\overline{D}^{\hat{\mathbf{R}}}$ is isomorphic to the maximal submodule of the module $P^{\hat{\mathbf{R}}}(\mathbf{w})$, which contains the socle of $P^{\hat{\mathbf{R}}}(\mathbf{w})$ and such that all other composition subquotients of $\overline{D}^{\hat{\mathbf{R}}}$ have the form $L(x)$, where $x <_{\mathtt{R}} \mathbf{w}$.*

(ii) We have $D^{\hat{\mathbf{R}}} \subset \overline{D}^{\hat{\mathbf{R}}}$ *and the condition of Theorem 4*(b) *is equivalent to the equality* $D^{\hat{\mathbf{R}}} = \overline{D}^{\hat{\mathbf{R}}}$.

Proof. As $P^{\hat{\mathbf{R}}}(\mathbf{w})$ is the injective envelope of $L(\mathbf{w})$, the statement (i) follows immediately from the definition of A.

The inclusion $D^{\hat{\mathbf{R}}} \subset \overline{D}^{\hat{\mathbf{R}}}$ follows from [MS2, Lemmata 6-8]. The rest of the statement (ii) now follows from (i) and the definition of A. □

Lemma 7. *For any* $w \in W$ *we have*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w L(\mathbf{w})) = \dim \mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}})$$

Proof. Since $L(\mathbf{w}) \in \mathcal{C}_2$, we have $\theta_w L(\mathbf{w}) \in \mathcal{C}_2$ by Lemma 3. Hence, by the definition of A, we have that A annihilates neither $L(\mathbf{w})$ nor any simple submodule of $\theta_w L(\mathbf{w})$. Applying A and using its definition we thus obtain an inclusion

$$\mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w L(\mathbf{w})) \subset \mathrm{Hom}_{\mathfrak{g}}(\mathrm{A}L(\mathbf{w}), \mathrm{A}\theta_w L(\mathbf{w})).$$

Using Lemma 5 and the definition of $\overline{D}^{\hat{\mathbf{R}}}$ we thus get the inclusion

$$\mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w L(\mathbf{w})) \subset \mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}). \tag{3}$$

On the other hand, consider the short exact sequence

$$0 \to L(\mathbf{w}) \to \overline{D}^{\hat{\mathbf{R}}} \to C \to 0, \tag{4}$$

where C is the cokernel. Applying the exact functor θ_w yields the short exact sequence

$$0 \to \theta_w L(\mathbf{w}) \to \theta_w \overline{D}^{\hat{\mathbf{R}}} \to \theta_w C \to 0. \tag{5}$$

Applying the bifunctor $\mathrm{Hom}_{\mathfrak{g}}({}_{-}, {}_{-})$ from the sequence (4) to the sequence (5) yields the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{g}}(C, \theta_w L(\mathbf{w})) \arrow[r, hook] \arrow[d, hook] & \mathrm{Hom}_{\mathfrak{g}}(C, \theta_w \overline{D}^{\hat{\mathbf{R}}}) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_{\mathfrak{g}}(C, \theta_w C) \arrow[d, hook] \\
\mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w L(\mathbf{w})) \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w C) \arrow[d] \\
\mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w L(\mathbf{w})) \arrow[r, hook] & \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w \overline{D}^{\hat{\mathbf{R}}}) \arrow[r] & \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w C).
\end{tikzcd}$$

We have $C, \theta_w C \in \mathcal{C}_1$ by definitions and Lemma 3, and $L(\mathbf{w}) \in \mathcal{C}_3$. This yields $\mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w C) = 0$, which implies

$$\mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w L(\mathbf{w})) = \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w \overline{D}^{\hat{\mathbf{R}}}).$$

Since $C \in \mathcal{C}_1$ while $\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}} \in \mathcal{C}_2$ by definitions and Lemma 3, we have $\mathrm{Hom}_{\mathfrak{g}}(C, \theta_w \overline{D}^{\hat{\mathbf{R}}}) = 0$, which yields the inclusion

$$\mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}) \subset \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w \overline{D}^{\hat{\mathbf{R}}}).$$

The latter, together with the equality, obtained in the previous paragraph, implies the opposite to (3) inclusion

$$\mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}) \subset \mathrm{Hom}_{\mathfrak{g}}(L(\mathbf{w}), \theta_w L(\mathbf{w}))$$

and the statement of the lemma follows. □

Lemma 8. *The inclusion* $L(\mathbf{w}) \subset \overline{D}^{\hat{\mathbf{R}}}$ *induces an isomorphism of* $\mathfrak{g}$*-bimodules as follows:* $\mathcal{L}(L(\mathbf{w}), L(\mathbf{w})) \cong \mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, \overline{D}^{\hat{\mathbf{R}}})$.

Proof. Applying the bifunctor $\mathcal{L}({}_{-}, {}_{-})$ to (4) we get the following commutative diagram with exact rows and columns:

(6)
$$\begin{tikzcd}
\mathcal{L}(C, L(\mathbf{w})) \arrow[r, hook] \arrow[d, hook] & \mathcal{L}(C, \overline{D}^{\hat{\mathbf{R}}}) \arrow[r] \arrow[d, hook] & \mathcal{L}(C, C) \arrow[d, hook] \\
\mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, L(\mathbf{w})) \arrow[r, hook] \arrow[d] & \mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, \overline{D}^{\hat{\mathbf{R}}}) \arrow[r] \arrow[d] & \mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, C) \arrow[d] \\
\mathcal{L}(L(\mathbf{w}), L(\mathbf{w})) \arrow[r, hook] & \mathcal{L}(L(\mathbf{w}), \overline{D}^{\hat{\mathbf{R}}}) \arrow[r] & \mathcal{L}(L(\mathbf{w}), C).
\end{tikzcd}$$

Since for any $w \in W$ we have $C, \theta_w C \in \mathcal{C}_1$ by definitions and Lemma 3, while $L(\mathbf{w}) \in \mathcal{C}_3$, from [Ja, 6.8(3)] we have $\mathcal{L}(L(\mathbf{w}), C) = 0$ implying $\mathcal{L}(L(\mathbf{w}), L(\mathbf{w})) \cong \mathcal{L}(L(\mathbf{w}), \overline{D}^{\hat{\mathbf{R}}})$.

Since for any $w \in W$ we have $\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}} \in \mathcal{C}_2$ by definitions and Lemma 3, while $C \in \mathcal{C}_1$, from [Ja, 6.8(3)] it follows that $\mathcal{L}(C, \overline{D}^{\hat{\mathbf{R}}}) = 0$ implying $\mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, \overline{D}^{\hat{\mathbf{R}}}) \subset \mathcal{L}(L(\mathbf{w}), \overline{D}^{\hat{\mathbf{R}}})$. Taking the above, Lemma 7 and [Ja, 6.8(3)] into account yields $\mathcal{L}(L(\mathbf{w}), L(\mathbf{w})) \cong \mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, \overline{D}^{\hat{\mathbf{R}}})$, which completes the proof. □

Proof of the implication (b)⇒(a) *in Theorem 4.* Because of the assumption Theorem 4(b), from Lemma 6(ii) we have $\overline{D}^{\hat{\mathbf{R}}} = D^{\hat{\mathbf{R}}}$. The module $D^{\hat{\mathbf{R}}}$ is a quotients of the dominant Verma module $\Delta(e)$ and hence $U(\mathfrak{g})$ surjects onto $\mathcal{L}(D^{\hat{\mathbf{R}}}, D^{\hat{\mathbf{R}}})$ by [Ja, 6.9(10)]. Lemma 8 and the diagram (6) now give the induced surjection of $U(\mathfrak{g})$ onto $\mathcal{L}(L(\mathbf{w}), L(\mathbf{w}))$. This completes the proof. □

Corollary 9. *If the condition Theorem 4*(b) *is satisfied, we have the equality* $\mathrm{Ann}_{U(\mathfrak{g})}(L(\mathbf{w})) = \mathrm{Ann}_{U(\mathfrak{g})}(D^{\hat{\mathbf{R}}})$.

Proof. From $L(\mathbf{w}) \subset D^{\hat{\mathbf{R}}}$ we have $\mathrm{Ann}_{U(\mathfrak{g})}(D^{\hat{\mathbf{R}}}) \subset \mathrm{Ann}_{U(\mathfrak{g})}(L(\mathbf{w}))$. On the other hand, from the previous proof we have

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(D^{\hat{\mathbf{R}}}) \cong \mathcal{L}(D^{\hat{\mathbf{R}}}, D^{\hat{\mathbf{R}}}) \cong \\ \cong \mathcal{L}(L(\mathbf{w}), L(\mathbf{w})) \cong U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(L(\mathbf{w})),$$

which implies the statement. □

Lemma 10. *(i)* $\mathrm{A}D^{\hat{\mathbf{R}}} \cong \overline{D}^{\hat{\mathbf{R}}}$.

(ii) $\mathrm{A}\overline{D}^{\hat{\mathbf{R}}} \cong \overline{D}^{\hat{\mathbf{R}}}$.

(iii) For any $w \in W$ there is an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(D^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}) \cong \mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}).$$

Proof. Consider the short exact sequence

$$0 \to D^{\hat{\mathbf{R}}} \to \overline{D}^{\hat{\mathbf{R}}} \to C \to 0, \tag{7}$$

where $C \in \mathcal{C}_1$ is the cokernel. From the definition of A we have $\mathrm{A}C = 0$. Applying now A to (7) and using the left exactness of A yields the statement (i). The statement (ii) follows immediately from the definition of A.

Since $C \in \mathcal{C}_1$ and $\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}} \in \mathcal{C}_2$, applying $\mathrm{Hom}_{\mathfrak{g}}({}_-, \theta_w \overline{D}^{\hat{\mathbf{R}}})$ to (7) yields the inclusion

$$\mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}) \subset \mathrm{Hom}_{\mathfrak{g}}(D^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}). \tag{8}$$

On the other hand, the functor A annihilates neither the socle of $D^{\hat{\mathbf{R}}}$ nor any submodule in the socle of $\theta_w \overline{D}^{\hat{\mathbf{R}}}$. Hence from the definition of A we have the inclusion

$$\mathrm{Hom}_{\mathfrak{g}}(D^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}) \subset \mathrm{Hom}_{\mathfrak{g}}(\mathrm{A}D^{\hat{\mathbf{R}}}, \mathrm{A}\theta_w \overline{D}^{\hat{\mathbf{R}}}).$$

Using (i), Lemma 5 and (ii) we obtain

$$\mathrm{Hom}_{\mathfrak{g}}(\mathrm{A}D^{\hat{\mathbf{R}}}, \mathrm{A}\theta_w \overline{D}^{\hat{\mathbf{R}}}) = \mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \mathrm{A}\overline{D}^{\hat{\mathbf{R}}}) = \mathrm{Hom}_{\mathfrak{g}}(\overline{D}^{\hat{\mathbf{R}}}, \theta_w \overline{D}^{\hat{\mathbf{R}}}),$$

which implies that the inclusion (8) is in fact an isomorphism. This completes the proof. □

Proof of the implication (a)⇒(b) *in Theorem 4.* The inclusion $L(\mathbf{w}) \subset D^{\hat{\mathbf{R}}}$ induces the inclusion $\mathrm{Ann}_{U(\mathfrak{g})}(D^{\hat{\mathbf{R}}}) \subset \mathrm{Ann}_{U(\mathfrak{g})}(L(\mathbf{w}))$, which, in turn, induces the surjection

$$U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(D^{\hat{\mathbf{R}}}) \twoheadrightarrow U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(L(\mathbf{w})). \tag{9}$$

Assume that the condition of Theorem 4(b) is not satisfied. As we have $\mathcal{L}(D^{\hat{\mathbf{R}}}, D^{\hat{\mathbf{R}}}) \cong U(\mathfrak{g})/\mathrm{Ann}_{U(\mathfrak{g})}(D^{\hat{\mathbf{R}}})$ by [Ja, 6.9(10)], from the latter formula and (9) it follows that the inequality

$$\mathcal{L}(D^{\hat{\mathbf{R}}}, D^{\hat{\mathbf{R}}}) \subsetneq \mathcal{L}(L(\mathbf{w}), L(\mathbf{w})) \cong \mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, \overline{D}^{\hat{\mathbf{R}}}). \tag{10}$$

would imply that the algebra $U(\mathfrak{g})$ does not surjects onto $\mathcal{L}(L(\mathbf{w}), L(\mathbf{w}))$. Hence we are now left to prove the inequality (10).

We apply the bifunctor $\mathcal{L}({}_{-}, {}_{-})$ to the short exact sequence (7), where the cokernel $C \neq 0$ by Lemma 6(ii). Since $C \in \mathcal{C}_1$ and $D^{\hat{\mathbf{R}}}$, $\theta_w D^{\hat{\mathbf{R}}}$, $\overline{D}^{\hat{\mathbf{R}}}$ and $\theta_w \overline{D}^{\hat{\mathbf{R}}}$ are in $\mathcal{C}_2$ for all $w \in W$, by [Ja, 6.8(3)] we obtain the following commutative diagram with exact rows and columns:

(11)
$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathcal{L}(C, C) \arrow[d, hook] \\
\mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, D^{\hat{\mathbf{R}}}) \arrow[r, hook] \arrow[d, hook] & \mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, \overline{D}^{\hat{\mathbf{R}}}) \arrow[r] \arrow[d, "\wr"] & \mathcal{L}(\overline{D}^{\hat{\mathbf{R}}}, C) \arrow[d] \\
\mathcal{L}(D^{\hat{\mathbf{R}}}, D^{\hat{\mathbf{R}}}) \arrow[r, hook] & \mathcal{L}(D^{\hat{\mathbf{R}}}, \overline{D}^{\hat{\mathbf{R}}}) \arrow[r, "\alpha"] & \mathcal{L}(D^{\hat{\mathbf{R}}}, C),
\end{tikzcd}$$

where the isomorphism in the second column follows from Lemma 10(iii). To complete the proof it is thus enough to show that the map α on the diagram (11) is non-zero.

Pick some simple submodule $L(x) \subset C$ (recall once more that $C \neq 0$ by Lemma 6(ii)). Then, using the adjointness and defining properties of projective functors, we have

(12)
$$\begin{array}{rcl}
\mathbb{C} & = & \mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(x), L(x)) \\
 & = & \mathrm{Hom}_{\mathfrak{g}}(\theta_x P^{\hat{\mathbf{R}}}(e), L(x)) \\
 & = & \mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(e), \theta_{x^{-1}} L(x)) \\
 & \subset & \mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(e), \theta_{x^{-1}} C).
\end{array}$$

Applying the bifunctor $\mathrm{Hom}_{\mathfrak{g}}({}_{-}, {}_{-})$ from the short exact sequence

$$0 \to K \to P^{\hat{\mathbf{R}}}(e) \to D^{\hat{\mathbf{R}}} \to 0,$$

where K is just the kernel of the natural projection $P^{\hat{\mathbf{R}}}(e) \twoheadrightarrow D^{\hat{\mathbf{R}}}$ (note that $K \in \mathcal{C}_1$ by [MS2, Lemmata 6-8]), to the short exact sequence

$$0 \to \theta_{x^{-1}} D^{\hat{\mathbf{R}}} \to \theta_{x^{-1}} \overline{D}^{\hat{\mathbf{R}}} \to \theta_{x^{-1}} C \to 0$$

we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{g}}(D^{\hat{\mathbf{R}}}, \theta_{x^{-1}} D^{\hat{\mathbf{R}}}) \arrow[r, hook] \arrow[d, "\wr"] & \mathrm{Hom}_{\mathfrak{g}}(D^{\hat{\mathbf{R}}}, \theta_{x^{-1}} \overline{D}^{\hat{\mathbf{R}}}) \arrow[r] \arrow[d, "\wr"] \arrow[dr, dashed, "\beta"] & \mathrm{Hom}_{\mathfrak{g}}(D^{\hat{\mathbf{R}}}, \theta_{x^{-1}} C) \arrow[d, hook] \\
\mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(e), \theta_{x^{-1}} D^{\hat{\mathbf{R}}}) \arrow[r, hook] \arrow[d] & \mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(e), \theta_{x^{-1}} \overline{D}^{\hat{\mathbf{R}}}) \arrow[r, two heads] \arrow[d] & \mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(e), \theta_{x^{-1}} C) \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Hom}_{\mathfrak{g}}(K, \theta_{x^{-1}} C),
\end{tikzcd}$$

where the second row is exact as $P^{\hat{\mathbf{R}}}(e)$ is projective in $\mathcal{O}_0^{\hat{\mathbf{R}}}$, and the zeros in the third row follow from the fact that $K \in \mathcal{C}_1$ while $\theta_{x^{-1}}\overline{D}^{\hat{\mathbf{R}}} \in \mathcal{C}_2$. From (12) it follows that the composition β is a surjection onto a non-zero vector space, hence is a non-zero map. From the definitions we have that $\beta \neq 0$ implies $\alpha \neq 0$. This completes the proof. □

3.2. **A sufficient condition for the negative answer.** Let Λ be the basic finite-dimensional associative algebra, whose module category is equivalent to $\mathcal{O}_0$. The algebra Λ is Koszul (see [So]) so we can fix the positive Koszul $\mathbb{Z}$-grading on Λ. Let $\Lambda-$gmod denote the category of finite-dimensional graded Λ-modules. For $x \in \hat{\mathbf{R}}$ let $\mathtt{P}^{\hat{\mathbf{R}}}(x)$ denote the standard graded lift of $P^{\hat{\mathbf{R}}}(x)$ with head concentrated in degree zero (see [MS1, 4.3]), and $\mathtt{L}(x)$ denote the standard graded lift of the corresponding simple quotient (concentrated in degree zero). For $w \in W$ we denote by $\hat{\theta}_w$ the standard graded lifts of the functors θ_w, see [St, Section 8]. Finally, let $\mathbf{a} : W \to \mathbb{Z}$ denote Lusztig's $\mathbf{a}$-function (see [Lu]), which is uniquely determined by the properties that it is constant on the two-sided cells of W and equals the length of w_0' on every w_0', which is the longest element of a parabolic subgroup of W.

If $\mathtt{M}$ is a graded module, then $\mathtt{M} = \oplus_{i\in\mathbb{Z}}\mathtt{M}_i$ is the decomposition of $\mathtt{M}$ into a direct sum of graded components. As usually, for $k \in \mathbb{Z}$ we denote by $\langle k\rangle : \Lambda-\text{gmod} \to \Lambda-\text{gmod}$ the functor, which shifts the grading such that $\mathtt{M}\langle k\rangle_i = \mathtt{M}_{i+k}$.

Lemma 11. *Let $\mathbf{w} \in W$ be an involution and $\mathtt{M} = \hat{\theta}_{\mathbf{w}}L(\mathbf{w})$. Then:*

(i) $\mathtt{M}_i = 0$ *for all i such that $|i| > \mathbf{a}(\mathbf{w})$.*
(ii) $\mathtt{M}_{\mathbf{a}(\mathbf{w})}$ *is the simple socle of* $\mathtt{M}$ *(which is isomorphic to the module* $\mathtt{L}(\mathbf{w})\langle -\mathbf{a}(\mathbf{w})\rangle$*).*

Proof. Since $\mathbf{a}$ is an invariant of two-sided cells, by [MS1, Theorem 18] we may without loss of generality assume that $\mathbf{w}$ is the maximal element of some parabolic subgroup. For such $\mathbf{w}$ the statement (i) follows immediately from [St, Theorem 8.2]. Moreover, the same argument implies $\mathtt{M}_{\mathbf{a}(\mathbf{w})} \neq 0$.

As Λ is positively graded and $\mathtt{M}$ is injective (the latter follows from [MS1, Section 5] and [MS2, Key statement]), $\mathtt{M}_{\mathbf{a}(\mathbf{w})} \neq 0$ must be the simple socle of $\mathtt{M}$. This completes the proof. □

Theorem 12. *Let $\mathbf{w} \in W$ be an involution and $\mathtt{M} = \hat{\theta}_{\mathbf{w}}\mathtt{L}(\mathbf{w})$. Assume that there exists $x \in W$ such that $x <_{\mathtt{R}} \mathbf{w}$ and*

$$[\mathtt{M} : \mathtt{L}(x)\langle 1 - \mathbf{a}(\mathbf{w})\rangle] > [\mathtt{P}^{\hat{\mathbf{R}}}(e) : \mathtt{L}(x)\langle 1 - \mathbf{a}(\mathbf{w})\rangle].$$

Then Kostant's problem has the negative answer for $L(\mathbf{w})$.

Proof. Let $\mathtt{N}$ be the quotient of $\mathtt{M}$ modulo $D^{\hat{\mathbf{R}}}$. As $D^{\hat{\mathbf{R}}}$ is non-zero, it must contain the socle of $\mathtt{M}$. Hence $\mathtt{N}_i = 0$ for all $i \geq \mathbf{a}(\mathbf{w})$ by

Lemma 11. By our assumption, $\mathtt{N}_{\mathbf{a}(\mathbf{w})-1}$ contains at least one copy of $\mathtt{L}(x)\langle 1-\mathbf{a}(\mathbf{w})\rangle$.

Since Λ is positively graded and $\mathtt{N}_i = 0$ for all $i \geq \mathbf{a}(\mathbf{w})$, the space $\mathtt{N}_{\mathbf{a}(\mathbf{w})-1}$ belongs to the socle of $\mathtt{N}$. Thus the condition of Theorem 4(b) is not satisfied and the answer to Kostant's problem for $L(\mathbf{w})$ is negative by Theorem 4. □

Remark 13. As $\mathtt{P}^{\hat{\mathbf{R}}}(e)$ is a quotient of the graded dominant Verma module $\Delta(e)$, in Lemma 12 one could use a stronger assumption

$$[\mathtt{M} : \mathtt{L}(x)\langle 1-\mathbf{a}(\mathbf{w})\rangle] > [\Delta(e) : \mathtt{L}(x)\langle 1-\mathbf{a}(\mathbf{w})\rangle]$$

with the same result.

Remark 14. The numerical condition of Theorem 12 is relatively easy to check for example using the computer, because it can be easily formulated in terms of Kazhdan-Lusztig combinatorics, [KL, BB]. Via the standard categorification approach to $\mathcal{O}$ (see for example [MS1, 3.4]), the characters of graded Λ-modules can be considered as elements of the Hecke algebra $\mathcal{H}$ of W (such that Verma modules correspond to the standrad basis of $\mathcal{H}$, projective modules correspond to the Kazhdan-Lusztig basis, and simple modules correspond to the dual Kazhdan-Lusztig basis). There are effective algorithms, which allow one to multiply elements of $\mathcal{H}$ and to transform them from one of the mentioned basis to the other. Some of the applications, presented in the next section are obtained using this approach.

Remark 15. The statement of Lemma 11 has a strong resemblance with [Ma2, Theorem 16], and is in some sense the Koszul dual of it (see the proof of [Ma2, Theorem 16] for details).

4. Applications

4.1. **Kostant's problem for the socle of the dominant Verma module in a parabolic category.** Let $\mathfrak{p} \subset \mathfrak{g}$ be a parabolic subcategory containing $\mathfrak{h} \oplus \mathfrak{n}_+$, and $\mathcal{O}_0^{\mathfrak{p}}$ be the corresponding parabolic subcategory of $\mathcal{O}_0$ in the sense of [RC]. Let $W' \subset W$ be the Weyl group of the Levi factor of $\mathfrak{p}$, w_0 be the longest element in W and w_0' be the longest element in W'. Then $\mathcal{O}_0^{\mathfrak{p}} = \mathcal{O}_0^{\hat{\mathbf{R}}}$, where $\mathbf{R}$ is the right cell of the element $w_0'w_0$, see [MS1, Remark 14]. Let $\mathbf{w}$ be the involution in $\mathbf{R}$.

Corollary 16. *Kostant's problem has the positive answer for* $L(\mathbf{w})$.

Proof. The category $\mathcal{O}_0^{\mathfrak{p}}$ is known to be a highest weight category in the sense of [CPS]. Thus any projective-injective module in $\mathcal{O}_0^{\mathfrak{p}}$ is tilting in the sense of [Ri], in particular, it has a filtration by standard modules (i.e. generalized Verma modules, induced from simple finite-dimensional $\mathfrak{p}$-modules). In particular, the dominant standard module

$P^{\hat{\mathbf{R}}}(e)$ is a submodule of $P^{\hat{\mathbf{R}}}(\mathbf{w})$, and the cokernel of this inclusion again has a filtration by standard modules. Since all standard modules belong to $\mathcal{C}_2$ by [Ir] (see also [MS3, Theorem 5.1] for a short argument), we obtain that the condition of Theorem 4(b) is satisfied and hence Kostant's problem has the positive answer for $L(\mathbf{w})$ by Theorem 4. $\square$

Remark 17. The fact $P^{\hat{\mathbf{R}}}(e) \subset P^{\hat{\mathbf{R}}}(\mathbf{w})$ implies that the statement of Conjecture 2 is true if $\mathbf{R}$ contains some $w_0'w_0$.

4.2. **Kostant's problem for $L(s)$, where s is a simple reflection.**

Corollary 18 ([Ma1])**.** *Let $s \in W$ be a simple reflection. Then Kostant's problem has the positive answer for $L(s)$.*

Proof. The only element of W, which is strictly smaller than s with respect to the order $<_{\mathtt{R}}$ is the identity element e. As, by adjointness,

$$\dim \mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(e), \theta_s L(s)) = \dim \mathrm{Hom}_{\mathfrak{g}}(P^{\hat{\mathbf{R}}}(s), L(s)) = 1,$$

the module $L(e)$ occurs in $\theta_s L(s)$ with multiplicity one, and hence $L(e)$ does not occur in the cokernel of the inclusion $D^{\hat{\mathbf{R}}} \subset \theta_s L(s)$ at all. Therefore the condition of Theorem 4(b) is obviously satisfied and hence Kostant's problem has the positive answer for $L(s)$ by Theorem 4. $\square$

4.3. **Kostant's problem for $L(st)$, where s and t are commuting simple reflections.** Here we generalize the counterexample, constructed in [MS2, Section 5]. Let $s_i = (i, i+1)$, $i = 1, \dots, n-1$, be the i-th simple reflection in W. We recall that for a simple reflection $s \in W$ and any $x \in W$ such that $xs < x$ with respect to the Bruhat order we have that the module $\hat{\theta}_s \mathtt{L}(x)$ is self-dual with simple head and socle and we moreover have the following graded picture (the middle row of which is in degree 0):

$$\hat{\theta}_s\mathtt{L}(x): \quad \begin{tikzcd} & \mathtt{L}(x)\langle 1\rangle \arrow[dl] \arrow[dr] & \\ \mathtt{L}(xs) \arrow[dr] & & X \arrow[dl] \\ & \mathtt{L}(x)\langle -1\rangle, & \end{tikzcd} \tag{13}$$

where X is a direct sum of $L(y)$'s such that $ys > y$ with multiplicity $\mu(x, y)$, where μ is Kazhdan-Lusztig's μ-function, [KL]. The formula (13) is a standard corollary of (now proved) Kazhdan-Lusztig's conjecture in equivalent Vogan's form (see [KL, GJ1, Vo]). We also refer to Remark 14 and to [St, Section 8] for the appropriate graded reformulation. The arrows on (13) schematically represent the action of the algebra Λ.

Corollary 19. *Let s_i and s_j be two commuting different simple reflections in W (i.e. $|i-j|>1$). Then Kostant's problem has the positive answer for $L(s_is_j)$ if and only if $|i-j|>2$.*

Proof. Without loss of generality we assume $j>i$. Let $\mathbf{R}_e=\{e\}$, $\mathbf{R}_i$ denote the right cell of s_i, $\mathbf{R}_j$ denote the right cell of s_j, and $\mathbf{R}$ denote the right cell of s_is_j. Then the Hasse diagram of $<_{\mathtt{R}}$ on the set $\{\mathbf{R}_e,\mathbf{R}_i,\mathbf{R}_j,\mathbf{R}\}$, where $\mathbf{R}$ is the maximum element, is as follows:

$$\begin{tikzcd} & \mathbf{R} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbf{R}_i \arrow[dr, dash] & & \mathbf{R}_j \arrow[dl, dash] \\ & \mathbf{R}_e, & \end{tikzcd}$$

and we further have

$$\begin{aligned} \mathbf{R}_i&=\{s_i,s_is_{i-1},\dots,s_is_{i-1}\dots s_1,s_is_{i+1},\dots,s_is_{i+1}\dots s_{n-1}\};\\ \mathbf{R}_j&=\{s_j,s_js_{j-1},\dots,s_js_{j-1}\dots s_1,s_js_{j+1},\dots,s_js_{j+1}\dots s_{n-1}\}. \end{aligned}$$

A direct calculation gives $\theta_{s_i}\theta_{s_j}=\theta_{s_is_j}=\theta_{s_j}\theta_{s_i}$.

Assume first that $j=i+2$. Since both s_is_{i+2} and $s_is_{i+1}s_{i+2}$ are Boolean elements of W (in the sense of [Mm]), we have that the Kazhdan-Lusztig polynomial $P_{s_is_{i+2},s_is_{i+1}s_{i+2}}(q)=1$ by [Mm, Theorem 5.4] and hence $\mu(s_is_{i+2},s_is_{i+1}s_{i+2})=1$ as well by definition. This yields that $\mathrm{Ext}^1_{\mathcal{O}}(L(s_is_{i+2}),L(s_is_{i+1}s_{i+2}))\neq 0$ and thus $L(s_is_{i+1}s_{i+2})$ occurs as a composition subquotient in $\theta_{s_i}L(s_is_{i+2})$ (as a direct summand of X in (13)). Applying (13) we get that $\mathtt{L}(s_is_{i+1}s_{i+2})\langle -1\rangle$ occurs as a composition subquotient in $\hat{\theta}_{s_is_{i+2}}\mathtt{L}(s_is_{i+2})$. Note that we have $s_is_{i+1}s_{i+2}<_{\mathtt{R}} s_is_{i+2}$. At the same time from [Di, Lemma 7.2.5] it follows that $\mathtt{P}^{\hat{\mathbf{R}}}(e)_1$ contains only composition subquotients of the form $\mathtt{L}(s_k)\langle -1\rangle$, $k=1,\dots,n-1$. Hence the numerical assumption of Theorem 12 is satisfied and therefore the answer to Kostant's problem for $L(s_is_{i+2})$ is negative by Theorem 12.

If $j>i+2$, a similar application of [Mm, Theorem 5.4] yields $\mu(s_is_j,s_is_{i+1}\dots s_{j-1}s_j)=0$ and also $\mu(s_is_j,s_js_{j-1}\dots s_{i+1}s_i)=0$. The only other elements of $\mathbf{R}_i$ and $\mathbf{R}_j$, comparable with s_is_j with respect to the Bruhat order, are s_i and s_j respectively. Because of (13), this means that

$$\hat{\theta}_{s_i}\mathtt{L}(s_is_j):\quad \begin{tikzcd} & \mathtt{L}(s_is_j)\langle 1\rangle \arrow[dl] \arrow[dr] & \\ \mathtt{L}(s_j) \arrow[dr] & & X \arrow[dl] \\ & \mathtt{L}(s_is_j)\langle -1\rangle, & \end{tikzcd}$$

where X is a direct sum of simple modules $L(y)$, $y\in\mathbf{R}$. Applying now $\hat{\theta}_{s_j}$ and using (13) again we obtain the following graded filtration for the module $\hat{\theta}_{s_is_j}\mathtt{L}(s_is_j)$:

(14)
$$\begin{tikzcd}
 & & \mathtt{L}(s_is_j)\langle 2\rangle \arrow[dll] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\mathtt{L}(s_j)\langle 1\rangle \arrow[d] \arrow[dr] \arrow[drrr] & & \mathtt{L}(s_i)\langle 1\rangle \arrow[d] & Y\langle 1\rangle \arrow[dl] & X'\langle 1\rangle \arrow[dl] \arrow[d] \\
\mathtt{L}(e) \arrow[d] & Z \arrow[dl] & \mathtt{L}(s_is_j) \arrow[dll] \arrow[drr] & \mathtt{L}(s_is_j) \arrow[dl] \arrow[d] & U \arrow[d] \\
\mathtt{L}(s_j)\langle -1\rangle \arrow[drr] & & \mathtt{L}(s_i)\langle -1\rangle \arrow[d] & Y\langle -1\rangle \arrow[dl] & X'\langle -1\rangle \arrow[dll] \\
 & & \mathtt{L}(s_is_j)\langle -2\rangle, & &
\end{tikzcd}$$

where Z is a direct sum of simples modules of the form $L(y)$, $y \in \mathbf{R}_j$; Y is a direct sum of simples modules of the form $L(y)$, $y \in \mathbf{R}$; and X' is a direct summand of X. Note that the arrows on (14) (which are supposed to schematically represent the action of Λ) show only the part of the action, which obviously comes from (13), but they do not show the whole action. From [Di, Lemma 7.2.5] it follows that the module $\mathtt{D}^{\hat{\mathbf{R}}}$ looks as follows:

$$\mathtt{D}^{\hat{\mathbf{R}}}: \qquad \begin{tikzcd}
 & \mathtt{L}(e) \arrow[dl] \arrow[dr] & \\
\mathtt{L}(s_i)\langle -1\rangle \arrow[dr] & & \mathtt{L}(s_j)\langle -1\rangle \arrow[dl] \\
 & \mathtt{L}(s_is_j)\langle -2\rangle &
\end{tikzcd}$$

Now we have to analyze (14) to determine the cokernel C of the inclusion $\mathtt{D}^{\hat{\mathbf{R}}} \subset \hat{\theta}_{s_is_j}\mathtt{L}(s_is_j)$. C obviously contains both $Y\langle -1\rangle$ and $X'\langle -1\rangle$, but all direct summands of these modules have the form $L(y)$, $y \in \mathbf{R}$, by above. None of the simple subquotients of U can be in C by (13). Similarly one excludes $\mathtt{L}(s_i)\langle 1\rangle$ and $\mathtt{L}(s_i)\langle 1\rangle$. All simple submodules in Z have the form $L(y)$, $y \in \mathbf{R}_j$. Considering $\hat{\theta}_{s_is_j}\mathtt{L}(s_is_j) = \hat{\theta}_{s_i}\hat{\theta}_{s_j}\mathtt{L}(s_is_j)$ and using the same arguments as above one shows that none of the simple submodules of Z belongs to C. Hence C contains only simple modules of the form $L(y)$, $y \in \mathbf{R}$. Thus the condition of Theorem 4(b) is satisfied and therefore Kostant's problem has the positive answer for $L(s_is_j)$ by Theorem 4. This completes the proof. $\square$

4.4. **Kostant's problem for $\mathfrak{sl}_n$, $n \leq 3$.**

Proposition 20. *Assume that $n \leq 3$ and $w \in W$. Then Kostant's problem has the positive answer for $L(w)$.*

Proof. The statement is trivial for $n = 1$. In the case $n = 2$ for $w = e$ the statement follows from [Ja, 6.9(10)] (as $L(e)$ is a quotient of the dominant Verma module) and for $w = s_1$ it follows from [Jo, Corollary 6.4] (as $L(s_1)$ is a Verma module).

Finally, in the case $n = 3$ for $w = e$ the statement follows, as above, from [Ja, 6.9(10)], for $w = s_1, s_2$ it follows from Corollary 18, for $w = s_1s_1, s_2s_1$ it follows from [GJ2, Theorem 4.4], and, finally, for $w = s_1s_2s_1$ it follows, as above, from [Jo, Corollary 6.4]. □

4.5. **Kostant's problem for $\mathfrak{sl}_4$.**

Proposition 21. *Assume that $n = 4$ and $w \in W$. Then Kostant's problem has the positive answer for $L(w)$ if and only if $w \neq s_1s_3, s_2s_1s_3$.*

Proof. The group S_4 has 10 involutions: e, s_1, s_2, s_3, s_1s_3, $s_1s_2s_1$, $s_3s_2s_3$, $s_2s_1s_3s_2$, $s_1s_2s_3s_2s_1$, and $s_2s_1s_2s_3s_2s_1$. The module $L(e)$ is a quotient of the dominant Verma module, and hence for $L(e)$ the claim follows from [Ja, 6.9(10)]. The module $L(s_2s_1s_2s_3s_2s_1)$ is a Verma module and hence for this module the claim follows from [Jo, Corollary 6.4]. For $L(s_1)$, $L(s_2)$, $L(s_3)$ the claim follows from Corollary 18. The left cell of each of the elements $s_1s_2s_1$, $s_3s_2s_3$, $s_2s_1s_3s_2$, $s_1s_2s_3s_2s_1$ contains an element of the form $w_0'w_0$, where w_0' is the longest element of some parabolic subgroup. Hence for $L(s_1s_2s_1)$, $L(s_3s_2s_3)$, $L(s_2s_1s_3s_2)$ and $L(s_1s_2s_3s_2s_1)$ the claim follows from [GJ2, Theorem 4.4] and [MS1, Theorem 60]. Finally, for $L(s_1s_3)$ the claim follows from Corollary 19 (or [MS2, Theorem 13]). Note that the answer is negative only in the case of $L(s_1s_3)$. The left cell of s_1s_3 contains one more element, namely $s_2s_1s_3$. The statement of the proposition now follows from [MS1, Theorem 60]. □

4.6. **Kostant's problem for $\mathfrak{sl}_5$.**

Proposition 22. *Assume that $n = 5$ and $w \in W$. Then Kostant's problem has the positive answer for $L(w)$ if and only if w does not belong to the left cells containing one of the following involutions: s_1s_3, s_2s_4, $s_2s_3s_2$, $s_1s_2s_1s_4$ or $s_1s_3s_4s_3$.*

Proof. The group S_5 has 26 involutions. As above, Kostant's problem has the positive answer for $L(e)$ since it is a quotient of the dominant Verma module. The answers for $L(s_1)$, $L(s_2)$, $L(s_3)$ and $L(s_4)$ are also positive by Corollary 18, and for $L(s_1s_2s_1s_3s_2s_1s_4s_3s_2s_1)$ the answer is positive as this module is a Verma module. The involutions

$$\begin{array}{lll}
s_1s_2s_1, & s_1s_2s_1s_3s_2s_1, & s_1s_2s_3s_4s_3s_2s_1,\\
s_3s_4s_3, & s_2s_3s_2s_4s_3s_2, & s_2s_1s_3s_2s_1s_4s_3s_2,\\
s_3s_2s_4s_3, & s_1s_3s_2s_1s_4s_3, & s_1s_2s_3s_2s_4s_3s_2s_1,\\
s_2s_1s_3s_2, & s_2s_1s_3s_4s_3s_2, & s_1s_2s_1s_3s_4s_3s_2s_1.
\end{array}$$

are all in left cells containing elements on the form $w_0'w_0$ where w_0' is the longest element of some parabolic subgroup of W. Hence Kostant's problem has the positive answer for the corresponding simple modules by [GJ2, Theorem 4.4] and [MS1, Theorem 60]. The involutions $s_2s_3s_4s_3s_2$ and $s_2s_4s_3s_2s_1$ are both in left cells containing elements on the form $sw_0'w_0$, where w_0' is the longest element of some parabolic subgroup, and s is a simple reflection of the same parabolic subgroup, so Kostant's problem has the positive answer for $L(s_2s_4s_3s_2s_1)$ and $L(s_2s_3s_4s_3s_2)$ by [Ma1, Theorem 1] and [MS1, Theorem 60]. Kostant's problem has the positive answer for $L(s_1s_4)$, and the negative answer for $L(s_1s_3)$ and $L(s_2s_4)$, by Corollary 19.

Finally, the fact that Kostant's problem has the negative answer for $L(s_2s_3s_2)$, $L(s_1s_3s_4s_3)$ and $L(s_1s_2s_1s_4)$ follows from Theorem 12 by a direct computation as described in Remark 14. Consider first the involution $s_2s_3s_2$ for which we have $\mathfrak{a}(s_2s_3s_2) = 3$. A direct calculation shows that the graded component $\mathtt{P}^{\hat{\mathbf{R}}}(s_2s_3s_2)_2$ has, after forgetting the grading, the following form:

$$L(s_3s_2) \oplus L(s_3s_2s_4s_3) \oplus L(s_2s_1s_3s_2s_4s_3) \oplus L(s_3s_2s_1s_4s_3s_2) \oplus \\ \oplus L(s_2s_3s_2s_1) \oplus L(s_2s_3) \oplus L(s_2s_1s_3s_2) \oplus L(s_2s_3s_2s_4).$$

Another calculation shows that the graded component $\Delta(e)_2$, after forgetting the grading, the following form:

$$L(s_3s_4) \oplus L(s_2s_4) \oplus L(s_2s_1) \oplus L(s_3s_2) \oplus L(s_1s_3) \oplus L(s_1s_4) \oplus \\ \oplus L(s_4s_3) \oplus L(s_1s_2) \oplus L(s_2s_3) \oplus L(s_2s_1s_3s_2) \oplus L(s_3s_2s_4s_3).$$

Hence the module $L(s_3s_2s_1s_4s_3s_2)$ occurs in $\mathtt{P}^{\hat{\mathbf{R}}}(s_2s_3s_2)_2$ but not in $\Delta(e)_2$. By Theorem 12 and Remark 13 this implies that Kostant's problem has the negative answer for $L(s_2s_3s_2)$.

For the involution $s_1s_2s_1s_4$ we have $\mathfrak{a}(s_1s_2s_1s_4) = 4$. A direct calculation shows that the module $L(s_1s_4s_3s_2s_1)$ occurs in $\mathtt{P}^{\hat{\mathbf{R}}}(s_1s_2s_1s_4)_3$ but not in $\Delta(e)_3$. Hence again Remark 13 implies that Kostant's problem has the negative answer for $L(s_1s_2s_1s_4)$. Applying the symmetry of the root system we obtain that the answer for $L(s_4s_3s_4s_1)$ is also negative and it remains to observe that $s_4s_3s_4s_1 = s_1s_3s_4s_3$. □

Figures 1, 2 and 3 show the three two-sided cells of S_5 which contain left cells for elements of which Kostant's problem has the negative answer. These left cells are columns, which are marked with an arrow. The rows and columns are indexed by the left and the right Young tableaux in the corresponding Robinson-Schensted pair. Also, each element is denoted simply by the sequence of indices in some shortest expression, i.e. $s_1s_3s_2$ is denoted by 132. There seems to exist some hidden symmetry in these pictures, but we do not understand it yet.

4.7. **Kostant's problem for $\mathfrak{sl}_6$.** We are not able yet to give a complete answer to Kostant's problem in the case $\mathfrak{g} = \mathfrak{sl}_6$. The group S_6 has 76 involutions. For 47 involutions one can use arguments analogous to the arguments above to show that Kostant's problem has the positive answer, for 20 involutions one can analogously show that Kostant's problem has the negative answer. This leaves 9 involutions for which the answer is still unclear.

	1 2 3 / 4 5	1 2 4 / 3 5	1 3 4 / 2 5	1 3 5 / 2 4	1 2 5 / 3 4
1 2 3 / 4 5	3243	324	3214	32143	321432
1 2 4 / 3 5	243	24	214	2143	21432
1 3 4 / 2 5	1321	124	14	143	1432
1 3 5 / 2 4	13243	1324	134	13	132
1 2 5 / 3 4	213243	21324	2134	213	2132
		↑		↑	

FIGURE 1

	1 2 3 / 4 / 5	1 2 4 / 3 / 5	1 2 5 / 3 / 4	1 3 4 / 2 / 5	1 3 5 / 2 / 4	1 4 5 / 2 / 3
1 2 3 / 4 / 5	343	3432	32432	34321	324321	3214321
1 2 4 / 3 / 5	2343	23432	2432	234321	24321	214321
1 2 5 / 3 / 4	23243	2324	232	23214	2321	21321
1 3 4 / 2 / 5	12343	123432	12432	1234321	124321	14321
1 3 5 / 2 / 4	123243	12324	1232	123214	12321	1321
1 4 5 / 2 / 3	1213243	121324	12132	12134	1213	121
			↑			

FIGURE 2

There are 44 involutions, which lie in left cells containing an element of the form $w_0'w_0$ or $sw_0'w_0$, and hence Kostant's problem has the positive answer for these involutions. The remaining three for which Kostant's problem for sure has a positive answer are $L(s_1s_4)$, $L(s_1s_5)$ and $L(s_2s_5)$ (this follows from Corollary 19). By the same corollary, Kostant's problem has the negative answer for $L(s_1s_2)$, $L(s_2s_4)$ and $L(s_3s_5)$. This and computations as described in Remark 14 show that

	1 2 / 3 4 / 5	1 2 / 3 5 / 4	1 3 / 2 4 / 5	1 3 / 2 5 / 4	1 4 / 2 5 / 3
1 2 / 3 4 / 5	213432	2132432	21343	21324321	2134321
1 2 / 3 5 / 4	2321432	21321432	232143	2132143	213214
1 3 / 2 4 / 5	13432	132432	1343	1324321	134321
1 3 / 2 5 / 3	12321432	1321432	1232143	132143	13214
1 4 / 2 5 / 3	1213432	121432	121343	12143	1214
			↑		↑

FIGURE 3

Kostant's problem has the negative answer for the following 17 involutions:

$$\begin{array}{llll}
s_1s_3, & s_1s_3s_5, & s_1s_4s_3s_5s_4, & s_1s_2s_1s_4s_5s_4,\\
s_3s_5, & s_1s_2s_1s_4 & s_2s_1s_3s_2s_5, & s_1s_2s_1s_3s_2s_1s_5,\\
s_2s_4, & s_1s_3s_4s_3, & s_1s_2s_3s_2s_1s_5, & s_1s_3s_4s_3s_5s_4s_3,\\
s_2s_3s_2, & s_2s_4s_5s_4, & s_1s_3s_4s_5s_4s_3, & s_1s_3s_2s_1s_4s_5s_4s_3,\\
s_3s_4s_3, & s_2s_3s_2s_5, & s_2s_3s_2s_4s_3s_2, & s_1s_2s_1s_3s_4s_3s_5s_4s_3s_2s_1,
\end{array}$$

The remaining 9 involutions, which are not covered by Theorem 12 are:

$$\begin{array}{lll}
s_3s_2s_4s_3, & s_2s_3s_4s_3s_2, & s_2s_3s_2s_4s_5s_4s_3s_2,\\
s_1s_4s_5s_4, & s_2s_1s_4s_3s_2s_5s_4, & s_1s_3s_2s_4s_3s_2s_1s_5s_4s_3,\\
s_1s_2s_1s_5, & s_1s_2s_3s_2s_4s_3s_2s_1, & s_2s_1s_3s_2s_1s_4s_5s_4s_3s_2.
\end{array}$$

For these involutions the answer is still unclear.

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