

# On finitistic dimension of stratified algebras

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## Abstract

In this survey we discuss the results on the finitistic dimension of various stratified algebras. We describe what is already known, present some recent estimates, and list some open problems.

## 1 Introduction and preliminaries

Let  $A$  be a finite-dimensional, associative, and unital algebra over an algebraically closed field  $\mathbb{k}$ , and  $A\text{-mod}$  be the category of finite-dimensional left  $A$ -modules. Assume that the isomorphism classes of simple  $A$ -modules are indexed by  $\Lambda = \{1, 2, \dots, n\}$  and denote by  $L(\lambda)$ ,  $P(\lambda)$ ,  $I(\lambda)$ ,  $\lambda \in \Lambda$ , the corresponding simple module, its projective cover, and its injective envelope respectively. Remark that the elements of  $\Lambda$  are ordered in the natural way. For  $\lambda \in \Lambda$  set  $P^{>\lambda} = \bigoplus_{\mu > \lambda} P(\mu)$  and define the *standard* module  $\Delta(\lambda) = P(\lambda)/\text{Trace}_{P^{>\lambda}}(P(\lambda))$ . Denote by  $\mathcal{F}(\Delta)$  the full subcategory of  $A\text{-mod}$ , which consists of all modules, having a filtration with subquotients isomorphic to standard modules. Call the algebra  $A$  *strongly standardly stratified* (or an *SSS-algebra*) if  ${}_A A \in \mathcal{F}(\Delta)$ . The class of SSS-algebras contains the very important subclass of quasi-hereditary algebras, and forms a subclass of the class of standardly stratified algebras, introduced in [CPS]. SSS-algebras (sometimes also called just standardly stratified in the literature, which makes everything somewhat confusing) were intensively studied during the last decade, see [AHLU1, AHLU2, Ma] and references therein. Such algebras arise naturally in Lie theory, see [Ma]. In [AHLU1] it has been shown that both the projectively and the injectively defined finitistic dimensions of such algebras do not exceed  $2n - 2$ . Though this bound is exact for certain algebras, in most cases this estimate is very rough. For example any hereditary algebra is stratified (even quasi-hereditary) with respect to any order on  $\Lambda$ , see [DR, Theorem 1], and has global dimension 1.

In the present paper we try to approach rather non-symmetric situations, i.e. the one for which projective and injective dimensions can be different. Let  $\mathcal{P}^{<\infty}(A)$  and  $\mathcal{I}^{<\infty}(A)$  denote the full subcategories of  $A\text{-mod}$ , which consists of all modules  $M$  having finite projective or injective dimension respectively. We denote by  $\text{fdim}(A)$  the *projectively defined finitistic dimension* of  $A$ , that is the supremum of  $\text{pd}(M)$ , taken over all  $M \in \mathcal{P}^{<\infty}(A)$ ; and by  $\text{ifdim}(A)$  the *injectively defined finitistic dimension* of  $A$ , that is the supremum of  $\text{id}(M)$ , taken over all  $M \in \mathcal{I}^{<\infty}(A)$ .

For  $\lambda \in \Lambda$  define the *proper standard* module  $\overline{\Delta}(\lambda) = \Delta(\lambda)/\text{Trace}_{P(\lambda)}(\text{rad } \Delta(\lambda))$ . Dually one defines the *costandard* modules  $\nabla(\lambda)$  and the *proper costandard* modules  $\overline{\nabla}(\lambda)$ ,  $\lambda \in \Lambda$ . The categories  $\mathcal{F}(\nabla)$ ,  $\mathcal{F}(\overline{\Delta})$  and  $\mathcal{F}(\overline{\nabla})$  are defined analogously to  $\mathcal{F}(\Delta)$ . For all modules indexed by  $\lambda \in \Lambda$  the notation without index will mean the direct sum over all  $\lambda \in \Lambda$ , for example  $L = \bigoplus_{\lambda=1}^n L(\lambda)$  etc. According to [Dl2, La], an alternative description of SSS-algebras can be given requiring  $I \in \mathcal{F}(\overline{\nabla})$ .

Varying the requirements one gets many other classes of stratified algebras. The ones, which are important for the present paper, are *properly stratified algebras*, defined in [Dl1] via  ${}_A A \in \mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\Delta})$ , or, alternatively, via  $I \in \mathcal{F}(\nabla) \cap \mathcal{F}(\overline{\nabla})$ ; and *quasi-hereditary algebras*, defined as those properly stratified algebras, for which  $\Delta(\lambda) = \overline{\Delta}(\lambda)$  for all  $\lambda$ , which is equivalent to requiring  $\nabla(\lambda) = \overline{\nabla}(\lambda)$  for all  $\lambda$  (see for example [DR]).

## 2 General approach via tilting modules

### 2.1 Tilting modules and finitistic dimension

Let us forget the stratified structure for a moment. So, let  $A$  just be a finite-dimensional, associative, and unital  $\mathbb{k}$ -algebra. Recall, see [Mi], that an  $A$ -module  $T$  is called a *generalized tilting module* if  $T$  has finite projective dimension, is ext-self-orthogonal, and its additive closure  $\text{Add}(T)$  coresolves  ${}_A A$  in a finite number of steps. The *generalized cotilting modules* are defined dually. Looking at the homomorphisms in  $D^b(A)$  from  $T^\bullet[i]$  to the tilting coresolution of  ${}_A A$  one easily derives that  $\text{pd}(T)$ , in fact, equals the length of the shortest tilting coresolution of  ${}_A A$ . Here for  $M \in A\text{-mod}$  we denote by  $M^\bullet$  the complex, whose only non-zero component is  $M$ , concentrated in degree zero.

The trivial example of a generalized tilting module is  $P$ . If  $\text{gldim}(A) < \infty$ , then  $I$  is a generalized tilting module as well. In general  $I$  need not be a tilting module, since it may have infinite projective dimension. However, if  $I$  is a generalized tilting module then, embedding any  $M \in \mathcal{P}^{<\infty}(A)$  into an injective module, and applying  $\text{Hom}_A(-, I)$ , one derives that  $\text{fdim}(A) = \text{pd}(I)$ . Moreover, in this case any  $M \in \mathcal{P}^{<\infty}(A)$  can be substituted in  $D^b(A)$  by its finite projective resolution, which then can be turned into a finite injective complex in  $D^b(A)$ , since  $I$  is a tilting module (see for example [MO, Lemma 4]). This implies that any  $M \in \mathcal{P}^{<\infty}(A)$  has finite injective coresolution, in particular,  $\mathcal{P}^{<\infty}(A)$  is contravariantly finite in  $A\text{-mod}$ , see [AR].

### 2.2 Using self-dual tilting modules

We have seen that finding non-trivial generalized tilting modules in  $A\text{-mod}$  can give some interesting information about the homological behavior of  $A\text{-mod}$ . Especially if such modules are self-dual with respect to some contravariant exact equivalence on  $A\text{-mod}$  (usually called a *duality*). A duality is called *simple preserving* if it preserve the isomorphism classes of simple modules. A careful study of the proof of [MO, Theorem 1] shows that what is actually proved there is the following statement:

**Theorem 1.** *Let  $A$  be a finite-dimensional, associative, and unital  $\mathbb{k}$ -algebra for which  $\text{fdim}(A) < \infty$ . Assume that there exists a duality on  $A\text{-mod}$ , and a generalized tilting  $A$ -module  $T$ , such that  $Q^* \cong Q$  for every indecomposable  $Q \in \text{Add}(T)$ . Then  $\text{fdim}(A) = 2 \cdot \text{pd}(T)$ .*

*Proof.* Applying  $\star$  to the tilting coresolution of  $P$  gives a tilting resolution of  $I$ , in particular,  $\text{pd}(I) < \infty$ . Since  $\text{fdim}(A) < \infty$  we can embed any  $M \in A\text{-mod}$  with  $\text{pd}(M) = \text{fdim}(A)$  into an injective module, apply  $\text{Hom}_A(-, L)$ , and obtain  $\text{pd}(I) = \text{pd}(M) = \text{fdim}(A)$ . Further,  $\text{pd}(I)$  is exactly the maximal degree  $l$ , for which  $\text{Ext}_A^l(I, P)$  does not vanish. The latter can be computed in  $D^b(A)$  studying homomorphisms from the shifted tilting resolution of  $I$  to the tilting coresolution of  $P$ . Under our assumptions we can apply [MO, Lemma 1] and the arguments from [MO, Appendix]. The statement of the theorem follows.  $\square$

## 2.3 Applications to stratified algebras

Assuming  $A$  has some sort of stratification makes it in many cases possible to ensure the assumptions of Theorem 1. Indeed, assume that  $A$  is an SSS-algebra having a simple preserving duality (i.e. a duality, which preserves the isomorphism classes of simple modules). Then  $A$  is in fact properly stratified, the category  $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$  equals  $\text{Add}(T)$  for some generalized tilting module  $T$  called the *characteristic tilting module*, and the category  $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla)$  equals  $\text{Add}(C)$  for some generalized tilting module  $C$ , called the *characteristic cotilting module*. Moreover, if  $T \cong C$ , then all indecomposable direct summands of  $T$  are self-dual. The condition  $T \cong C$  is satisfied, for example, for quasi-hereditary algebras. Hence we obtain (see [MO, Theorem 1 and Corollary 1]).

**Corollary 1.** *Let  $A$  be an algebra having a simple preserving duality.*

1. *If  $A$  is an SSS-algebra and  $T \cong C$  then  $\text{fdim}(A) = 2 \cdot \text{pd}(T)$ .*
2. *If  $A$  is quasi-hereditary then  $\text{gldim}(A) = 2 \cdot \text{pd}(T)$ .*

## 3 Using tilting and various filtration dimensions

### 3.1 Filtration (co)dimensions

Let  $\mathcal{M}$  be a class of  $A$ -modules and  $\mathcal{F}(\mathcal{M})$  be the full subcategory in  $A\text{-mod}$ , which consists of all modules having a filtration with subquotients, isomorphic to modules from  $\mathcal{M}$ . For an  $A$ -module  $N$  we say that  $N$  has  $\mathcal{M}$ -filtration dimension (resp. codimension)  $l \in \{0, 1, \dots, \infty\}$  if there exists a resolution (resp. coresolution) of  $N$  by modules from  $\mathcal{F}(\mathcal{M})$  and  $l$  is the length of the shortest such resolution. For properly stratified algebras and SSS-algebras the following filtration (co)dimensions appear in a natural way: the *Weyl or standard filtration dimension*  $\text{dim}_\Delta(N)$  for  $\mathcal{M} = \{\Delta(\lambda), \lambda \in \Lambda\}$ ; the *proper standard filtration dimension*  $\text{dim}_{\overline{\Delta}}(N)$  for  $\mathcal{M} = \{\overline{\Delta}(\lambda), \lambda \in \Lambda\}$ ; the *good or costandard filtration*

*codimension*  $\text{codim}_{\nabla}(N)$  for  $\mathcal{M} = \{\nabla(\lambda), \lambda \in \Lambda\}$ ; and the *proper costandard filtration codimension*  $\text{codim}_{\overline{\nabla}}(N)$  for  $\mathcal{M} = \{\overline{\nabla}(\lambda), \lambda \in \Lambda\}$ . If  $A$  is an SSS-algebra, then both  $\dim_{\Delta}(N)$  and  $\text{codim}_{\overline{\nabla}}(N)$  are well-defined for all  $N \in A\text{-mod}$ . In [MP, Lemma 1] it is shown that  $\dim_{\Delta}(N) = \max\{l \mid \text{Ext}_A^l(N, \overline{\nabla}) \neq 0\}$ , and  $\text{codim}_{\overline{\nabla}}(N) = \max\{l \mid \text{Ext}_A^l(\Delta, N) \neq 0\}$ . In particular,  $\text{codim}_{\overline{\nabla}}(N) \leq \text{pd}(\Delta)$  for all  $N$ , whereas  $\dim_{\Delta}(N) < \infty$  is obviously equivalent to  $\text{pd}(M) < \infty$  as  $P \in \mathcal{F}(\Delta)$ . We define  $\dim_{\Delta}(A)$ ,  $\dim_{\overline{\Delta}}(A)$ ,  $\text{codim}_{\nabla}(A)$ ,  $\text{codim}_{\overline{\nabla}}(A)$ ,  $\text{fdim}_{\Delta}(A)$  and  $\text{fcodim}_{\nabla}(A)$  in the natural way and for an SSS-algebra we obtain  $\text{codim}_{\overline{\nabla}}(A) = \text{pd}(\Delta) = \text{pd}(T)$  by [MP, Lemma 1]. For properly stratified algebras we dually have  $\dim_{\overline{\Delta}}(A) = \text{id}(\nabla) = \text{id}(C)$ . Moreover, by [MP, Lemma 2] we also have  $\text{fdim}_{\Delta}(A) \leq \text{pd}(\nabla) = \text{pd}(C)$  and  $\text{fcodim}_{\nabla}(A) \leq \text{id}(\Delta) = \text{id}(T)$ .

These filtration (co)dimensions were reinterpreted in [MO, Subsection 4.3] in terms of tilting complexes. Thus we have that  $\dim_{\Delta}(N) \leq l$  if and only if  $N^{\bullet} \in D^b(A)$  is quasi-isomorphic to a tilting complex  $\mathcal{T}^{\bullet}$  such that  $\mathcal{T}^i = 0$  for all  $i < -l$ .

### 3.2 An “old” upper bound for $\text{fdim}(A)$

The following upper bound for  $\text{fdim}(A)$  is stated in [MP] for properly stratified algebras. Here we formulate the result for SSS-algebras and present a different proof based on tilting resolutions (see also [MO, Corollary 5]).

**Theorem 2.** *Let  $A$  be an SSS-algebra. Then  $\text{fdim}(A) \leq \text{fdim}_{\Delta}(A) + \text{pd}(T)$ .*

*Proof.* If  $M \in \mathcal{P}^{<\infty}(A)$ , then  $\text{pd}(M) = \max\{l \mid \text{Ext}_A^l(M, P) \neq 0\}$ . We substitute  $M^{\bullet} \in D^b(A)$  by a quasi-isomorphic tilting complex  $\mathcal{T}^{\bullet}$  satisfying  $\mathcal{T}^i = 0$  for all  $i < -\dim_{\Delta}(M)$ , and we substitute  $P$  by its tilting coresolution of length  $\text{pd}(T)$  (see Subsection 2.1). Since for the tilting complexes the homomorphisms in  $D^b(A)$  can be computed in the homotopic category, it is straightforward that  $\text{pd}(M) \leq \dim_{\Delta}(M) + \text{pd}(T)$  and the statement follows.  $\square$

If  $A$  is properly stratified, as an immediate consequence we have  $\text{fdim}(A) \leq \text{id}(C) + \text{pd}(T)$ , which is left-right symmetric and hence works for  $\text{ifdim}(A)$  as well. If  $A$  has a duality, everything reduces to  $\text{fdim}(A) \leq 2 \cdot \text{pd}(T)$ . As we have already seen in Subsection 2.3, the last bound is exact for quite a wide class of quasi-hereditary and stratified algebras, including Schur algebras, algebras associated with the BGG-category  $\mathcal{O}$  and its parabolic analogues.

### 3.3 $\text{fdim}(A)$ if one can control $\text{End}_A(T)$

Let  $A$  be an SSS-algebra. The endomorphism algebra  $R = \text{End}_A(T)$  of the characteristic tilting module  $T$  is called the *Ringel dual* of  $A$ . The algebra  $\text{End}_A(T)^{\text{opp}}$  is always an SSS-algebra with respect to the opposite order on  $\Lambda$ , see [AHLU2]. However,  $R$  does not need to be properly stratified, even in the case when  $A$  itself is properly stratified. The algebra  $R$  comes together with the *Ringel duality* functor  $F(-) = \text{Hom}_A(T, -) : A\text{-mod} \rightarrow R\text{-mod}$ ,

which induces an exact equivalence between the category of  $A$ -modules having a proper costandard filtration and the category of  $R$ -modules having a proper standard filtration.

The Ringel dual  $R$  is properly stratified if and only if the module  $T$  has a filtration with subquotients isomorphic to  $N(\lambda) = T(\lambda)/\text{Trace}_{T^{<\lambda}}(T(\lambda))$ , where  $T^{<\lambda} = \bigoplus_{\mu < \lambda} T(\mu)$  (see [FM]). In the case when  $R$  is properly stratified we denote by  $H(\lambda)$ ,  $\lambda \in \Lambda$ , the preimage under  $F$  of the indecomposable tilting  $R$ -module corresponding to  $\lambda$ , and by  $H$  the preimage under  $F$  of the characteristic tilting  $R$ -module  $T^{(R)}$ . The module  $H$  is called the *two-step tilting module* for  $A$  (since it is a tilting module for the Ringel dual of  $A$ ). The following properties of  $H$  were obtained in [FM]:

**Theorem 3.** *Assume that  $R$  is properly stratified and  $H$  is the two-step tilting module for  $A$ . Then*

1.  $H$  is a generalized tilting module;
2.  $\text{pd}(H) = \text{fdim}(A)$ ;
3.  $\mathcal{P}^{<\infty}(A)$  coincides with the category of  $A$ -modules, which admit a finite coresolution by modules from  $\text{Add}(H)$ , in particular,  $\mathcal{P}^{<\infty}(A)$  is contravariantly finite.

In particular, the module  $H$  is a good test module for  $\text{fdim}(A)$  and it completely describes  $\mathcal{P}^{<\infty}(A)$  in the homological sense. It is also shown in [FM] that the existence of  $H$  makes it possible to relate  $\text{fdim}(A)$  with the projective dimension of the characteristic tilting module:

**Theorem 4.** *Let  $A$  be a properly stratified algebra having a simple preserving duality. Assume  $R$  is properly stratified. Then*

1.  $\text{fdim}(A) = 2 \cdot \text{pd}(T^{(R)})$ .
2.  $\text{fdim}(A) = 2 \cdot \text{pd}(T)$ , in particular,  $\text{pd}(T) = \text{pd}(T^{(R)})$ , if  $R$  has a simple preserving duality itself.

### 3.4 A new lower bound for $\text{fdim}(A)$

Carefully combining the results of [MO] and [FM] one can deduce the following lower bound for the finitistic dimension of properly stratified algebras having a simple preserving duality.

**Theorem 5.** *Let  $A$  be properly stratified with a simple preserving duality  $\star$ . Then we have  $\text{fdim}(A) \geq 2 \cdot \text{fdim}_{\Delta}(A)$ .*

*Proof.* We have to produce a module from  $\mathcal{P}^{<\infty}(A)$  of projective dimension at least  $2 \cdot \text{fdim}_{\Delta}(A)$ . For this it is enough to show that any  $A$ -module  $M$ , such that  $\dim_{\Delta}(M) = \text{fdim}_{\Delta}(A)$ , satisfies  $\text{pd}(M) \geq 2 \cdot \text{fdim}_{\Delta}(A)$ . Set  $k = \text{fdim}_{\Delta}(A)$ . By [MO, Lemma 6],  $M^{\bullet}$  is quasi-isomorphic to a finite tilting complex,  $\mathcal{T}^{\bullet}$ , satisfying  $\mathcal{T}^i = 0$  for all  $i < -k$ .

Applying  $\star$  gives a finite cotilting complex  $\mathcal{C}^\bullet$  satisfying  $\mathcal{C}^i = 0$  for all  $i > k$ . Using [FM, Lemma 11] one finds a (possibly infinite) tilting complex  $\mathcal{Q}^\bullet$ , which is quasi-isomorphic to  $\mathcal{C}^\bullet$ , and which satisfies  $\mathcal{Q}^i = 0$  for all  $i > k$ . Moreover, using [FM, Lemma 12] one can also guarantee that  $\mathcal{T}^{-k}$  is non-trivial and is a direct summand of  $\mathcal{Q}^k$ . Using the arguments as in [MO, Section 3] one shows that there is a non-zero morphism from  $\mathcal{T}^\bullet[-2k]$  to  $\mathcal{Q}^\bullet$ , implying  $\text{pd}(M) \geq 2k$ .  $\square$

It is interesting to compare the bound, given in Theorem 5, with the results, described in Subsection 3.3. For this we will need the following lemma:

**Lemma 1.** *Let  $A$  be an SSS-algebra and  $M \in \mathcal{F}(\overline{\nabla})$  such that  $\text{pd}(M) < \infty$ . Then  $\dim_\Delta(M) = \text{pd}(F(M))$ .*

*Proof.* Taking the minimal projective resolution  $\mathcal{P}^\bullet$  of  $M$  and applying [MO, Lemma 4.1] we obtain a finite tilting complex  $\mathcal{T}^\bullet$ , which is quasi-isomorphic to  $M^\bullet \in D^b(A)$ . Using the arguments from the proof of [MO, Lemma 5] one even shows that  $\mathcal{T}^\bullet$  is quasi-isomorphic to a finite minimal (in the sense of [MO]) tilting complex  $\mathcal{Q}^\bullet$  satisfying  $\mathcal{Q}^i = 0$ ,  $i > 0$ . In other words, the module  $M$  admits a finite tilting resolution. Applying  $F$  gives a projective resolution of  $F(M)$  and we see that the length of the minimal tilting resolution of  $M$  is exactly  $\text{pd}(F(M))$ . From [MO, Lemma 6] it also follows that the length of the minimal tilting resolution of  $M$  equals  $\dim_\Delta(M)$ , completing the proof.  $\square$

**Corollary 2.** *Let  $A$  be properly stratified and assume that  $R$  is also properly stratified. Then  $\text{pd}(T^{(R)}) = \text{fdim}_\Delta(A)$ .*

*Proof.* By Lemma 1 we have  $\text{pd}(T^{(R)}) = \dim_\Delta(H)$ . Further, let  $M \in \mathcal{P}^{<\infty}(A)$  be such that  $l = \dim_\Delta(M) = \text{fdim}_\Delta(A)$ . By Theorem 3, we have a short exact sequence  $M \hookrightarrow H_1 \twoheadrightarrow K$ , where  $H_1 \in \text{Add}(H)$  and  $K \in \mathcal{P}^{<\infty}(A)$ . In particular,  $\dim_\Delta(H_1)$  and  $\dim_\Delta(K)$  do not exceed  $\dim_\Delta(M)$ . Applying  $\text{Hom}_A(-, \overline{\nabla})$  we obtain that  $\text{Ext}_A^l(H_1, \overline{\nabla})$  surjects onto  $\text{Ext}_A^l(M, \overline{\nabla}) \neq 0$  and hence  $\dim_\Delta(H_1) = l$  by [MP, Lemma 1]. This implies  $\dim_\Delta(H) = l$  and completes the proof.  $\square$

An immediate corollary of Theorem 4 and Corollary 2 is:

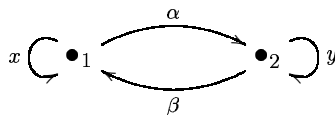
**Corollary 3.** *Let  $A$  be properly stratified having a simple preserving duality. Assume that  $R$  is also properly stratified. Then  $\text{fdim}(A) = 2 \cdot \text{fdim}_\Delta(A)$ .*

## 4 A counterexample

In [MP, Conjecture 1] it was conjectured that the finitistic dimension of a properly stratified algebra having a simple preserving duality always equals twice the projective dimension of the characteristic tilting module. As we saw above this is true under assumptions that  $R$  is properly stratified and has a simple preserving duality, which includes, in particular, the cases of quasi-hereditary algebras, and properly stratified algebras whose tilting modules are also cotilting. Unfortunately, in the full generality the statement of the conjecture is

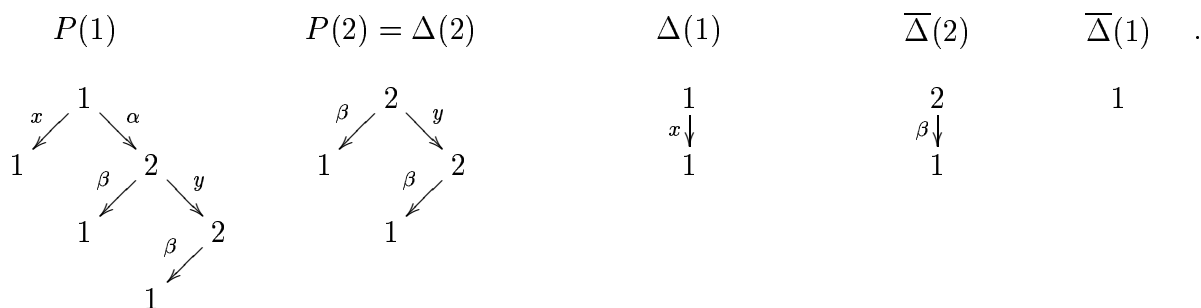
wrong. As a counter example one can consider the following algebra (the first counter example was constructed by the author, computed by Birge Huisgen-Zimmermann, and simplified by Steffen König).

Let  $A$  be the path algebra of the quiver

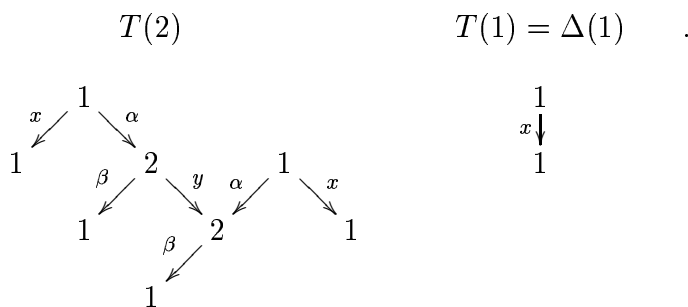


modulo the relations  $\alpha\beta = x^2 = y^2 = x\beta = \alpha x = 0$ . The map  $\alpha \mapsto \beta, \beta \mapsto \alpha$  extends to an anti-involution on  $A$  and hence gives rise to a duality on  $A\text{-mod}$ .

The radical filtrations of the projective, standard, and proper standard modules look as follows:



It follows that  $A$  is properly stratified. Since  $A$  has a duality, the socle filtrations of injective, costandard and proper costandard modules are duals of the radical filtrations of the corresponding projective, standard and proper standard modules above. The indecomposable tilting  $A$ -modules have the following radical filtration:



Neither  $T(2)$  nor  $\Delta(1)$  are projective, which implies that  $\text{fdim}(A) \geq 1$ . Further it is easy to see that there are the following minimal projective resolutions of tilting modules:  $0 \rightarrow P(2) \rightarrow P(1) \rightarrow T(1) \rightarrow 0$  and  $0 \rightarrow P(2) \rightarrow P(1) \oplus P(1) \rightarrow T(2) \rightarrow 0$ , and hence  $\text{pd}(T) = 1$ . It is also easy to see that any injection between tilting modules is an

isomorphism. This and [MO, Lemma 6] implies  $\text{fdim}_\Delta(A) = 0$ . Hence, by Theorem 2 we obtain  $\text{fdim}(A) \leq \text{fdim}_\Delta(A) + \text{pd}(T) = 1$  and thus  $\text{fdim}(A) = 1$ . In particular, in this example we have  $2 \cdot \text{fdim}_\Delta(A) = 0 < \text{fdim}(A) = 1 < 2 \cdot \text{pd}(T) = 2$ .

The example together with Theorem 2 motivates to make the following correction to [MP, Conjecture 1]:

**Corrected conjecture.** *Let  $A$  be a properly stratified algebra with a simple preserving duality. Then  $\text{fdim}(A) = \text{fdim}_\Delta(A) + \text{pd}(T)$ .*

Remark that for algebras having a simple preserving duality we always have  $\text{fdim}_\Delta(A) \leq \text{pd}(T) = \text{codim}_{\overline{\nabla}}(A)$ , see [MP].

## 5 A bound for $\text{ifdim}(A)$ in the case of an SSS-algebra

Up to this point all the results mentioned were about the projectively defined finitistic dimension. A natural question is: what can one say about the injectively defined version? In the case of an algebra having a (simple preserving) duality the answer is very easy: the injectively and the projectively defined finitistic dimensions obviously coincide. But what can be said in the general case? This question is still more or less open, see Section 6. Here we just present an easy upper bound for the case, when we have enough information about the Ringel dual of the algebra.

**Theorem 6.** *Let  $A$  be and SSS-algebra and assume that  $R$  is properly stratified. Then  $\text{ifdim}(A) \leq \text{fdim}(A) \leq \text{fdim}_\Delta(A) + \text{pd}(T)$ .*

*Proof.* Because of Theorem 2 it is enough to prove that  $\text{ifdim}(A) \leq \text{fdim}(A)$ . To do this we will show that for any  $M$  with  $\text{id}(M) = k < \infty$  we have  $\text{Ext}_A^k(H, M) \neq 0$ . From the definition of  $H(\lambda)$  it follows that there is an injection  $\overline{\nabla}(\lambda) \hookrightarrow H(\lambda)$ , and hence there is an injection  $L \hookrightarrow H$  with cokernel,  $K$  say. Applying  $\text{Hom}_A(-, M)$  to the short exact sequence  $L \hookrightarrow H \twoheadrightarrow K$  and using  $\text{id}(M) = k$  we get a surjection of  $\text{Ext}_A^k(H, M)$  onto  $\text{Ext}_A^k(L, M) \neq 0$ . This completes the proof.  $\square$

## 6 Some comments and questions

Summarizing the results of the paper we can say the following: if  $A$  is a properly stratified algebra having a simple preserving duality, then we have the following bounds for  $\text{fdim}(A)$ :

$$2 \cdot \text{fdim}_\Delta(A) \leq \text{fdim}(A) \leq \text{fdim}_\Delta(A) + \text{pd}(T) \leq 2 \cdot \text{pd}(T). \quad (1)$$

Most of the components of (1) have equivalent reformulations in other homological terms for  $A$  or for the Ringel dual  $R$  of  $A$ . In many cases, for example for quasi-hereditary algebras, we know that all inequalities in (1) are in fact equalities. We also know that the first and the third inequalities can be strict. This gives rise to the following question:



1. Let  $A$  be an SSS-algebra. How different can  $\text{pd}(T)$  and  $\text{fdim}_\Delta(A)$  be? The same for properly stratified algebras and for properly stratified algebras with duality.
2. Describe the class of SSS-algebras with properly stratified Ringel duals, satisfying  $\text{pd}(T) = \text{pd}(T^{(R)})$  (remark that the last condition immediately makes all inequalities of (1) into equalities). The same for properly stratified algebras and for properly stratified algebras with duality.

We saw that the module  $H$ , which appears in the case when  $R$  is properly stratified, can be used as a test module for  $\text{fdim}(A)$ . It was shown in [FM] that  $\text{End}_A(H)^{\text{opp}}$  is always an SSS-algebra. Hence, very natural questions are:

3. Find, in terms of  $A$ -mod, necessary and sufficient conditions for  $\text{End}_A(H)$  to be properly stratified.
4. Is there any relation between  $\text{fdim}(A)$  and  $\text{fdim}(\text{End}_A(H))$ ?

As we have already mentioned, much less is known about the injectively defined finitistic dimension, so even the following very general question is not answered:

5. Let  $A$  be an SSS-algebra or a properly stratified algebra. Can one use tilting modules to estimate or compute  $\text{ifdim}(A)$ ?

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