

# KOSZUL DUALITY FOR STRATIFIED ALGEBRAS I. QUASI-HEREDITARY ALGEBRAS

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ABSTRACT. We give a complete picture of the interaction between Koszul and Ringel dualities for quasi-hereditary algebras admitting linear tilting (co)resolutions of standard and costandard modules. We show that such algebras are Koszul, that the class of these algebras is closed with respect to both dualities and that on this class these two dualities commute. All arguments reduce to short computations in the bounded derived category of graded modules.

## 1. INTRODUCTION

Let  $A$  be a positively graded quasi-hereditary algebra. Then there exist two classical duals for  $A$ : the Ringel dual  $R(A)$  ([Ri]), which is the endomorphism algebra of the characteristic tilting  $A$ -module, and the Koszul dual  $E(A)$  ([ADL2]), which is the extension algebra of the direct sum of all simple  $A$ -modules. The algebra  $R(A)$  is always quasi-hereditary, while the algebra  $E(A)$  is quasi-hereditary only under some additional assumptions. For example,  $E(A)$  is quasi-hereditary if both, projective resolutions of all standard  $A$ -modules and injective coresolutions of all costandard  $A$ -modules, are linear (see [ADL2]). Such algebras were called *standard Koszul* in [ADL2].

The natural question to ask is whether  $R(E(A)) \cong E(R(A))$ . This question was addressed in [MO], where it was shown that this is the case under some assumptions, which, roughly speaking, mean that the algebras  $A$ ,  $R(A)$ ,  $E(A)$ ,  $E(R(A))$  and  $R(E(A))$  are standard Koszul with respect to the grading, induced from the grading on  $A$ . The main disadvantage of this result was that the condition was not formulated in terms of  $A$ -modules and hence was very difficult to check.

The main motivation for the present paper was to find an easier condition which would guarantee the isomorphism  $R(E(A)) \cong E(R(A))$ . For this we further develop the approach of [MO], based on the category of linear complexes of tilting  $A$ -modules. The main point of the paper is that we find an easy way to check Koszulity of  $A$  and quasi-heredity of  $E(A)$  based on direct computations in the derived category. This looks much easier than, for example, the subtle analysis of the structure of projective resolutions, carried out in [ADL2].

A part of the condition, used in [MO], was formulated as follows: all standard and costandard  $A$ -modules have linear tilting (co)resolutions.

We call such algebras balanced. Using our computational approach we show that already this is enough to ensure that all algebras in the list  $A$ ,  $R(A)$ ,  $E(A)$ ,  $E(R(A))$  and  $R(E(A))$  are standard Koszul with respect to the induced grading and derive as a corollary that Koszul and Ringel dualities on such  $A$  commute. Under our assumptions we reprove main results from [ADL2] and strengthen the main result from [MO]. Our main result is the following:

**Theorem 1.** *For every balanced quasi-hereditary algebra  $A$  we have:*

- (i) *The algebra  $A$  is Koszul and standard Koszul.*
- (ii) *The algebras  $A$ ,  $R(A)$ ,  $E(A)$ ,  $E(R(A))$  and  $R(E(A))$  are balanced.*
- (iii) *Every simple  $A$ -module is represented by a linear complex of tilting modules.*
- (iv)  *$R(E(A)) \cong E(R(A))$  as graded quasi-hereditary algebras.*

By [BGS, MOS] we also have equivalences of the corresponding bounded derived categories of graded modules for the algebras  $A$ ,  $E(A)$ ,  $R(A)$  and  $R(E(A)) \cong E(R(A))$ . Another advantage of our approach is that it admits a straightforward generalization to stratified algebras, both in the sense of [ADL1] and [CPS]. There is, however, a technical complication in this generalization: In the case when a stratified algebra is not quasi-hereditary, it has infinite global dimension and hence the Koszul dual is infinite-dimensional. Thus to apply our approach one has first to develop a sensible tilting theory for infinite-dimensional stratified algebras. This is an extensive technical work, which will be carried out in the separate paper [Ma2]. In the present paper we avoid these technicalities to make our approach clearer. Another advantage of our approach is that it generalizes to infinite-dimensional quasi-hereditary algebras of finite homological dimension.

The paper is organized as follows: In Section 2 we collect all necessary preliminaries about graded quasi-hereditary algebras. In Section 3 we prove our main result. We complete the paper with some examples in Section 4.

**Acknowledgments.** This research is partially supported by the Swedish Research Council. Most of the results of the paper were obtained during the visit of the author to Department of Algebra and Number Theory, Eötvös University, Budapest in September 2008. The hospitality of Eötvös University is gratefully acknowledged. The author also thanks István Ágoston and Erzsébet Lukács for their hospitality and many stimulating discussions.

## 2. GRADED QUASI-HEREDITARY ALGEBRAS

By  $\mathbb{N}$  we denote the set of all positive integers. By a module we always mean a *graded left* module, and by grading we always mean

*$\mathbb{Z}$ -grading.* Let  $\mathbb{k}$  be an algebraically closed field and  $A$  be a basic, finite-dimensional, positively graded and quasi-hereditary  $\mathbb{k}$ -algebra. Let  $\Lambda = \{1, \dots, n\}$  and  $\{e_\lambda : \lambda \in \Lambda\}$  be a complete set of pairwise orthogonal primitive idempotents for  $A$  such that the natural order on  $\Lambda$  is the one which defines the quasi-hereditary structure on  $A$ . Then  $A = \bigoplus_{i \geq 0} A_i$ ,  $A_0 \cong \mathbb{k}e_1 \oplus \dots \oplus \mathbb{k}e_n$  and  $\text{rad}(A) = \bigoplus_{i > 0} A_i$ .

Let  $A\text{-gmod}$  denote the category of all finite-dimensional graded  $A$ -modules. Morphisms in this category are homogeneous morphism of degree zero between graded  $A$ -modules. This is an abelian category with enough projectives and enough injectives. For  $i \in \mathbb{Z}$  we denote by  $\langle i \rangle$  the autoequivalence of  $A\text{-gmod}$ , which shifts the grading as follows:  $(M\langle i \rangle)_j = M_{i+j}$ ,  $j \in \mathbb{Z}$ . We adopt the notation  $\text{hom}_A$  and  $\text{ext}_A^i$  to denote homomorphisms and extensions in  $A\text{-gmod}$ .

For  $\lambda \in \Lambda$  we consider the graded indecomposable projective module  $P(\lambda) = Ae_\lambda$ , its graded simple quotient  $L(\lambda) = P(\lambda)/\text{rad}(A)P(\lambda)$  and the graded indecomposable injective envelop  $I(\lambda)$  of  $L(\lambda)$ . Let  $\Delta(\lambda)$  be the standard quotient of  $P(\lambda)$  and  $\nabla(\lambda)$  be the costandard submodule of  $I(\lambda)$ . By [MO, Corollary 5], there exists a graded lift  $T(\lambda)$  of the indecomposable tilting module corresponding to  $\lambda$  such that  $\Delta(\lambda)$  is a submodule of  $T(\lambda)$  and  $\nabla(\lambda)$  is a quotient of  $T(\lambda)$ .

For every  $i \in \mathbb{Z}$  we will say that *centroids* of the modules  $L(\lambda)\langle i \rangle$ ,  $\Delta(\lambda)\langle i \rangle$ ,  $\nabla(\lambda)\langle i \rangle$ ,  $P(\lambda)\langle i \rangle$ ,  $T(\lambda)\langle i \rangle$  and  $T(\lambda)\langle i \rangle$  belong to  $-i$ . Simple, projective, injective, standard, costandard and tilting  $A$ -modules will be called *structural* modules. A complex  $\mathcal{X}^\bullet$

$$(\mathcal{X}^\bullet, d_\bullet) : \dots \xrightarrow{d_{i-2}} \mathcal{X}^{i-1} \xrightarrow{d_{i-1}} \mathcal{X}^i \xrightarrow{d_i} \mathcal{X}^{i+1} \xrightarrow{d_{i+1}} \dots$$

of structural  $A$ -modules is called *linear* provided that for every  $i \in \mathbb{Z}$  centroids of all indecomposable direct summands of  $\mathcal{X}^i$  belong to  $-i$ .

The algebra  $A$  is called *standard Koszul* provided that all standard modules have linear projective resolutions and all costandard modules have linear injective coresolutions (see [ADL2]). The algebra  $A$  is called *balanced* provided that all standard modules have linear tilting coresolutions and all costandard modules have linear tilting resolutions (see [MO], where a stronger condition was imposed, however, we will show that both conditions are equivalent). The algebra  $A$  is called *Koszul* provided that projective resolutions of simple  $A$ -modules are linear (see [Pr, BGS, MOS]). Denote by  $E(A)$  the opposite of the Yoneda extension algebra of the direct sum of all simple  $A$ -modules. If  $A$  is Koszul, the algebra  $E(A)$  is called the *Koszul dual* of  $A$  and we have that  $E(A)$  is Koszul as well and  $E(E(A)) \cong A$ .

Let  $\mathcal{D}^b(A)$  denote the bounded derived category of  $A\text{-gmod}$ . For  $i \in \mathbb{Z}$  we denote by  $[i]$  the autoequivalence of  $\mathcal{D}^b(A)$ , which shifts the position of the complex as follows:  $\mathcal{X}[i]^j = \mathcal{X}^{i+j}$ ,  $j \in \mathbb{Z}$  and  $\mathcal{X}^\bullet \in \mathcal{D}^b(A)$ . As usual, we identify  $A$ -modules with complexes concentrated

in position 0. If  $A$  is Koszul, then the Koszul duality functor

$$K = \mathcal{R}\mathrm{hom}_A(\oplus_{i \in \mathbb{Z}} \mathcal{P}\langle i \rangle[-i]^\bullet, -),$$

where  $\mathcal{P}^\bullet$  is the projective resolution of the direct sum of simple  $A$ -modules (see [BGS, MOS]), is well-defined and gives rise to an equivalence from  $\mathcal{D}^b(A)$  to  $\mathcal{D}^b(E(A))$ .

Denote by  $\mathfrak{LT}$  the full subcategory of  $\mathcal{D}^b(A)$ , which consists of all linear complexes of tilting  $A$ -modules. The category  $\mathfrak{LT}$  is equivalent to  $E(R(A))\text{-gmod}$  and the simple objects of  $\mathfrak{LT}$  have the form  $T(\lambda)\langle -i \rangle[i]$ ,  $\lambda \in \Lambda$ ,  $i \in \mathbb{Z}$  ([MO]).

Let  $R(A)$  denote the *Ringel dual* of  $A$ , which is the opposite of the (graded) endomorphism algebra of the characteristic tilting module  $T = \oplus_{\lambda \in \Lambda} T(\lambda)$ . The algebra  $R(A)$  is quasi-hereditary with respect to the opposite order on  $\Lambda$ . The first Ringel duality functor

$$F = \mathcal{R}\mathrm{hom}_A(\oplus_{i \in \mathbb{Z}} T\langle i \rangle, -)$$

induces an equivalence from  $\mathcal{D}^b(A)$  to  $\mathcal{D}^b(R(A))$ , which maps tilting modules to projectives, costandard modules to standard and injective modules to tilting. The second Ringel duality functor

$$G = \mathcal{R}\mathrm{hom}_A(-, \oplus_{i \in \mathbb{Z}} T\langle i \rangle)^*$$

where  $*$  denotes the usual duality, induces an equivalence from  $\mathcal{D}^b(A)$  to  $\mathcal{D}^b(R(A))$ , which maps tilting modules to injectives, standard modules to costandard and projective modules to tilting.

### 3. THE MAIN RESULT

The aim of this section is to prove Theorem 1. For this we fix a balanced algebra  $A$  throughout. For  $\lambda \in \Lambda$  we denote by  $\mathcal{S}_\lambda^\bullet$  and  $\mathcal{C}_\lambda^\bullet$  the linear tilting coresolution of  $\Delta(\lambda)$  and resolution of  $\nabla(\lambda)$ , respectively. We will need the following easy observation from [MO] and include the proof for the sake of completeness.

**Lemma 2** ([MO]). *The natural grading on  $R(A)$ , induced from  $A\text{-gmod}$ , is positive.*

*Proof.* Let  $\lambda, \mu \in \Lambda$ . Then  $T(\lambda)$  has a standard filtration and  $T(\mu)$  has a costandard filtration ([Ri]). As standard modules are left orthogonal to costandard modules ([Ri]), every morphism from  $T(\lambda)$  to  $T(\mu)\langle j \rangle$ ,  $j \in \mathbb{Z}$ , is induced by a morphism from some standard module from a standard filtration of  $T(\lambda)$  to some costandard module from a costandard filtration of  $T(\mu)$ . Hence to prove our claim it is enough to show that every standard module occurring in the standard filtration of  $T(\lambda)$  and different from  $\Delta(\lambda)$  has the form  $\Delta(\nu)\langle j \rangle$  for some  $j > 0$ ; and that every costandard module occurring in the costandard filtration of  $T(\mu)$  and different from  $\nabla(\mu)$  has the form  $\nabla(\nu)\langle j \rangle$  for some  $j < 0$ .

We will prove the result for  $T(\lambda)$  and for  $T(\mu)$  the proof is similar. We use induction on  $\lambda$ . For  $\lambda = 1$  the claim is trivial. For  $\lambda > 1$  we consider the first two terms of the linear tilting coresolution of  $\Delta(\lambda)$ :

$$0 \rightarrow \Delta(\lambda) \rightarrow T(\lambda) \rightarrow X.$$

By linearity of our resolution, all direct summands of  $X$  have the form  $T(\nu)\langle 1 \rangle$  for some  $\nu < \lambda$ . All modules from the standard filtration of  $T(\lambda)$ , except for  $\Delta(\lambda)$ , occur in a standard filtration of  $X$ . Hence the necessary claim follows from the inductive assumption.  $\square$

From Lemma 2 we directly have the following:

**Corollary 3.** *We have  $\text{hom}_A(T(\lambda)\langle i \rangle, T(\mu)) = 0$ ,  $\lambda, \mu \in \Lambda$ ,  $i \in \mathbb{N}$ .*

Corollary 3 allows us to formulate the following main technical tool of our analysis. Let  $\mathcal{X}^\bullet$  and  $\mathcal{Y}^\bullet$  be two bounded complexes of tilting modules. We will say that  $\mathcal{X}^\bullet$  *dominates*  $\mathcal{Y}^\bullet$  provided that for every  $i \in \mathbb{Z}$  the following holds: if the centroid of an indecomposable summand of  $\mathcal{X}^i$  belongs to  $j$  and the centroid of an indecomposable summand of  $\mathcal{Y}^i$  belongs to  $j'$ , then  $j < j'$ .

**Corollary 4.** *Let  $\mathcal{X}^\bullet$  and  $\mathcal{Y}^\bullet$  be two bounded complexes of tilting modules. Assume that  $\mathcal{X}^\bullet$  dominates  $\mathcal{Y}^\bullet$ . Then  $\text{Hom}_{\mathcal{D}^b(A)}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet) = 0$ .*

*Proof.* Since tilting modules are self-orthogonal, by [Ha, Chapter III(2), Lemma 2.1] the necessary homomorphism space can be computed already in the homotopy category. Since  $\mathcal{X}^\bullet$  dominates  $\mathcal{Y}^\bullet$ , from Corollary 3 we obtain  $\text{Hom}_A(\mathcal{X}^i, \mathcal{Y}^i) = 0$  for all  $i$ . The claim follows.  $\square$

**Proposition 5.** *For every  $\lambda \in \Lambda$  the module  $L(\lambda)$  is isomorphic in  $\mathcal{D}^b(A)$  to a linear complex  $\mathcal{L}_\lambda^\bullet$  of tilting modules.*

*Proof.* Consider a minimal projective resolution  $\mathcal{P}^\bullet$  of  $L(\lambda)$ . Since  $A$  is positively graded, for every  $i \in \mathbb{Z}$  centroids of all indecomposable projective modules in  $\mathcal{P}^i$  belong to some  $j$  such that  $j \geq -i$ . Each projective has a standard filtration. Hence all centroids of standard subquotients in any standard filtration of an indecomposable projective module in  $\mathcal{P}^i$  also belong to some  $j$  such that  $j \geq -i$ .

Resolving each standard subquotient  $\Delta(\lambda)\langle j \rangle$  in every  $\mathcal{P}^i$  using  $\mathcal{S}_\lambda\langle j \rangle[i]^\bullet$ , we obtain a complex  $\overline{\mathcal{P}}^\bullet$  of tilting modules, which is isomorphic to  $L(\lambda)$  in  $\mathcal{D}^b(A)$ . By construction and the previous paragraph, for each  $i$  all centroids of indecomposable summands in  $\overline{\mathcal{P}}^i$  belong to some  $j$  such that  $j \geq -i$ .

Similarly, we consider a minimal injective coresolution  $\mathcal{Q}^\bullet$  of  $L(\lambda)$ . Since  $A$  is positively graded, for every  $i \in \mathbb{Z}$  centroids of all indecomposable injective modules in  $\mathcal{Q}^i$  belong to some  $j$  such that  $j \leq -i$ . Resolving each standard subquotient  $\nabla(\lambda)\langle j \rangle$  in every  $\mathcal{Q}^i$  using  $\mathcal{C}_\lambda\langle j \rangle[-i]^\bullet$ ,

we obtain another complex,  $\overline{\mathcal{Q}}^\bullet$ , of tilting modules, which is isomorphic to  $L(\lambda)$  in  $\mathcal{D}^b(A)$ . By construction, for each  $i$  all centroids of indecomposable summands in  $\overline{\mathcal{Q}}^i$  belong to some  $j$  such that  $j \leq -i$ .

Because of the uniqueness of the minimal tilting complex  $\mathcal{L}_\lambda^\bullet$ , representing  $L(\lambda)$  in  $\mathcal{D}^b(A)$ , we thus conclude that for all  $i \in \mathbb{Z}$  centroids of all indecomposable summands in  $\mathcal{L}_\lambda^i$  belong to  $-i$ . This means that  $\mathcal{L}_\lambda^\bullet$  is linear and completes the proof.  $\square$

**Corollary 6.** *The algebra  $A$  is Koszul.*

*Proof.* Assume that  $\text{ext}_A^i(L(\lambda), L(\mu)\langle j \rangle) \neq 0$  for some  $\lambda, \mu \in \Lambda$  and  $j \in \mathbb{Z}$ . Then  $j \leq -i$  as  $A$  is positively graded. By Proposition 5, such a nonzero extension corresponds to a non-zero homomorphism from  $\mathcal{L}_\lambda^\bullet$  to  $\mathcal{L}_\mu\langle j \rangle[i]^\bullet$ . Since both  $\mathcal{L}_\lambda^\bullet$  and  $\mathcal{L}_\mu\langle j \rangle[i]^\bullet$  are linear, the complex  $\mathcal{L}_\lambda^\bullet$  dominates  $\mathcal{L}_\mu\langle j \rangle[i]^\bullet$  for  $j < -i$  and the homomorphism space vanish by Corollary 4. Therefore  $j = -i$  and the claim follows.  $\square$

**Corollary 7.** *The algebra  $A$  is standard Koszul.*

*Proof.* That the minimal projective resolution of  $\Delta(\lambda)$  is linear, is proved similarly to Corollary 6. To prove that the minimal injective coresolution of  $\nabla(\mu)$  is linear we assume that  $\text{ext}_A^i(L(\lambda)\langle j \rangle, \nabla(\mu)) \neq 0$  for some  $\lambda, \mu \in \Lambda$  and  $j \in \mathbb{Z}$ . Then  $j \geq i$  as  $A$  is positively graded. As both  $L(\lambda)$  and  $\nabla(\mu)$  are represented in  $\mathcal{D}^b(A)$  by linear complexes of tilting modules, one obtains that for  $j > i$  the complex  $\mathcal{L}_\lambda\langle j \rangle[-i]^\bullet$  dominates  $\mathcal{C}_\mu^\bullet$ , and thus the extension must vanish by Corollary 4. Therefore  $j = i$  and the claim follows.  $\square$

**Corollary 8.** *The algebra  $R(A)$  is balanced.*

*Proof.* By Lemma 2, the algebra  $R(A)$  is positively graded with respect to the grading, induced from  $A\text{-gmod}$ . The functor  $F$  maps linear injective coresolutions of costandard  $A$ -modules to linear tilting coresolutions of standard  $R(A)$ -modules. The functor  $G$  maps linear projective resolutions of standard  $A$ -modules to linear tilting resolutions of costandard  $R(A)$ -modules. The claim follows.  $\square$

**Remark 9.** A standard Koszul quasi-hereditary algebra  $A$  is balanced if and only if  $R(A)$  is positively graded with respect to the grading induced from  $A\text{-gmod}$ , see [MO, Theorem 7].

**Corollary 10.** *The algebra  $R(A)$  is Koszul.*

*Proof.* This follows from Corollaries 6 and Corollaries 8.  $\square$

**Proposition 11.** (i) *The objects  $\mathcal{S}_\lambda^\bullet$ ,  $\lambda \in \Lambda$ , are standard objects in  $\mathfrak{L}\mathfrak{T}$  with respect to the natural order on  $\Lambda$ .*

(ii) *The objects  $\mathcal{C}_\lambda^\bullet$ ,  $\lambda \in \Lambda$ , are costandard objects in  $\mathfrak{L}\mathfrak{T}$  with respect to the natural order on  $\Lambda$ .*

*Proof.* We prove the claim (i), the claim (ii) is proved similarly. Let  $\lambda, \mu \in \Lambda$  be such that  $\lambda > \mu$ . Every first extension  $\xi$  from  $\mathcal{S}_\lambda^\bullet$  to  $T(\mu)\langle -i \rangle[i]$ ,  $i \in \mathbb{Z}$ , is a complex and hence is obtained as the cone of some morphism  $\varphi$  from  $\mathcal{S}[-1]_\lambda^\bullet$  to  $T(\mu)\langle -i \rangle[i]$ . The homology of the former complex is  $\Delta(\lambda)$  and the homology of the latter is  $T(\mu)$ , which has a costandard filtration, where  $\nabla(\lambda)$  does not occur (since  $\mu < \lambda$ ). Since standard modules are left orthogonal to costandard modules, we get that all homomorphisms and extensions from  $\Delta(\lambda)$  to  $T(\mu)$  vanish. Therefore  $\varphi$  is homotopic to zero, which splits  $\xi$ . The claim follows.  $\square$

**Proposition 12.** *For all  $\lambda, \mu \in \Lambda$  and  $i, j \in \mathbb{Z}$  we have*

$$(1) \quad \text{Hom}_{\mathcal{D}^b(\mathcal{E}\mathfrak{T})}(\mathcal{S}_\lambda^\bullet, \mathcal{C}_\mu\langle j \rangle[-i]^\bullet) = \begin{cases} \mathbb{k}, & \lambda = \mu, i = j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Via the equivalence  $\text{K} \circ \text{F}$ , the equality (1) reduces to the equality

$$\text{Hom}_{\mathcal{D}^b(A)}(\Delta(\lambda)^\bullet, \nabla(\mu)\langle j \rangle[-i]^\bullet) = \begin{cases} \mathbb{k}, & \lambda = \mu, i = j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

The latter equality is true as standard modules are left orthogonal to costandard modules (see [Ri]).  $\square$

**Corollary 13.** *The algebra  $E(R(A))$  is quasi-hereditary with respect to the natural order on  $\Lambda$ .*

*Proof.* By Propositions 11 and 12, standard  $E(R(A))$ -modules are left orthogonal to costandard. Now the claim follows from [DR, Theorem 1] (or [ADL1, Theorem 3.1]).  $\square$

**Corollary 14.** *The complexes  $\mathcal{L}_\lambda^\bullet$ ,  $\lambda \in \Lambda$ , are tilting objects in  $\mathcal{E}\mathfrak{T}$ .*

*Proof.* Because of [ADL1, Theorem 3.1] (or [DR, Ri]), we just need to show that any first extension from a standard object to  $\mathcal{L}_\lambda^\bullet$  splits, and that any first extension from  $\mathcal{L}_\lambda^\bullet$  to a costandard object splits. We prove the first claim and the second one is proved similarly.

Any first extension  $\xi$  from  $\mathcal{S}_\mu\langle -i \rangle[i]^\bullet$ ,  $\mu \in \Lambda$ ,  $i \in \mathbb{Z}$ , to  $\mathcal{L}_\lambda^\bullet$  is a cone of some homomorphism  $\varphi$  from  $\mathcal{S}_\mu\langle -i \rangle[i-1]^\bullet$  to  $\mathcal{L}_\lambda^\bullet$ . Thus  $\varphi$  corresponds to a (nonlinear) extension of degree  $1-i$  from  $\Delta(\mu)\langle -i \rangle$  to  $L(\lambda)$ . As  $A$  is standard Koszul by Corollary 7, we get that  $\varphi$  is homotopic to zero, and thus the extension  $\xi$  splits. The claim follows.  $\square$

**Corollary 15.** *There is an isomorphism  $E(A) \cong R(E(R(A)))$  of graded algebras, both considered with respect to the natural grading induced from  $\mathcal{D}^b(A)$ . In particular, we have  $R(E(A)) \cong E(R(A))$ .*

*Proof.* By Corollary 14, the algebra  $R(E(R(A)))$  is the opposite of the endomorphism algebra of  $\bigoplus_{\lambda \in \Lambda} \mathcal{L}_\lambda^\bullet$ . Since  $\mathcal{L}_\lambda^\bullet$  is isomorphic to  $L(\lambda)$  in  $\mathcal{D}^b(A)$ , from [Ha, Chapter III(2), Lemma 2.1] it follows that the same algebra is isomorphic to  $E(A)$ . The claim follows.  $\square$

**Corollary 16.** *Both  $E(A)$  and  $R(E(A))$  are positively graded with respect to the natural grading induced from  $\mathcal{D}^b(A)$ .*

*Proof.* For  $E(A)$  the claim is obvious. By Corollary 15, we have  $R(E(A)) \cong E(R(A))$ . As  $R(A)$  is positively graded with respect to the grading induced from  $\mathcal{D}^b(A)$  (Lemma 2), the algebra  $E(R(A))$  is positively graded with respect to the induced grading as well.  $\square$

**Proposition 17.** *The positively graded algebras  $E(A)$  and  $R(E(A))$  are balanced.*

*Proof.* Because of Corollary 8, it is enough to prove the claim for the algebra  $E(A)$ . Consider the algebra  $E(R(A))$ , whose module category is realized via  $\mathfrak{L}\mathfrak{T}$ .

**Lemma 18.** *The algebra  $E(R(A))$  is standard Koszul.*

*Proof.* We already know that  $E(R(A))$  is positively graded with respect to the grading, induced from  $\mathcal{D}^b(A)$ . Let us show that projective resolutions of standard  $E(R(A))$ -modules are linear. For injective resolutions of costandard modules the argument is similar.

We have to compute

$$(2) \quad \text{hom}_{\mathcal{D}^b(\mathfrak{L}\mathfrak{T})}(\mathcal{S}_\lambda^\bullet, T(\mu)\langle j \rangle[i])$$

for all  $\lambda, \mu \in \Lambda$  and  $i, j \in \mathbb{Z}$ . Via the equivalence  $K \circ F$ , the space (2) is isomorphic to the space  $\text{hom}_{\mathcal{D}^b(A)}(\Delta(\lambda), T(\mu)\langle j \rangle[i])$ . As  $T(\mu)$  has a costandard filtration and standard modules are left orthogonal to costandard, we get that the later space is non-zero only if  $i = 0$ . As  $R(A)$  is positively graded, we also get that  $j < 0$ . Applying [MOS, Theorem 22] we obtain that the standard  $E(R(A))$ -module  $\mathcal{S}_\lambda^\bullet$  has only linear extensions with simple  $E(R(A))$ -modules. This completes the proof.  $\square$

Using Lemma 18, the proof of Proposition 17 is completed similarly to the proof of Corollary 8.  $\square$

*Proof of Theorem 1.* The claim (i) follows from Corollaries 6 and 7. The claim (ii) follows from Corollary 8 and Proposition 17. The claim (iii) follows from Proposition 5. Finally, the claim (iv) follows from Corollary 15.  $\square$

#### 4. EXAMPLES

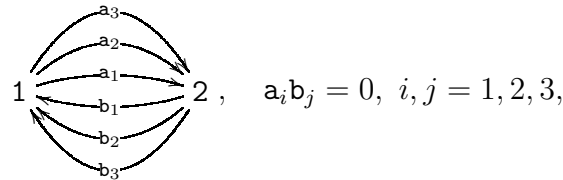
**Example 19.** Graded quasi-hereditary algebras, associated with blocks of the usual BGG category  $\mathcal{O}$  and the parabolic category  $\mathcal{O}$  for a semi-simple complex finite-dimensional Lie algebra, are balanced by [Ma1].

**Example 20.** The algebra  $A$  is called directed if either all standard or all costandard  $A$ -modules are simple (this is equivalent to the requirement that the quiver of  $A$  is directed with respect to the natural order

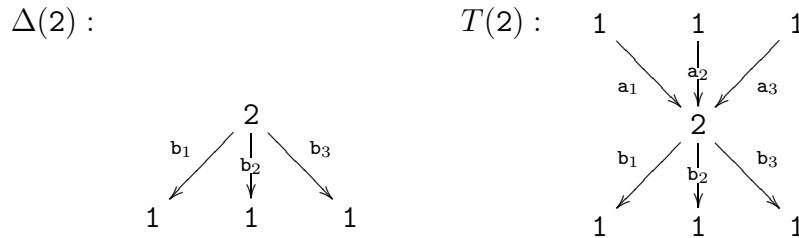
on  $\Lambda$ ). For a directed algebra  $A$  tilting modules are either injective (if standard modules are simple) or projective (if costandard modules are simple). Hence any directed Koszul algebra is balanced.

**Example 21.** Finite truncations  $V_{\mathcal{T}}$  of Cubist algebras from [CT, Section 6] are balanced. Indeed,  $V_{\mathcal{T}}$  is standard Koszul by [CT, Proposition 46], and that the Ringel dual of  $V_{\mathcal{T}}$  is positively graded with respect to the induced grading follows from [CT, Corollary 71]. So, the fact that  $V_{\mathcal{T}}$  is balanced follows from Remark 9.

**Example 22.** One explicit example. Consider the path algebra  $A$  of the following quiver with relations:



We have  $\Delta(1) \cong T(1) \cong L(1) \cong \nabla(1)$  and for  $\lambda = 2$  we have the following standard and tilting modules:

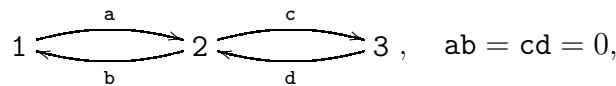


Hence we have the following linear tilting coresolution of  $\Delta(2)$ :

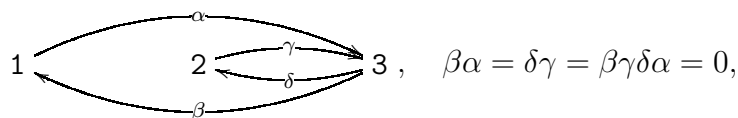
$$0 \rightarrow \Delta(2) \rightarrow T(2) \rightarrow T(1)\langle 1 \rangle \oplus T(1)\langle 1 \rangle \oplus T(1)\langle 1 \rangle \rightarrow 0.$$

Swapping  $a_i$  and  $b_i$ ,  $i = 1, 2, 3$ , defines an antiinvolution on  $A$ , which preserves the primitive idempotents. Hence there is a duality on  $A\text{-gmod}$ , which preserves isomorphism classes of simple modules. Applying this duality to the above resolution gives a linear tilting resolution of  $\nabla(2)$ . Thus  $A$  is balanced. In this example one can also arbitrarily increase or decrease the number of arrows.

**Example 23.** One computes that the path algebra of the following quiver with relations



is standard Koszul but not balanced. In fact, the Ringel dual of this algebra is the path algebra of the following quiver with relations



which is not Koszul (not even quadratic). So, our results can not be extended to all standard Koszul algebras.

**Remark 24.** Directly from the definition it follows that if the algebra  $A$  is balanced, then the algebra  $A/Ae_nA$  is balanced as well. It is also easy to see that if  $A$  and  $B$  are balanced, then both  $A \oplus B$  and  $A \otimes_{\mathbb{k}} B$  are balanced.

#### REFERENCES

- [ADL1] I. Ágoston, V. Dlab, E. Lukács, Stratified algebras. *C. R. Math. Acad. Sci. Soc. R. Can.* **20** (1998), no. 1, 22–28.
- [ADL2] I. Ágoston, V. Dlab, E. Lukács, Quasi-hereditary extension algebras. *Algebr. Represent. Theory* **6** (2003), no. 1, 97–117.
- [BGS] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory. *J. Amer. Math. Soc.* **9** (1996), no. 2, 473–527.
- [CT] J. Chuang, W. Turner, Cubist algebras. *Adv. Math.* **217** (2008), no. 4, 1614–1670.
- [CPS] E. Cline, B. Parshall, L. Scott, Stratifying endomorphism algebras. *Mem. Amer. Math. Soc.* **124** (1996), no. 591.
- [DR] V. Dlab, C. Ringel, The module theoretical approach to quasi-hereditary algebras. *Representations of algebras and related topics (Kyoto, 1990)*, 200–224, *London Math. Soc. Lecture Note Ser.*, **168**, Cambridge Univ. Press, Cambridge, 1992.
- [Ha] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*. *London Mathematical Society Lecture Note Series*, 119. Cambridge University Press, Cambridge, 1988.
- [Ma1] V. Mazorchuk, Applications of the category of linear complexes of tilting modules associated with the category  $\mathcal{O}$ , preprint arXiv:math/0501220, to appear in *Alg. Rep. Theory*.
- [Ma2] V. Mazorchuk, Koszul duality for stratified algebras II. Standardly stratified algebras, to appear.
- [MO] V. Mazorchuk, S. Ovsienko, A pairing in homology and the category of linear complexes of tilting modules for a quasi-hereditary algebra. With an appendix by Catharina Stroppel. *J. Math. Kyoto Univ.* **45** (2005), no. 4, 711–741.
- [MOS] V. Mazorchuk, S. Ovsienko, C. Stroppel, Quadratic duals, Koszul dual functors, and applications, preprint arXiv:math/0603475, to appear in *Trans. AMS*.
- [Pr] S. Priddy, Koszul resolutions. *Trans. Amer. Math. Soc.* **152** (1970), 39–60.
- [Ri] C. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.* **208** (1991), no. 2, 209–223.

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