CATEGORY $\mathcal{O}$ FOR QUANTUM GROUPS

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Abstract. In this paper we study of the BGG-categories $\mathcal{O}_q$ associated to quantum groups. We prove that many properties of the ordinary BGG-category $\mathcal{O}$ for a semisimple complex Lie algebra carry over to the quantum case.

Of particular interest is the case when q is a complex root of unity. Here we prove a tensor decomposition for both simple modules, projective modules, and indecomposable tilting modules. Using the known Kazhdan-Lusztig conjectures for $\mathcal{O}$ and for finite dimensional U_q-modules we are able to determine all irreducible characters as well as the characters of all indecomposable tilting modules in $\mathcal{O}_q$.

As a consequence of our study of the root of unity case we deduce that the non-root of unity case (including the generic case) behaves like $\mathcal{O}$.

1. Introduction

Let $\mathfrak{g}$ denote a semisimple Lie algebra over $\mathbb{Q}$. The corresponding BGG-category $\mathcal{O}$, defined in [BGG], has been studied intensively over the last decades, see the recent monograph [Hu] for details.

In this paper we study similar categories for quantum groups. We let v denote an indeterminate and set U_v equal to the quantum group (or rather the quantized enveloping algebra) for $\mathfrak{g}$ over $\mathbb{Q}(v)$. The subcategory $\mathcal{O}_v$ of the module category for U_v is then defined in complete analogy with $\mathcal{O}_{int}$, the subcategory of $\mathcal{O}$ consisting of modules with integral weight. One of the consequences of our results in this paper is that the combinatorics (i.e. the composition factor multiplicities of simple modules in Verma modules, and the multiplicities of Verma modules in Verma flags of indecomposable tilting modules) for $\mathcal{O}_v$ coincide with the corresponding combinatorics for $\mathcal{O}$ (this statement is a part of the mathematical folklore in the area but we have been unable to locate a proof in the literature).

Set $A = \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}(v)$ and let U_A be the Lusztig A-form of U_v, cf. [Lu90b]. For any non-zero $q \in \mathbb{C}$ we set $U_q = U_A \otimes_A \mathbb{C}$, where $\mathbb{C}$ is made into an A-algebra by the specialization $v \mapsto q$. Then again we have a BGG-category $\mathcal{O}_q$. As always, these categories are especially interesting in the case where q is a root of unity.

For all q we denote by $\mathcal{F}_q$ the subcategory of $\mathcal{O}_q$ consisting of all finite dimensional U_q-modules of type $\mathbf{1}$. When q is not a root of unity this category is semisimple and its "combinatorics" is exactly the same as that of the category of finite dimensional modules for (the complexification of) $\mathfrak{g}$, i.e. the characters of the simple modules are given by Weyl's character formula, see e.g. [APW91].

Suppose now q is a root of unity of odd order l and assume that l is prime to 3 if $\mathfrak{g}$ contains a copy of type G_2. Then $\mathcal{F}_q$ has a much more complicated structure. However, its "combinatorics" has been worked out: Lusztig stated the conjecture [Lu89] that the irreducible characters in $\mathcal{F}_q$ should be given by the values at 1 of the Kazhdan-Lusztig polynomials associated to the affine Weyl group for $\mathfrak{g}$. Kazhdan and Lusztig proved that the category of finite dimensional modules (of type $\mathbf{1}$) for U_q is equivalent to a category of modules for the corresponding affine Kac-Moody algebra [KL94] (in the non-simply laced case this has to be supplemented by Lusztig's later work [Lu94]) and then Kashiwara and Tanisaki proved the corresponding conjecture for affine Kac-Moody algebras, [KT95] and [KT96]. Soergel has determined the characters of indecomposable tilting modules in $\mathcal{F}_q$, [So97b] and [So99].

We prove that the fundamental modules like simple modules, indecomposable projective modules, indecomposable injective modules, and indecomposable tilting modules in $\mathcal{O}_q$ have a tensor product decomposition in a part which "comes from" $\mathcal{F}_q$ and a part which is a q-Frobenius twist of a corresponding module in $\mathcal{O}_{int}$, see Sections 3 and 4 for the precise statements. In the process of establishing these results we prove that many of the properties of $\mathcal{O}$, e.g. finite lengths of all modules, the existence of enough projectives and injectives, existence of tilting modules, and Ringel self-duality all carry over to $\mathcal{O}_q$.

One of the main features of $\mathcal{O}_q$ is that it contains a copy of $\mathcal{O}_{int}$, namely we may identify $\mathcal{O}_{int}$ with the direct sum of all "special blocks" in $\mathcal{O}_q$, see Theorem 3.11 below. Once we have established this and the above mentioned properties of $\mathcal{O}_q$ we return to the generic category $\mathcal{O}_v$. Using specialization of v at 1 on the one hand side and at large order roots of unity q on the other hand we are able to identify the combinatorics in $\mathcal{O}_v$ with that of $\mathcal{O}_{int}$.

The paper is organized as follows. In Section 2 we recall some basic facts about quantum groups at roots of unity. Then in Sections 3-4 we establish the results about $\mathcal{O}_q$ mentioned above. In particular, the tensor decompositions of simple modules, indecomposable projective or injective modules, and of tilting modules are found in Theorem 3.1, Theorem 3.15, Theorem 3.18, and Corollary 4.8, respectively. Then we deduce in Section 5 the combinatorics of $\mathcal{O}_q$ before we conclude the paper in Section 6 by proving that in the non-root of unity case (including the generic case) the combinatorics of $\mathcal{O}_v$ is the same as that of $\mathcal{O}_{int}$.

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2. Preliminaries on quantum groups

2.1. **Quantum groups at roots of** 1. For an indeterminate v denote by U_v the quantum group over $\mathbb{Q}(v)$ corresponding to a complex simple Lie algebra $\mathfrak{g}$. This is the $\mathbb{Q}(v)$-algebra with generators $E_i, F_i, K_i^{\pm 1}$, $i = 1, 2, \ldots, n = \mathrm{rank}(\mathfrak{g})$ and relations as given in [Ja, Chapter 5].

Set $A = \mathbb{Z}[v, v^{-1}]$. Then A contains the quantum numbers $[r]_d = \frac{v^{dr} - v^{-rd}}{v^d - v^{-d}}$ for any $r, d \in \mathbb{Z}$, $d \neq 0$ as well as the corresponding q binomials $\left[\begin{smallmatrix} m \\ t \end{smallmatrix}\right]_d$, $m \in \mathbb{Z}, t \in \mathbb{N}$. When $r \geq 0$ we set $[r]_d! = [r]_d[r-1]_d \cdots [1]_d$. In the following we will often need these elements for $d = 1$ in which case we will omit it from the notation.

Let C be the Cartan matrix associated with $\mathfrak{g}$. We denote by D a diagonal matrix whose entries are relatively prime natural numbers d_i with the property that DC is symmetric. Then we set $E_i^{(r)} = E_i^r/[r]_{d_i}!$. With a similar expression for $F_i^{(r)}$ we define now the A-form of U_v to be the A-subalgebra of U_v generated by the elements $E_i^{(r)}, F_i^{(r)}, K_i^{\pm 1}$, $i = 1, \ldots, n$, $r \geq 0$. This is the Lusztig divided power quantum group.

In this paper we fix throughout a primitive root of unity $q \in \mathbb{C}$ of odd order l. We assume that l is prime to 3 if $\mathfrak{g}$ has type G_2. The corresponding quantum group is then the specialization $U_q = U_A \otimes_A \mathbb{C}$ where $\mathbb{C}$ is considered an A-module via $v \mapsto q$, cf. [Lu90a], [Lu90b]. We abuse notation and write $E_i^{(r)}$ also for the element $E_i^{(r)} \otimes 1 \in U_q$ and similarly for $F_i^{(r)}$.

We have a triangular decomposition $U_q = U_q^- U_q^0 U_q^+$ with U_q^- and U_q^+ being the subalgebra generated by $F_i^{(r)}$ or $E_i^{(r)}$, $i = 1, \ldots, n$, $r \geq 0$, respectively. The "Cartan part" U_q^0 is the subalgebra generated by $K_i^{\pm 1}$ and $\left[\begin{smallmatrix} K_i \\ t \end{smallmatrix}\right]$, $i = 1, \ldots, n$, $t \geq 0$, where

$$\begin{bmatrix} K_i \\ t \end{bmatrix} = \prod_{j=1}^{t} \frac{K_i v^{d_i(1-j)} - K_i v^{-d_i(1-j)}}{v^{d_i j} - v^{-d_i j}}.$$

We denote the "Borel subalgebra" $U_q^0 U_q^+$ by B_q.

Recall that U_v is a Hopf algebra with comultiplication Δ, counit ϵ and antipode S, see [Ja, 4.11]. It is easy to see that their restrictions give U_A the structure of a Hopf algebra over A. Then U_q also gets an induced Hopf algebra structure.

2.2. **The small quantum group.** We also have the small quantum group $u_q \subset U_q$, defined as the subalgebra of U_q generated by $E_i, F_i, K_i^{\pm 1}$, $i = 1, \ldots, n$. It is also a Hopf subalgebra. Note that U_q is generated by u_q and $E_i^{(l)}, F_i^{(l)}$, $i = 1, \ldots, n$, as follows from [Lu89, Proposition 3.2(a)].

The small quantum group also has a triangular decomposition $u_q = u_q^- u_q^0 u_q^+$ with the obvious definitions of the three parts. We write $b_q = u_q^0 u_q^+$. Note that u_q^- and u_q^+ are finite dimensional. In fact, the PBW basis for U_q^- (resp. U_q^+) leads to a basis for u_q^- (resp. u_q^+): we just have to take PBW-monomials where each "root vector" has degree at most l, [Lu90b, Theorem 8.3]. Also u_q^0 is finite dimensional. In fact, $K_i^{2l} = 1$ for all i, see [Lu90a, 5.7].

2.3. **The quantum Frobenius homomorphism.** Let $U_{\mathbb{C}}$ denote the enveloping algebra of $\mathfrak{g}$. It has generators e_i, f_i and h_i, $i = 1, \ldots, n$. Lusztig has then defined in [Lu90b, Section 8], see also [Lu, Part V], a quantum Frobenius homomorphism $Fr_q : U_q \to U_{\mathbb{C}}$ by

$$E_i^{(r)} \mapsto \begin{cases} e_i^{(r/l)} & \text{if } l \text{ divides } r; \\ 0 & \text{if not.} \end{cases}, \qquad K_i \mapsto 1,$$
$$F_i^{(r)} \mapsto \begin{cases} f_i^{(r/l)} & \text{if } l \text{ divides } r; \\ 0 & \text{if not.} \end{cases}, \qquad \begin{bmatrix} K_i \\ t \end{bmatrix} \mapsto \binom{h_i}{t}.$$

Here $\binom{h_i}{t} = \prod_{s=1}^{t} \frac{(h_i - s + 1)}{s}$.

2.4. **Representations of U_q.** Set $X = \mathbb{Z}^n$. Then for $\lambda \in \mathbb{Z}^n$ we define $\chi_\lambda : U_q^0 \to \mathbb{C}$ by $\chi_\lambda(K_i^{\pm 1}) = q^{\pm d_i \lambda_i}$ and $\chi_\lambda(\begin{bmatrix} K_i \\ t \end{bmatrix}) = \begin{bmatrix} \lambda_i \\ t \end{bmatrix}_{d_i}$. This is a well-defined character of U_q^0 (see e.g. [APW91, Lemma 1.1]) and it extends to B_q by mapping $E_i^{(r)}$ to 0 for all $r > 0$, $i = 1, \ldots, n$.

If M is a U_q^0-module, then the λ weight space of M is defined as follows:

$$M_\lambda = \{m \in M \mid um = \chi_\lambda(u)m \text{ for all } u \in U_q^0\}. \tag{2.1}$$

The module M is called a *weight module of type* **1** provided that M decomposes into a direct sum of weight spaces of the form (2.1). In this paper we consider only weight modules of type **1** and will simply call them *weight modules.*

If N is a $U_{\mathbb{C}}$-module then we may consider N also as a U_q-module via Fr_q. To distinguish it from N we denote this U_q-module by $N^{[l]}$

and call it the (q-Frobenius) twist of N. Note that u_q acts trivially on $N^{[l]}$. Conversely, if M is a weight U_q-module on which u_q acts trivially, then there exists a $U_{\mathbb{C}}$-module N such that $M = N^{[l]}$, [Lu90b, 8.16]. In this case we also write $N = M^{[-l]}$. Note that $N = M$ as $\mathbb{C}$-spaces and the action of e_i (resp. f_i) on a vector $v \in N$ is given by $e_i v = E_i^{(l)} v$ (resp. $f_i v = F_i^{(l)} v$).

Note that Fr_q restricts to homomorphisms $U_q^0 \to U_{\mathbb{C}}^0$ and $B_q \to B_{\mathbb{C}}$, where $U_{\mathbb{C}}^0$ is the enveloping algebra of the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ generated by the h_i's and $B_{\mathbb{C}}$ is the enveloping algebra of the Borel subalgebra of $\mathfrak{g}$ generated by the h_i's and e_i's. We also denote these homomorphisms by Fr_q. Using them we can twist both $U_{\mathbb{C}}^0$- and $B_{\mathbb{C}}$-modules. For instance, the 1-dimensional U_q^0- (or B_q-) module $\mathbb{C}_{l\lambda}$ is the twist of the 1-dimensional $U_{\mathbb{C}}^0$- (or $B_{\mathbb{C}}$-) module $\mathbb{C}_\lambda$ determined by $\lambda \in X$ (we identify X with the set of integral weights in $\mathfrak{h}^*$ in the usual way).

3. The category $\mathcal{O}_q$

3.1. **Definition.** Similarly to [BGG] we define the category $\mathcal{O}_q$ as the full subcategory of U_q-mod consisting of those U_q-modules M which satisfy the following conditions:

(I) M is finitely generated as a U_q-module,
(II) M is a weight module,
(III) $\dim U_q^+ m < \infty$ for all $m \in M$.

Remark. Let $\mathcal{O}_{\text{int}}$ denote the integral block of the usual BGG category $\mathcal{O}$ for $\mathfrak{g}$ (see [BGG]). If $M \in \mathcal{O}_{\text{int}}$ then $M^{[l]} \in \mathcal{O}_q$.

For $\lambda \in X$ the Verma U_q-module with highest weight λ is given by the usual recipe:

$$\Delta_q(\lambda) = U_q \otimes_{B_q} \mathbb{C}_\lambda.$$

The standard arguments (see e.g. [Di, Chapter 7]) show that $\Delta_q(\lambda)$ has the following universal property:

$$\mathrm{Hom}_{\mathcal{O}_q}(\Delta_q(\lambda), M) = \{m \in M_\lambda \mid E_i^{(r)} m = 0 \text{ for all } r > 0, i = 1, \dots, n\}.$$

Moreover, it is easily seen that $\Delta_q(\lambda)$ has a unique maximal proper submodule. The corresponding simple quotient is denoted $L_q(\lambda)$. Then the set $\{L_q(\lambda) : \lambda \in X\}$ is a complete and irredundant set of representatives of isomorphism classes of simple modules in $\mathcal{O}_q$.

3.2. **Infinitesimal modules.** Replacing U_q by the small quantum group u_q we get *baby Verma modules* defined by:

$$\bar{\Delta}_q(\lambda) = u_q \otimes_{b_q} \mathbb{C}_\lambda, \qquad \lambda \in X.$$

If we here replace u_q by the subalgebra $u_qU_q^0$ of U_q and b_q by $U_q^0b_q = U_q^0u_q^+$, then we have similarly

$$\hat{\Delta}_q(\lambda) = u_qU_q^0 \otimes_{U_q^0b_q} \mathbb{C}_\lambda.$$

The baby Verma module $\hat{\Delta}_q(\lambda)$ restricted to u_q coincides with $\bar{\Delta}_q(\lambda)$ and is a finite dimensional module. It has dimension l^N where N is the number of positive roots (because as a vector space we may identify it with u_q^-). It has a universal property similar to the one enjoyed by $\Delta_q(\lambda)$ and it has a unique simple quotient which we denote $\hat{L}_q(\lambda)$.

Set now $X_l = \{\lambda \in X \mid 0 \leq \lambda_i < l, i = 1, \dots, n\}$. Then each $\lambda \in X$ has an "l-adic expansion" $\lambda = \lambda^0 + l\lambda^1$ with $\lambda^0 \in X_l$, $\lambda^1 \in X$. In the following upper indices 0 and 1 on a weight will always refer to the components of the weight in this expansion.

We set $X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all positive roots } \alpha\}$. The elements of X^+ are called the *dominant weights*. An *antidominant* weight is a $\lambda \in X$ for which $\lambda + \rho \in -X^+$.

We have the following remarkable fact about these infinitesimal simple modules, see [AW, Theorem 1.9].

$$\hat{L}_q(\lambda) \simeq L_q(\lambda^0) \otimes \mathbb{C}_{\lambda^1}^{[l]}. \tag{3.1}$$

The most "special" infinitesimal simple module is the one with highest weight $(l-1)\rho$. Here, as usual, ρ is the half of the sum of all positive roots. We call this module the *quantum Steinberg module* and denote it by St_l. Note that by (3.1) it is in fact a simple U_q-module, moreover, we have

$$St_l = \hat{L}_q((l-1)\rho) = \hat{\Delta}_q((l-1)\rho) = L_q((l-1)\rho).$$

Remark. Above we could also replace u_q by u_qB_q. Then we get baby Verma modules for u_qB_q defined by $\tilde{\Delta}_q(\lambda) = u_qB_q \otimes_{B_q} \mathbb{C}_\lambda$ with simple quotient $\tilde{L}_q(\lambda)$. When restricted to $u_qU_q^0$ these modules coincide with $\hat{\Delta}_q(\lambda)$ and $\hat{L}_q(\lambda)$, respectively. Note, in particular, that the Steinberg module St_l is also a simple u_qB_q-module as it extends, in fact, to U_q.

The composition factor multiplicities of $\tilde{\Delta}_q(\lambda)$ as well as the multiplicities with which $\tilde{\Delta}_q(\lambda)$ occurs in a baby Verma flag of an indecomposable projective u_qB_q-module coincide with the corresponding numbers for the Weyl module in $\mathcal{F}_q$ with highest weight λ when λ is sufficiently dominant. This follows from (3.1), cf. [APW92, Theorem 4.6]. This fact allows us to apply the combinatorics of $\mathcal{F}_q$ mentioned in the introduction to the category of u_qB_q-modules.

3.3. **Tensor product formula for simple modules in $\mathcal{O}_q$.** Recall that U_q is a Hopf algebra. In particular, its comultiplication allows us to make the tensor product (over $\mathbb{C}$) of two modules for U_q into a

U_q-module. When we tensor $\mathbb{C}$-modules we omit $\mathbb{C}$ from the notation. Note that $\mathcal{O}_q$ is stable under tensoring with finite dimensional modules (of type **1**).

Let $M, N \in \mathcal{O}_q$. Then we consider $\mathrm{Hom}_{\mathbb{C}}(M, N)$ as a U_q-module in the usual way, see e.g. [APW91, Section 2.9]. The u_q-fixed points $\mathrm{Hom}_{u_q}(M, N)$ then form a U_q-submodule on which u_q acts trivially, cf. [APW92, Section 3.2]. Hence by Section 2.4 there exists a $U_{\mathbb{C}}$-module $P = \mathrm{Hom}_{u_q}(M, N)^{[-l]}$ with $P^{[l]} = \mathrm{Hom}_{u_q}(M, N)$.

Let us also record the following observation valid whenever in addition to $M, N \in \mathcal{O}_q$ we have a module $Q \in \mathcal{O}$

$$\mathrm{Hom}_{\mathcal{O}_q}(Q^{[l]} \otimes M, N) \simeq \mathrm{Hom}_{\mathcal{O}}(Q, \mathrm{Hom}_{u_q}(M, N)^{[-l]}). \tag{3.2}$$

Theorem 3.1. *Let $\lambda \in X$. Then $L_q(\lambda) \simeq L_{\mathbb{C}}(\lambda^1)^{[l]} \otimes L_q(\lambda^0)$.*

Proof. Let L be any simple u_q-module (of type **1**). Recall from the previous subsection that L is the restriction of a simple U_q-module (which we also denote by L). Then for any $M \in \mathcal{O}_q$ the natural map $\mathrm{Hom}_{u_q}(L, M) \otimes L \to M$ which takes $f \otimes m$ to $f(m)$ is a U_q-homomorphism. It is in fact an injection which identifies $\mathrm{Hom}_{u_q}(L, M) \otimes L$ with the L-isotypic component of the u_q-socle of M. By the above this U_q-module is equal to $N^{[l]}$ for some $N \in \mathcal{O}_{\mathrm{int}}$.

Applying these observations to $M = L_q(\lambda)$ we get therefore $L_q(\lambda) \simeq \mathrm{Hom}_{u_q}(L, L_q(\lambda)) \otimes L$ for some such L, i.e. $L_q(\lambda) \simeq L_1^{[l]} \otimes L$ with $L_1 \in \mathcal{O}$. Clearly L_1 must be irreducible, i.e. $L_1 = L_{\mathbb{C}}(\mu)$ for some $\mu \in X$. By Section 3.2 we have $L \simeq L_q(\nu)$ for some $\nu \in X_l$. By weight considerations and the uniqueness of the l-adic expansion of λ we get $\mu = \lambda^1$ and $\nu = \lambda^0$. □

3.4. **Verma modules in $\mathcal{O}_q$.** We now want to study the composition factors of Verma modules. If $M \in \mathcal{O}_q$ and $\mu \in X$, we denote by $[M : L_q(\mu)]$ the multiplicity of $L_q(\mu)$ as a composition factor of M. We use similar notation for modules in $\mathcal{O}$ and for $u_q B_q$-modules.

Lemma 3.2. *Let M be a $B_{\mathbb{C}}$-module. Then we have an isomorphism of U_q-modules as follows:*

$$U_q \otimes_{u_q B_q} M^{[l]} \simeq (U_{\mathbb{C}} \otimes_{B_{\mathbb{C}}} M)^{[l]}.$$

Proof. We claim that the map taking $u \otimes m$ into $Fr_q(u) \otimes m$ is an isomorphism. To see this we note that $u_q B_q = u_q^- B_q$ and that the restriction of Fr_q to U_q^- is a surjection onto $U_{\mathbb{C}}^-$ with kernel generated by the augmentation ideal of u_q^-. It follows that the two modules in question are both isomorphic as $\mathbb{C}$-spaces to $U_{\mathbb{C}}^- \otimes M$ with the claimed map identifying the two. □

Proposition 3.3. *For $\lambda \in X$ the Verma module $\Delta_q(\lambda)$ has a filtration in $\mathcal{O}_q$ with quotients of the form $\Delta_{\mathbb{C}}(\mu^1)^{[l]} \otimes \tilde{L}_q(\mu^0)$, $\mu \in X$. Each quotient $\Delta_{\mathbb{C}}(\mu^1)^{[l]} \otimes \tilde{L}_q(\mu^0)$ occurs $[\tilde{\Delta}_q(\lambda) : \tilde{L}_q(\mu)]$ times.*

Proof. Consider a composition series of $\tilde{\Delta}_q(\lambda)$

$$0 = F^r \subset F^{r-1} \subset \cdots \subset F^0 = \tilde{\Delta}_q(\lambda)$$

with quotients $F^{i-1}/F^i \simeq \tilde{L}_q(\mu_i)$. When we apply the exact functor $U_q \otimes_{u_q B_q} -$ we obtain the filtration

$$0 = U_q \otimes_{u_q B_q} F^r \subset U_q \otimes_{u_q B_q} F^{r-1} \subset \cdots \subset U_q \otimes_{u_q B_q} F^0$$

of $U_q \otimes_{u_q B_q} \tilde{\Delta}_q(\lambda) \simeq \Delta_q(\lambda)$ with quotients $U_q \otimes_{u_q B_q} \tilde{L}_q(\mu_i)$. Now we recall from Section 3.2 that for any $\mu \in X$ we have $\tilde{L}_q(\mu) \simeq \mathbb{C}_{l\mu^1} \otimes L_q(\mu^0)$. By the tensor identity we have

$$U_q \otimes_{u_q B_q} \tilde{L}_q(\mu) \simeq (U_q \otimes_{u_q B_q} \mathbb{C}_{l\mu}) \otimes L_q(\mu^0).$$

Finally, Lemma 3.2 shows that $U_q \otimes_{u_q B_q} \mathbb{C}_{l\mu} \simeq \Delta_{\mathbb{C}}(\mu^1)^{[l]}$ and the proposition follows. □

Recall that modules in $\mathcal{O}$ have finite composition series (see [Di, Chapter 7]). Moreover, by Proposition 3.1 the composition factors of $\Delta_{\mathbb{C}}(\mu^1)^{[l]} \otimes L_q(\mu^0)$ are $L_{\mathbb{C}}(\nu^1)^{[l]} \otimes L_q(\mu^0) \simeq L_q(l\nu^1 + \mu^0)$ (occurring $[\Delta_{\mathbb{C}}(\mu^1) : L_{\mathbb{C}}(\nu^1)]$ times). We thus have

Corollary 3.4. *For every $\lambda \in X$ the Verma module $\Delta_q(\lambda)$ has finite length. Moreover, for $\mu \in X$ we have*

$$[\Delta_q(\lambda) : L_q(\mu)] = \sum_{\nu \geq \mu^1} [\tilde{\Delta}_q(\lambda) : \tilde{L}_q(l\nu + \mu^0)][\Delta_{\mathbb{C}}(\nu) : L_{\mathbb{C}}(\mu^1)].$$

Corollary 3.5. *All modules in $\mathcal{O}_q$ have finite length.*

Proof. By Condition (I) of $\mathcal{O}_q$ it is enough to establish this for cyclic modules M, i.e. we assume $M = U_q m$ for some $m \in M$. By Conditions (II) and (III), m is contained in a finite dimensional B_q-submodule $E \subset M$. This means that M is a quotient of $U_q \otimes_{B_q} E$ which has a finite Verma flag (take a B_q-filtration of E with 1-dimensional quotients and apply the exact functor $U_q \otimes_{B_q} -$). It is therefore enough to check that Verma modules in $\mathcal{O}_q$ have finite length. We did this in Corollary 3.4. □

For later use we record the following consequence of Corollary 3.4

Corollary 3.6. *Let $\lambda, \mu \in X$. Then for $l \gg 0$ we have*

$$[\Delta_q(\lambda) : L_q(\mu)] = [\tilde{\Delta}_q(\lambda) : \tilde{L}_q(\mu)].$$

Proof. Choose l so big that $\lambda - \mu \not\geq l\nu$ for any $\nu > 0$. Then the sum on the right hand side of the formula in Corollary 3.4 contains only one term, namely the term with $\nu = \mu^1$. □

3.5. **Special modules in $\mathcal{O}_q$.**

Proposition 3.7. *Let $\lambda \in X$. Then $\Delta_q(l\lambda + (l-1)\rho) \simeq \Delta_{\mathbb{C}}(\lambda)^{[l]} \otimes St_l$.*

Proof. We have $\tilde{\Delta}_q((l-1)\rho) \simeq St_l$, see [APW92, Lemma 2.6]. Just as in the proof of Proposition 3.3 we then get

$$\Delta_q(l\lambda + (l-1)\rho) \simeq U_q \otimes_{u_q B_q} \tilde{\Delta}_q(l\lambda + (l-1)\rho) \simeq (U_q \otimes_{u_q B_q} \otimes \mathbb{C}_{l\lambda}) \otimes St_l \simeq \Delta_{\mathbb{C}}(\lambda)^{[l]} \otimes St_l.$$

□

Corollary 3.8. *If λ is antidominant then $\Delta_q(l\lambda + (l-1)\rho)$ is simple. In particular, $\Delta_q(-\rho)$ is simple.*

Proof. It is well known (see e.g. [Di, Chapter 7]) that $\Delta_{\mathbb{C}}(\lambda)$ is simple in $\mathcal{O}$ when λ is antidominant. □

3.6. **The special block in $\mathcal{O}_q$.** The considerations at the beginning of Section 3.3 allow us to define a functor $\mathrm{F} : \mathcal{O}_q \to \mathcal{O}_{\mathrm{int}}$ by

$$\mathrm{F}\, N = (\mathrm{Hom}_{u_q}(St_l, N))^{[-l]}.$$

Since St_l is projective as a u_q-module F is exact.

Note that the map $f \otimes s \mapsto f(s)$ is a homomorphism and in fact an inclusion $(\mathrm{F}\, N)^{[l]} \otimes St_l \to N$. The considerations in Section 3.3 proves the following:

Proposition 3.9. *Let $\lambda \in X$. Then*

$$\mathrm{F}\, L_q(\lambda) \simeq \begin{cases} L_{\mathbb{C}}(\lambda^1), & \text{if } \lambda = l\lambda^1 + (l-1)\rho; \\ 0, & \text{if } l \text{ does not divide } \lambda + \rho. \end{cases}$$

This is a key ingredient in the following:

Proposition 3.10. *Let $\lambda, \mu \in X$. Suppose $\lambda^0 = (l-1)\rho \neq \mu^0$. Then $\mathrm{Ext}^i_{\mathcal{O}_q}(L_q(\lambda), L_q(\mu)) = 0$ for all i.*

Proof. As $L_q(\lambda) = L_{\mathbb{C}}(\lambda^1)^{[l]} \otimes St_l$ and St_l is projective as a u_q-module, for any $M \in \mathcal{O}_q$ we get via (3.2)

$$\mathrm{Ext}^i_{\mathcal{O}_q}(L_q(\lambda), M) \simeq \mathrm{Ext}^i_{\mathcal{O}}(L_{\mathbb{C}}(\lambda^1), \mathrm{F}\, M).$$

When $M = L_q(\mu)$ we have $\mathrm{F}\, M = 0$ by Proposition 3.9 and the desired vanishing follows. □

Proposition 3.10 allows us to define $\mathcal{O}_q^{\mathrm{spec}}$ to be the block in $\mathcal{O}_q$ consisting of those $M \in \mathcal{O}_q$ whose composition factors all belong to $lX + (l-1)\rho$. We call this the *special block* in $\mathcal{O}_q$ and its objects *special modules* in $\mathcal{O}_q$.

Define the functor $\mathrm{G} : \mathcal{O} \to \mathcal{O}_q^{\mathrm{spec}}$ by

$$\mathrm{G}\, N = N^{[l]} \otimes St_l.$$

Note that for $N \in \mathcal{O}$ we have indeed that $\mathrm{G}\, M$ is a special module in $\mathcal{O}_q$.

Clearly, G is exact and is in fact adjoint (left and right) to F. It is also immediate that $\mathrm{F} \circ \mathrm{G}$ is the identity functor on $\mathcal{O}$. Moreover, by Proposition 3.1 we have that $\mathrm{G} \circ \mathrm{F}$ is naturally the identity on simple modules and hence on $\mathcal{O}_q^{\mathrm{spec}}$. We have thus proved the following:

Theorem 3.11. *There is an equivalence of categories $\mathcal{O}_{\mathrm{int}} \cong \mathcal{O}_q^{\mathrm{spec}}$ given by the mutually inverse functors* F *and* G.

3.7. **Projective modules in $\mathcal{O}_q$.** Recall that in $\mathcal{O}_{int}$ the Verma module $\Delta_{\mathbb{C}}(\lambda)$ is projective whenever $\lambda+\rho$ is dominant, cf. [Hu, Proposition 3.8]. Hence Theorem 3.11 gives

Corollary 3.12. *If $\lambda+\rho$ is dominant, then $\Delta_q(l\lambda+(l-1)\rho)$ is projective in $\mathcal{O}_q$. In particular, $\Delta_q(-\rho)$ is projective.*

More generally, let $\mu \in X$ and denote by $P_{\mathbb{C}}(\mu) \in \mathcal{O}$ the projective cover of $L_{\mathbb{C}}(\mu)$. Then Theorem 3.11 gives the following:

Proposition 3.13. *For each $\lambda \in X$ the module $P_{\mathbb{C}}(\lambda)^{[l]} \otimes St_l$ is a projective cover of $L_q(l\lambda + (l-1)\rho)$ in $\mathcal{O}_q$.*

Having these projectives allow us to deduce the following:

Theorem 3.14. *The category $\mathcal{O}_q$ has enough projectives.*

Proof. This is a standard argument, cf. [Hu, 3.8]: By induction with respect to length we reduce the problem to proving that each simple module can be covered by a projective. Given $\lambda \in X$, we set $\nu = w_0\lambda^0 + (l-1)\rho$ where w_0 denotes the longest element in the Weyl group W for $\mathfrak{g}$. Then $w_0\nu = \lambda^0 - (l-1)\rho$ is the lowest weight of the finite dimensional simple module $L_q(\nu)$. Therefore $\Delta_q(l\lambda^1 + (l-1)\rho) \otimes L_q(\nu)$ surjects onto $\Delta_q(l\lambda^1 + (l-1)\rho + w_0\nu) = \Delta_q(\lambda)$. Now it is an easy consequence of Proposition 3.13 that $P_{\mathbb{C}}(\lambda^1)^{[l]} \otimes St_l$ surjects onto $\Delta_q((l\lambda^1 + (l-1)\rho)$. So we see that the projective module $P_{\mathbb{C}}(\lambda^1)^{[l]} \otimes St_l \otimes L_q(\nu)$ surjects onto $L_q(\lambda)$. □

Define $P_q(\lambda) \in \mathcal{O}_q$ as the projective cover of $L_q(\lambda)$. Then Corollary 3.12 says that $P_q(\lambda) = \Delta_q(\lambda)$ for all λ such that $\lambda + \rho \in X^+ \cap lX$. Moreover, Proposition 3.13 says that $P_q(l\lambda + (l-1)\rho) \simeq P_{\mathbb{C}}(\lambda)^{[l]} \otimes St_l$ for all $\lambda \in X$. We shall now generalize this by showing that all indecomposable projectives in $\mathcal{O}_q$ have a tensor factorization.

Recall that the subcategory $\mathcal{F}_q$ consisting of all finite dimensional modules in $\mathcal{O}_q$ also has enough projectives, see [APW92, Section 4]. Let us denote by $Q_q(\mu) \in \mathcal{F}_q$ the projective cover of $L_q(\mu)$ for $\mu \in X^+$.

Theorem 3.15. *For any $\lambda \in X$ we have $P_q(\lambda) \simeq P_{\mathbb{C}}(\lambda^1)^{[l]} \otimes Q_q(\lambda^0)$.*

Proof. By [APW92, Theorem 4.6] the restriction to u_q of $Q_q(\lambda^0)$ is the projective cover of $L_q(\lambda^0)$, i.e. for $\mu \in X_l$ we have

$$\mathrm{Hom}_{u_q}(Q(\lambda^0), L_q(\mu)) = \begin{cases} \mathbb{C} & \text{if } \mu = \lambda^0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence, using 3.2 and Proposition 3.1, for any $\mu \in X$ we get

$$\begin{aligned} &\mathrm{Ext}^i_{\mathcal{O}_q}(P(\lambda^1)^{[l]} \otimes Q(\lambda^0), L_q(\mu)) = \\ &\qquad \mathrm{Ext}^i_{\mathcal{O}_{\mathbb{C}}}(P_{\mathbb{C}}(\lambda^1), L_q(\mu^1) \otimes \mathrm{Hom}_{u_q}(Q(\lambda^0), L_q(\mu^0))^{[-l]}) = \\ &\qquad\qquad \begin{cases} \mathbb{C} & \text{if } \mu = \lambda \text{ and } i = 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

□

Let us also record the following important consequence of the constructions in the proof of Theorem 3.14

Corollary 3.16. *Projective modules in $\mathcal{O}_q$ all possess Verma flags.*

Proof. Let $\lambda \in X$. We shall prove that the corollary holds for $P_q(\lambda)$. When $\lambda \in lX + (l-1)\rho$ this follows from the fact that the corresponding statement is true in $\mathcal{O}$ combined with Theorem 3.11. But then the result follows in general because the construction in the proof of Theorem 3.14 reveals that $P_q(\lambda)$ may be obtained as a summand of a projective in $\mathcal{O}_q^{\mathrm{spec}}$ tensored by a finite dimensional module. □

3.8. **Injective modules in $\mathcal{O}_q$.** Let M be an arbitrary U_q-module. Since the antipode S on U_q is an antihomomorphism, the dual space $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ has the natural structure of a U_q-module given by $uf(m) = f(S(u)m)$, $u \in U_q$, $f \in M^*$, $m \in M$. Now U_v has an automorphism ω which interchanges E_i and F_i and inverts K_i , see [Ja, 4.6]. Clearly, ω gives rise to an automorphism of U_q. Twisting M^* by ω we get the U_q-module ${}^{\omega}M^*$ and when $M \in \mathcal{O}_q$ we set

$$M^\star = \oplus_{\lambda \in X} ({}^{\omega}M^*)_\lambda.$$

Then $(\_)^\star$ is an endofunctor on $\mathcal{O}_q$, called *duality*, with the property that for each $\lambda \in X$ we have $\dim(M^\star)_\lambda = \dim M_\lambda$. Hence $L_q^\star(\lambda) \simeq L_q(\lambda)$ (i.e. $\star$ is *simple preserving*). The existence of $\star$ gives immediately:

Theorem 3.17. *$\mathcal{O}_q$ has enough injectives.*

We set $I_q(\lambda) = P_q^\star(\lambda)$. This is the injective envelope of $L_q(\lambda)$ in $\mathcal{O}_q$ and if we denote by $I_{\mathbb{C}}(\mu)$ the injective envelope of $L_{\mathbb{C}}(\mu)$ in $\mathcal{O}_{int}$ then Theorem 3.15 implies:

Theorem 3.18. *For any $\lambda \in X$ we have $I_q(\lambda) \simeq I_{\mathbb{C}}(\lambda^1)^{[l]} \otimes Q_q(\lambda^0)$.*

3.9. **Projective-injective modules in $\mathcal{O}_q$.** By a projective-injective module we understand a module which is both projective and injective. We have

Theorem 3.19. *Let $\lambda \in X$. Then the following assertions are equivalent:*

(*a*) *$P_q(\lambda) \simeq I_q(\lambda)$.*
(*b*) *$L_q(\lambda)$ occurs in the socle of a projective-injective module in $\mathcal{O}_q$.*
(*c*) *$L_q(\lambda)$ occurs in the top of a projective-injective module in $\mathcal{O}_q$.*
(*d*) *$L_q(\lambda)$ occurs in the socle of some $\Delta_q(\mu)$, $\mu \in X$.*
(*e*) *λ is antidominant.*

Proof. The corresponding statement for $\mathcal{O}_{\text{int}}$ is well-known, see e.g. [Ir]. Hence Theorem 3.11 implies the claim for $\lambda \in lX + (l-1)\rho$.

Note that $Q_q(\lambda^0)$ is self-dual. Hence by Theorem 3.15 and 3.18 we see that (a) holds if and only if $P_{\mathbb{C}}(\lambda^1) \simeq I_{\mathbb{C}}(\lambda^1)$.

Now it is clear that (a) implies (b) and (b) implies (c). Because of Corollary 3.16 we have that (d) is a consequence of (c). Suppose $L_q(\lambda)$ is a submodule of $\Delta_q(\mu)$ for some $\mu \in X$. Then Proposition 3.3 and Proposition 3.1 show that $L_{\mathbb{C}}(\lambda^1)$ is a submodule of $\Delta_{\mathbb{C}}(\nu)$ for some $\nu \in X$. By the $\mathcal{O}$-result this implies that λ^1 is antidominant. But this is equivalent to (e). Finally, (c) implies (a) by the observations in the beginning of the proof. □

4. BGG reciprocity, Struktursatz and Ringel self-duality

4.1. **BGG reciprocity in $\mathcal{O}_q$.** The dual Verma module $\Delta_q^\star(\lambda)$ is denoted $\nabla_q(\lambda)$.

Theorem 4.1. *Let $\lambda, \mu \in X$ be arbitrary. Then*

$$\operatorname{Ext}^i_{\mathcal{O}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \simeq \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } \lambda = \mu; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that $\operatorname{Ext}^i_{\mathcal{O}_q}(\Delta_q(\lambda), \nabla_q(\mu)) \simeq \operatorname{Ext}^i_{\mathcal{O}_q}(\Delta_q(\mu), \nabla_q(\lambda))$ by duality. This allows us to assume that $\lambda \not< \mu$. Easy weight arguments show that the theorem holds for $i = 0$. Now by Corollary 3.16 all projectives in $\mathcal{O}_q$ have Verma filtrations. Moreover, we have a short exact sequence $0 \to K \to P_q(\lambda) \to \Delta_q(\lambda) \to 0$ with K having a Verma filtration where all subfactors $\Delta_q(\lambda')$ have $\lambda' > \lambda$. The $i > 0$ part of the theorem follows then from this sequence by a dimension shift argument. □

As a consequence we see that if $M \in \mathcal{O}_q$ has a Verma (resp. dual Verma) filtration then the number of occurrences $(M : \Delta_q(\lambda))$ (resp. $(M : \nabla_q(\lambda))$ of $\Delta_q(\lambda))$ (resp. $\nabla_q(\lambda))$ in this filtration equals the dimension of $\mathrm{Hom}_{\mathcal{O}_q}(M, \nabla_q(\lambda))$ (resp. $\mathrm{Hom}_{\mathcal{O}_q}(\Delta_q(\lambda), M)$). This immediately leads to the following BGG-reciprocity laws:

Corollary 4.2. *Let $\lambda, \mu \in X$. Then*

$$(P_q(\lambda) : \Delta_q(\mu)) = [\Delta_q(\mu) : L_q(\lambda)] = (I_q(\lambda) : \nabla_q(\mu)).$$

In other words, the above means that $\mathcal{O}_q$ is a highest weight category in the sense of [CPS] (with infinitely many isomorphism classes of simple modules).

4.2. **The category $\mathscr{C}$.** Let $\mathscr{C}$ denote the full subcategory of $\mathcal{O}_q$ with objects $P_q(\lambda)$, $\lambda \in X$. For simplicity we will identify objects of $\mathscr{C}$ with elements in X. Then Proposition 3.5 implies that $\mathscr{C}$ is a locally finite dimensional $\mathbb{C}$-linear category. Moreover, from Proposition 3.5 and Theorems 3.14 and 3.17 it follows that for any $\lambda \in X$ there exists only finitely many $\mu \in X$ such that $\mathscr{C}(\lambda, \mu) \neq 0$ and that for any $\lambda \in X$ there exists only finitely many $\mu \in X$ such that $\mathscr{C}(\mu, \lambda) \neq 0$.

Let $\mathscr{C}$-mod (resp. mod-$\mathscr{C}$) denote the category of finite dimensional left (resp. right) $\mathscr{C}$-modules, that is covariant (resp. contravariant) functors $\mathrm{M} : \mathscr{C} \to \mathbb{C}$-mod (the latter being the category of finite dimensional complex vector spaces) satisfying $\sum_{\lambda \in X} \dim \mathrm{M}(\lambda) < \infty$. Then the abstract nonsense (see e.g. [Ga]) implies that $\mathcal{O}_q$ is equivalent to mod-$\mathscr{C}$ and the latter is equivalent to $\mathscr{C}$-mod by duality.

4.3. **Dominance dimension and Soergel's Struktursatz.**

Proposition 4.3. *The category $\mathcal{O}_q$ has dominance dimension at least two with respect to projective-injective modules, that is for any projective module $P \in \mathcal{O}_q$ there exists an exact sequence*

$$0 \to P \to X_1 \to X_2, \tag{4.1}$$

where both X_1 and X_2 are projective-injective.

Proof. This claim is well-known for $\mathcal{O}_{\mathrm{int}}$, see e.g. [KSX, 3.1]. Hence Theorem 3.11 implies the claim for $P \in \mathcal{O}_q^{\mathrm{spec}}$. By Theorem 3.15, every indecomposable projective can be obtained by tensoring an indecomposable projective from $\mathcal{O}_q^{\mathrm{spec}}$ with a finite dimensional module and taking direct summand. As this tensoring is both left and right adjoint to an exact functor, it preserves projective-injective modules. Hence such tensoring maps a sequence of the form (4.1) to a sequence of the form (4.1) and the claim follows. $\square$

Denote by $\mathscr{C}^{PI}$ the full subcategory of $\mathscr{C}$ whose objects are all antidominant $\lambda \in X$, that is those $\lambda \in X$ for which the projective module $P_q(\lambda)$ is also injective (see Theorem 3.19). For $\lambda \in X$ define

$$\mathrm{M}_\lambda := \mathrm{Hom}_{\mathcal{O}_q}(\_, P_q(\lambda)) \in \text{mod-}\mathscr{C}^{PI}.$$

Let $\overline{\mathscr{C}}$ denote the full subcategory of mod-$\mathscr{C}^{PI}$ with objects M_λ, $\lambda \in X$.

Define a functor $\Phi : \mathscr{C} \to \overline{\mathscr{C}}$ as follows: on objects we set $\Phi(\lambda) := \mathrm{M}_\lambda$, $\lambda \in X$; if $\lambda, \mu \in X$ and $\varphi \in \mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu))$, then set

$$\Phi(\varphi) := \varphi \circ \_ : \mathrm{Hom}_{\mathcal{O}_q}(\_, P_q(\lambda)) \to \mathrm{Hom}_{\mathcal{O}_q}(\_, P_q(\mu)).$$

The following result generalizes [So90, Struktursatz].

Theorem 4.4. *The functor Φ is an isomorphism of categories.*

Proof. By definition, Φ induces a bijection on objects. So we need only to check that it induces a bijection on morphisms, that is that for any $\lambda, \mu \in X$ the map $\Phi_{\lambda,\mu} : \mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu)) \to \overline{\mathscr{C}}(\mathrm{M}_\lambda, \mathrm{M}_\mu)$ is an isomorphism. This is clear if both $P_q(\lambda)$ and $P_q(\mu)$ are injective.

By Proposition 4.3, the injective envelop of $P_q(\mu)$ is projective. Observe that, if $\varphi \in \mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu))$ is nonzero, then the image of φ contains a simple submodule L in the socle of $P_q(\mu)$. By Theorem 3.19, L is a homomorphic image of some projective-injective module P. By the projectivity of P, the surjection $f : P \to L$ lifts to a map $f' : P \to P_q(\lambda)$ such that $f = \varphi \circ f'$. This implies that $\Phi_{\lambda,\mu}(\varphi)$ is nonzero and hence $\Phi_{\lambda,\mu}$ is injective.

To prove surjectivity let $\lambda, \mu \in X$ and $f \in \overline{\mathscr{C}}(\mathrm{M}_\lambda, \mathrm{M}_\mu)$. By Proposition 4.3, there are exact sequences

$$0 \to P_q(\lambda) \to X_1 \to X_2 \qquad 0 \to P_q(\mu) \to Y_1 \to Y_2$$

in $\mathcal{O}_q$ such that X_1, X_2, Y_1 and Y_2 are projective-injective. Applying the covariant functor $\mathrm{Hom}_{\mathcal{O}_q}(\_, \_)$ to these exact sequence yields injective resolution for both M_λ and M_μ in mod-$\mathscr{C}^{PI}$. The map f admits lifts giving the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{M}_\lambda & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q}(_, X_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q}(_, X_2) \\
 & & \downarrow f & & \downarrow f' & & \downarrow f'' \\
0 & \longrightarrow & \mathrm{M}_\mu & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q}(_, Y_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_q}(_, Y_2)
\end{array}$$

As X_1, X_2, Y_1 and Y_2 are projective-injective, the right hand square of the latter diagram is the image of the right hand square of some commutative diagram of the form

$$\begin{array}{ccccccc}
0 & \longrightarrow & P_q(\lambda) & \longrightarrow & X_1 & \longrightarrow & X_2 \\
 & & \vdots\, \varphi & & \downarrow & & \downarrow \\
0 & \longrightarrow & P_q(\mu) & \longrightarrow & Y_1 & \longrightarrow & Y_2
\end{array}$$

As both rows are exact, the commutative right hand square of the latter diagram induces a unique $\varphi : P_q(\lambda) \to P_q(\mu)$ making the digram commutative and we have $\Phi_{\lambda,\mu}(\varphi) = f$. This proves surjectivity and completes the proof. □

Making a parallel with the results of [MM], we propose the following conjecture:

Conjecture 4.5. The category $\mathscr{C}^{PI}$ is symmetric, i.e. the bimodules $\mathscr{C}^{PI}(\_,\_)^*$ and $\mathscr{C}^{PI}(\_,\_)$ are isomorphic.

4.4. **Tilting modules in $\mathcal{O}_q$.** A module $M \in \mathcal{O}_q$ is called *tilting* if M has both a Verma filtration and a dual Verma filtration. In $\mathcal{O}_{int}$ there exists, for each $\lambda \in X$, a unique indecomposable tilting module $T_{\mathbb{C}}(\lambda)$ which has λ as its unique highest weight. The same is true in $\mathcal{O}_q$:

Theorem 4.6. *For each $\lambda \in X$ there exists an indecomposable tilting module $T_q(\lambda)$ with λ as its unique highest weight. Every indecomposable tilting module in $\mathcal{O}_q$ is isomorphic to $T_q(\lambda)$ for some $\lambda \in X$.*

Proof. The functor $G : \mathcal{O}_{int} \to \mathcal{O}_q^{spec}$ clearly takes tilting modules in $\mathcal{O}_{int}$ to tilting modules in $\mathcal{O}_q$, see Proposition 3.7. Hence for $\mu \in X$ we set $T_q(l\mu + (l-1)\rho) = T_{\mathbb{C}}(\mu)^{[l]} \otimes St_l$.

For general $\lambda \in X$ we set $\mu = \lambda^1 - \rho$ and consider $T = T_{\mathbb{C}}(\mu)^{[l]} \otimes St_l \otimes L_q(\lambda^0 + \rho)$. Then T is a tilting module and its highest weight is λ occurring with multiplicity 1. So we set $T_q(\lambda)$ equal to the unique indecomposable summand of T which has a non-zero λ-weight space.

This gives the existence of $T_q(\lambda)$. The second statement is then seen by standard arguments, see [Hu, Theorem 11.2]. □

For $N \in \mathcal{O}_q$ we denote by $\mathrm{Tr}_{PI}(N)$ the *trace* in N of all projective-injective modules, that is the sum of the images of all homomorphisms from M to N, where M is projective-injective. Note that for every finite dimensional $V \in \mathcal{O}_q$ the functor $V \otimes \_$ preserves the category of projective-injective modules. This implies that for any $N \in \mathcal{O}_q$ we have $\mathrm{Tr}_{PI}(V \otimes N) \cong V \otimes \mathrm{Tr}_{PI}(N)$. Titling modules in $\mathcal{O}_q$ can be alternatively described as follows:

Theorem 4.7. (*i*) *For every $\lambda \in X$ the module $\mathrm{Tr}_{PI}(P_q(\lambda))$ is an indecomposable tilting module.*

(*ii*) *Every indecomposable tilting module is isomorphic to $\mathrm{Tr}_{PI}(P_q(\lambda))$ for some $\lambda \in X$.*

(*iii*) (Ringel self-duality) For every $\lambda, \mu \in X$ we have

$$\mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu)) \cong \mathrm{Hom}_{\mathcal{O}_q}(\mathrm{Tr}_{PI}(P_q(\lambda)), \mathrm{Tr}_{PI}(P_q(\mu))).$$

Proof. This is well-known for $\mathcal{O}_\mathbb{C}$, see e.g. [So97b, FKM]. Hence Theorem 3.11 implies the claim for $\mathcal{O}_q^{\mathrm{spec}}$. Using translation and Theorem 3.15 we obtain that $\mathrm{Tr}_{PI}(P_q(\lambda))$ is a tilting module for every $\lambda \in X$.

For every $\lambda, \mu \in X$ from Theorem 4.4 it follows that the restriction map

$$\mathrm{Hom}_{\mathcal{O}_q}(P_q(\lambda), P_q(\mu)) \to \mathrm{Hom}_{\mathcal{O}_q}(\mathrm{Tr}_{PI}(P_q(\lambda)), \mathrm{Tr}_{PI}(P_q(\mu)))$$

is bijective. This proves (iii) and implies that every $\mathrm{Tr}_{PI}(P_q(\lambda))$ is indecomposable, proving (i). Claim (ii) follows from the fact that every tilting module occurs as a direct summand of a simple tilting module from $\mathcal{O}_q^{\mathrm{spec}}$ tensored with a finite dimensional module. □

In the classical case Ringel self-duality is due to Soergel, see [So97b]. Theorem 4.7(i) combined with Theorem 3.15 implies a tensor product formula for indecomposable tilting modules similar to Theorem 3.15 and Theorem 3.17. Namely, let $\lambda^0 \in X_l$ and write $\tilde{\lambda}^0 = l\rho + w_0 \cdot \lambda^0$.

Corollary 4.8. *For each $\lambda \in X$ we have $T_q(\lambda) \simeq T_\mathbb{C}(\lambda^1 - \rho)^{[l]} \otimes Q(\tilde{\lambda}^0)$.*

5. Characters and Kazhdan-Lusztig data

5.1. **Character formulas.** Consider the group ring $\mathbb{Z}[X]$ in which we denote the basis element corresponding to $\lambda \in X$ by e^λ. The multiplication is then determined by $e^\lambda e^\mu = e^{\lambda+\mu}$.

We extends this ring by defining its "completion" $\widehat{\mathbb{Z}[X]}$ to consist of all expressions $\sum_\lambda c_\lambda e^\lambda$ where $c_\lambda \in \mathbb{Z}$ for all λ and there exist $\lambda_1, \dots, \lambda_r \in X$ such that $c_\lambda = 0$ unless $\lambda \leq \lambda_i$ for some i (here $\leq$ is the usual order on X). Alternatively, this is the set of $\mathbb{Z}$-valued functions on X whose support is contained in a finite union of subsets of the form $X_{\leq \mu} = \{\lambda \in X \mid \lambda \leq \mu\}$. Clearly, the multiplication on $\mathbb{Z}[X]$ extends to $\widehat{\mathbb{Z}[X]}$.

If $f = \sum a_\lambda e^\lambda \in \widehat{\mathbb{Z}[X]}$, we set $f^{[l]} = \sum a_\lambda e^{l\lambda}$. If $M \in \mathcal{O}_{\mathrm{int}}$ or $M \in \mathcal{O}_q$, we set $\mathrm{ch}\, M = \sum_\mu (\dim M_\mu) e^\mu \in \mathbb{Z}[\hat{X}]$ and call this the character of M. Then for $M \in \mathcal{O}_{\mathrm{int}}$ we get $\mathrm{ch}(M^{[l]}) = (\mathrm{ch}\, M)^{[l]}$.

Using the notation from Section 3.4 for $M \in \mathcal{O}_{\mathrm{int}}$ we have

$$\mathrm{ch}\, M = \sum_\mu [M : L_\mathbb{C}(\mu)] \,\mathrm{ch}\, L_\mathbb{C}(\mu), \tag{5.1}$$

and similarly for $M \in \mathcal{O}_q$ we have

$$\mathrm{ch}\, M = \sum_\mu [M : L_q(\mu)] \,\mathrm{ch}\, L_q(\mu). \tag{5.2}$$

These sums are finite, cf. Corollary 3.5. If we take $M = \Delta_\mathbb{C}(\lambda)$, then the sum in (5.1) has a unique highest term, namely $1 \cdot \mathrm{ch}\, L_\mathbb{C}(\lambda)$. We can therefore "invert" these equations and obtain

$$\operatorname{ch} L_{\mathbb{C}}(\lambda) = \sum_{\mu} p^{\mathbb{C}}_{\mu,\lambda} \operatorname{ch} \Delta_{\mathbb{C}}(\mu) \tag{5.3}$$

for some unique $p^{\mathbb{C}}_{\mu,\lambda} \in \mathbb{Z}$. Similarly, we get

$$\operatorname{ch} L_q(\lambda) = \sum_{\mu} p^{q}_{\mu,\lambda} \operatorname{ch} \Delta_q(\mu) \tag{5.4}$$

for some unique $p^{q}_{\mu,\lambda} \in \mathbb{Z}$.

Note that whereas the sum in (5.3) is finite for all $\lambda \in X$ (we have $p^{\mathbb{C}}_{\mu,\lambda} = 0$ unless $\mu \in W \cdot \lambda$), this is not so in (5.4). For instance in the $\mathfrak{sl}_2$-case we have

$$\operatorname{ch} L_q(0) = \sum_{m \le 0} (\operatorname{ch} \Delta_q(ml) - \operatorname{ch} \Delta_q(ml - 2)). \tag{5.5}$$

Similarly, we may consider the characters of (finite dimensional) $u_q B_q$-modules. Here we obtain the analogous formulas

$$\operatorname{ch} M = \sum_{\mu} [M : \tilde{L}_q(\mu)] \operatorname{ch} \tilde{L}_q(\mu). \tag{5.6}$$

and

$$\operatorname{ch} \tilde{L}_q(\lambda) = \sum_{\mu} \tilde{p}^{q}_{\mu,\lambda} \operatorname{ch} \tilde{\Delta}_q(\mu) \tag{5.7}$$

for some unique $\tilde{p}^{q}_{\mu,\lambda} \in \mathbb{Z}$. Again, (5.6) clearly involves only finite sums for any finite dimensional M (and is in fact a formula in $\mathbb{Z}[X]$) whereas the sum in (5.7) may well be infinite.

Finally, we observe the following obvious identities

$$(\operatorname{ch} \Delta_C(\lambda)) e^{\mu} = \operatorname{ch} \Delta_{\mathbb{C}}(\lambda + \mu); \tag{5.8}$$

$$(\operatorname{ch} \Delta_q(\lambda)) e^{\mu} = \operatorname{ch} \Delta_q(\lambda + \mu); \tag{5.9}$$

and

$$(\operatorname{ch} \tilde{\Delta}_q(\lambda)) e^{\mu} = \operatorname{ch} \tilde{\Delta}_q(\lambda + \mu) \tag{5.10}$$

valid for all $\lambda, \mu \in X$.

5.2. Characters of simple modules in $\mathcal{O}_q$.

Using the terminology from Section 5.1 we have:

Theorem 5.1. *For all $\lambda \in X$ we have the following:*

(*i*) $\operatorname{ch} L_q(\lambda) = \sum_{\nu,\eta} p^{\mathbb{C}}_{\nu,\lambda^1} \tilde{p}^{q}_{\eta,\lambda^0} \operatorname{ch} \Delta_q(l\nu + \eta)$;
(*ii*) $p^{q}_{\mu,\lambda} = \sum_{l\nu+\eta=\mu} p^{\mathbb{C}}_{\nu,\lambda^1} \tilde{p}^{q}_{\eta,\lambda^0} = \sum_{w \in W} p^{\mathbb{C}}_{w\cdot\lambda^1,\lambda^1} \tilde{p}^{q}_{\mu - lw\cdot\lambda^1,\lambda^0}$.

Proof. By Proposition 3.1 combined with (5.3) and (5.7) we find

$$\begin{aligned}\operatorname{ch} L_q(\lambda) &= (\operatorname{ch} L_{\mathbb{C}}(\lambda^1)^{[l]} \operatorname{ch} L_q(\lambda^0) = \\ &= \sum_\nu p^{\mathbb{C}}_{\nu,\lambda^1} (\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} \sum_\eta \tilde{p}^q_{\eta,\lambda^0} \operatorname{ch} \tilde{\Delta}_q(\eta) = \\ &= \sum_\mu \left(\sum_{l\nu+\eta=\mu} p^{\mathbb{C}}_{\nu,\lambda^1} \tilde{p}^q_{\eta,\lambda^0} \right) (\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} \operatorname{ch} \tilde{\Delta}_q(\eta).\end{aligned}$$

So, to establish (i) we should only check that

$$(\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} \operatorname{ch} \tilde{\Delta}_q(\eta) = \operatorname{ch} \Delta_q(l\nu + \eta). \tag{5.11}$$

However, by (5.10) we have $\operatorname{ch} \tilde{\Delta}_q(\eta) = (\operatorname{ch} St_l) e^{\eta-(l-1)\rho}$ (because $St_l = \tilde{\Delta}_q((l-1)\rho)$). Hence using Proposition 3.7 we find

$$\begin{aligned}(\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} \operatorname{ch} \tilde{\Delta}_q(\eta) &= (\operatorname{ch} \Delta_{\mathbb{C}}(\nu))^{[l]} (\operatorname{ch} St_l) e^{\eta-(l-1)\rho} = \\ &= (\operatorname{ch} \Delta_q(l\nu + (l-1)\rho)) e^{\eta-(l-1)\rho} = \operatorname{ch} \Delta_q(l\nu + \eta).\end{aligned}$$

Here we have used (5.9) for the last equality.

The first equality in (ii) is immediate from (i) and the second comes from the fact that $p^{\mathbb{C}}_{\nu,\lambda^1} = 0$ unless $\nu \in W \cdot \lambda^1$. □

5.3. **Characters of indecomposable tilting modules in $\mathcal{O}_q$.** By Corollary 4.8 we get for any $\lambda \in X$

$$\operatorname{ch} T_q(\lambda) = \sum_{\nu,\eta} (T_{\mathbb{C}}(\lambda^1 - \rho) : \Delta_{\mathbb{C}}(\nu))(Q_q(\tilde{\lambda}^0) : \tilde{\Delta}_q(\eta)) \operatorname{ch}(\Delta_{\mathbb{C}}(\nu)^{[l]}) \operatorname{ch} \tilde{\Delta}_q(\eta).$$

Applying (5.11) in this formula we get

Theorem 5.2. *For all $\lambda, \mu \in X$ we have*

$$(T_q(\lambda) : \Delta_q(\mu)) = \sum_{\nu,\eta,l\nu+\eta=\mu} (T_{\mathbb{C}}(\lambda^1 - \rho) : \Delta_C(\nu))(Q_q(\tilde{\lambda}^0) : \tilde{\Delta}_q(\eta)).$$

5.4. **Kazhdan-Lusztig theory for $\mathcal{O}_q$.** Fix an antidominant weight $\lambda \in X$. For each $\mu \in W \cdot \lambda$ we pick $w \in W$ minimal such that $w \cdot \lambda = \mu$. Then the Kazhdan-Lusztig conjecture [KL] proved independently by Beilinson and Bernstein [BB], and by Brylinsky and Kashiwara [BK] says (for each such minimal $y, w \in W$)

$$p^{\mathbb{C}}_{y\cdot\lambda,w\cdot\lambda} = (-1)^{l(yw)} P_{y,w}(1). \tag{5.12}$$

Here $P_{y,w}$ is the Kazhdan-Lusztig polynomial associated to y, w, see [KL].

There is a similar conjecture in the category $\mathcal{F}_q$ of finite dimensional U_q-modules considered in Section 3.7. Here we formulate it in the corresponding category of u_qB_q-modules:

Let W_l be the affine Weyl group. Set

$$A_l^- = \{\lambda \in X \mid -l < \langle \lambda + \rho, \alpha^\vee \rangle < 0 \text{ for all positive roots } \alpha\}.$$

This is the top antidominant alcove. Fix $\lambda \in \bar{A}_l^-$ and choose for each $\mu \in W_l \cdot \lambda$ a minimal $x \in W_l$ such that $\mu = x \cdot \lambda$. Then in analogy with (5.12) for all such minimal $z, x \in W_l$ for which $x \cdot \lambda \in \overline{X^+}$ we have

$$\tilde{p}^q_{z\cdot\lambda, x\cdot\lambda} = (-1)^{l(zx)} P_{z,x}(1). \tag{5.13}$$

Here $P_{z,x}$ is again the Kazhdan-Lusztig polynomial associated to the pair (z, x) in the affine Weyl group W_l.

These two formulas allow us to formulate Theorem 5.1 as follows

Corollary 5.3. *Let $\lambda \in X$ and suppose $w \in W$, resp. $x \in W_l$ is minimal such that $w^{-1} \cdot \lambda^1$ is antidominant, resp. $x^{-1} \cdot \lambda^0 \in \bar{A}_l^-$. Then the character of $L_q(\lambda)$ equals*

$$\sum_{y\in W,\, z\in W_l} (-1)^{l(yw)+l(zx)} P_{y,w}(1) P_{z,x}(1) \operatorname{ch} \Delta_q\big(l(yw^{-1}\cdot\lambda^1) + zx^{-1}\cdot\lambda^0)\big)$$

Similarly, the result in Theorem 5.2 leads to the following formula for the characters of indecomposable tilting modules in $\mathcal{O}_q$.

Corollary 5.4. *Let $\lambda \in X$. We assume that $\lambda^1 - \rho$ is regular i.e. belongs to the interior of a chamber so that there is a unique $w \in W$ with $w^{-1} \cdot (\lambda^1 - \rho)$ antidominant. Likewise we assume that λ^0 is l-regular so that there is a unique $x \in W_l$ with $x^{-1} \cdot \lambda^0 \in A_l^+ = w_0 \cdot A_l^-$. Then*

$$(T_q(\lambda) : \Delta_q(\mu)) = \sum_{y,z} P_{y,w}(1) Q_{z,x}(1)$$

where the sum runs over those $y \in W$, $z \in W_l$ for which $\mu = lyw^{-1} \cdot (\lambda^1 - \rho) + zx^{-1} \cdot \lambda^0$ and where $Q_{z,x}$ denotes the "inverse" Kazhdan-Lusztig polynomial associated to (z, x).

Proof. According to [So97a, Conjecture 7.1] (proved in [So97b]) we have $(T_{\mathbb{C}}(\lambda^1-\rho) : \Delta_{\mathbb{C}}(yw^{-1}\cdot(\lambda^1-\rho)) = P_{y,w}(1)$ and $(Q_q(\tilde{\lambda}^0) : \tilde{\Delta}_q(zx^{-1}\cdot\lambda^0) = Q_{z,x}(1)$. □

Remark. Using [So97a, Remark 7.2.2] it is possible to generalize this corollary to include weights λ without the stated regularity assumptions.

6. The generic case

6.1. **The category $\mathcal{O}_v$.** We define the category $\mathcal{O}_v$ to be the full subcategory of the category of U_v-modules consisting of those modules which satisfy the analogues of (I)–(III) in Section 3.1.

Among the objects in $\mathcal{O}_v$ we have the generic Verma modules $\Delta_v(\lambda)$, $\lambda \in X$, defined in the usual way. They have unique simple quotients

$L_v(\lambda)$ and these are up to isomorphism a complete set of simple modules in $\mathcal{O}_v$.

The category $\mathcal{O}_v$ has properties completely analogous to $\mathcal{O}_{int}$, see e.g. [C-P, Chapters 9-10]. In particular all modules in $\mathcal{O}_v$ have finite length, the Verma module $\Delta_v(\lambda)$ has composition factors $L_v(\mu)$ with $\mu \in W \cdot \lambda$, and in fact $\mathcal{O}_v$ splits into blocks

$$\mathcal{O}_v = \oplus_\lambda \mathcal{O}_v^\lambda$$

where the block $\mathcal{O}_v^\lambda$ consists of those modules from $\mathcal{O}_v$ whose composition factors have highest weights in $W \cdot \lambda$, and where the sum runs over the set of all λ for which $\lambda + \rho$ are dominant.

In analogy to $\mathcal{O}_q$ we have a duality $\_^\star$ on $\mathcal{O}_v$ which fixes simple modules. The dual Verma module $\Delta_v^\star(\lambda)$ is denoted $\nabla_v(\lambda)$.

6.2. **A-lattices.** Clearly the Verma module $\Delta_v(\lambda)$ has an A-lattice, namely the Verma module for U_A defined by

$$\Delta_A(\lambda) = U_A \otimes_{B_A} A_\lambda.$$

Here B_A is the Borel subalgebra of U_A defined in analogy with B_q and A_λ denotes the free rank one A-module with B_A-action given by the analogue over A of the character χ_λ from Section 2.4.

Similarly, $\nabla_v(\lambda)$ has an A-lattice $\nabla_A(\lambda)$ defined as the A-dual of $\Delta_A(\lambda)$ (with the appropriate U_A-structure).

Note that $\mathrm{Hom}_{U_A}(\Delta_A(\lambda), \nabla_A(\lambda)) \simeq A$. We let c_λ denote a generator of this module and set $K_A(\lambda)$, respectively $L_A(\lambda)$, respectively $C_A(\lambda)$, equal to the kernel, respectively the image, respectively the cokernel, of c_λ. Then we get the following two short exact sequences in $\mathcal{O}_v$:

$$0 \to K_A(\lambda) \to \Delta_A(\lambda) \to L_A(\lambda) \to 0,$$

and

$$0 \to L_A(\lambda) \to \nabla_A(\lambda) \to C_A(\lambda) \to 0.$$

Tensoring by the fraction field $\mathbb{Q}(v)$ of A we see that $L_A(\lambda) \otimes_A \mathbb{Q}(v) \simeq L_v(\lambda)$ because $L_v(\lambda)$ is the image of $c_\lambda \otimes 1 : \Delta_v(\lambda) \to \nabla_v(\lambda)$. On the other hand, if we specialize to a root of unity $q \in \mathbb{C}$ (i.e. apply $- \otimes_A \mathbb{C}_q$ with $\mathbb{C}_q$ denoting $\mathbb{C}$ made into an A-module by mapping v to q) then we obtain the following two exact sequences in $\mathcal{O}_v$

$$K_A(\lambda) \otimes_A \mathbb{C}_q \to \Delta_q(\lambda) \to L_A(\lambda) \otimes_A \mathbb{C}_q \to 0,$$

and

$$0 \to \mathrm{Tor}_1^A(C_A(\lambda), \mathbb{C}_q) \to L_A(\lambda) \otimes_A \mathbb{C}_q \to \nabla_q(\lambda) \to C_A(\lambda) \otimes_A \mathbb{C}_q \to 0.$$

As $L_A(\lambda) \otimes_A \mathbb{C}_q$ is a non-zero quotient of $\Delta_q(\lambda)$ it has $L_q(\lambda)$ as a quotient but it may be bigger.

Proposition 6.1. *Let $\lambda, \mu \in X$ be fixed. Then*

$$\dim_{\mathbb{C}} L_q(\lambda)_\mu \geq \dim_{\mathbb{Q}(v)} L_v(\lambda)_\mu$$

for all l. Equality holds if $l \gg 0$.

Proof. The inequality follows from the above considerations. They also show that we have equality if and only if $\mathrm{Tor}_1^A(C_A(\lambda)_\mu, \mathbb{C}_q) = 0$. But $C_A(\lambda)_\mu$ is a finitely generated A-module so this Tor vanishes for all but at most finitely many q. □

6.3. **Generic multiplicities.**

Theorem 6.2. *Let $\lambda \in X$. Then we have for all $\mu \in X$*

$$[\Delta_v(\lambda) : L_v(\mu)] = [\Delta_{\mathbb{C}}(\lambda) : L_{\mathbb{C}}(\mu)].$$

Proof. Recall that both sides are 0 unless $\mu \in W \cdot \lambda$. Let therefore $\mu \in W \cdot \lambda$. Choose l so large that we have equality in Proposition 6.1 for all these finitely many μ's. Then it follows that we have

$$[\Delta_v(\lambda) : L_v(\mu)] = [\Delta_q(\lambda) : L_q(\mu)].$$

By Corollary 3.6 we see that this number equals $[\tilde{\Delta}_q(\lambda) : \tilde{L}_q(\mu)]$ (for large l). But if we also assume that l is so large that both λ and μ belong to l-alcoves adjacent to $-\rho$ then we claim that the composition factor multiplicities in these (baby) Verma modules for $u_q B_q$ agree with their counterparts in $\mathcal{O}$. This is a consequence of Corollary 5.3 because when $x \in W$ the Kazhdan-Lusztig polynomials $P_{z,x}$ clearly vanish unless $z \in W$. □

Remark. Set $A' = \mathbb{Q}[v, v^{-1}]$. According to [C-P, Proposition 10.1.10] it is possible to compute the determinant of the homomorphism $c_\lambda : \Delta_{A'}(\lambda) \to \nabla_{A'}(\lambda)$ restricted to any given weight space. The result implies that the cokernel of any such restriction becomes free when we localize A' at sufficiently many cyclotomic polynomials. Hence the dimensions of the weight spaces of $L_q(\lambda)$ at any non-root of unity $q \in \mathbb{C}$ coincide with those of $L_v(\lambda)$. It follows that the composition factor multiplicities of Verma modules in specializations away from roots of unity are identical to those of U_v. Hence the result in Theorem 6.2 holds not only for an indeterminate v but also for any specialization $v \mapsto q$ where $q \in \mathbb{C}$ satisfies $q^l \neq 1$ for all l.

After the above results it is natural to complete the paper with the following:

Conjecture 6.3. *If $q \in \mathbb{C}$ is a non-zero non-root of unity, then for every dominant $\lambda \in X$ the categories $\mathcal{O}_q^\lambda$ and $\mathcal{O}_{\mathrm{int}}^\lambda$ are equivalent.*

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