Representation type of ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$

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Abstract

For a semi-simple finite-dimensional complex Lie algebra $\mathfrak{g}$ we classify the representation type of the category ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ of Harish-Chandra bimodules for $\mathfrak{g}$.

1 The result

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra with a fixed triangular decoposition, $\mathfrak{g} = \mathfrak{n}_{-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+}$, λ and μ be two dominant integral (but not necessarily regular) weights, $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Let χ_λ and χ_μ denote the central characters of the Verma modules $M(\lambda)$ and $M(\mu)$ respectively. Let further ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ denote the full subcategory of the category of all $U(\mathfrak{g})$-bimodules, which consists of all X satisfying the following conditions (see [Ja, Kapitel 6]):

(1) X is finitely generated as a bimodule;

(2) X is algebraic, that is X is a direct sum of finite-dimensional $\mathfrak{g}$-modules with respect to the diagonal action $g \mapsto (g, \sigma(g))$, where σ is the Chevalley involution on $\mathfrak{g}$;

(3) $x(z - \chi_\mu(z)) = 0$ for all $x \in X$ and $z \in Z(\mathfrak{g})$;

(4) for every $x \in X$ and $z \in Z(\mathfrak{g})$ there exists $k \in \mathbb{N}$ such that $(z - \chi_\lambda(z))^k x = 0$.

Let $\mathbf{W}$ be the Weyl group of $\mathfrak{g}$ and ρ be the half of the sum of all positive roots of $\mathfrak{g}$. Then $\mathbf{W}$ acts on $\mathfrak{h}^*$ in the usual way and we recall the following *dot-action* of $\mathbf{W}$ on $\mathfrak{h}^*$: $w \cdot \nu = w(\nu + \rho) - \rho$. Let $\mathbf{G} \subset \mathbf{W}$ be the stabilizer of λ with respect to the dot-action, and $\mathbf{H} \subset \mathbf{W}$ be the stabilizer of μ with respect to the dot-action. We will say that the triple $(\mathbf{W}, \mathbf{G}, \mathbf{H})$ is associated to ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$. In the present paper we classify the categories ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ according to their representation type in terms of the associated triples, thus extending the results of [FNP, BKM, GP]. It is well-known, see [BeGe, So], that the categories ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ and ${}^{\infty}_{\lambda'}\mathcal{H}^{1}_{\mu'}$ are equivalent if the triples associated to them are the same. By the *Coxeter type* of a triple, $(\mathbf{W}, \mathbf{G}, \mathbf{H})$, we mean the triple that consists of the Coxeter types of all components of $(\mathbf{W}, \mathbf{G}, \mathbf{H})$. Note that the Coxeter type of the triple does not determine the triple in the unique way in general. Our main result in is the following statement:

Theorem 1.1. *(1) The category* ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ *has finite representation type if and only if the Coxeter type of the associated triple is*

- *(a) any and* $\mathbf{W} = \mathbf{G}$*;*
- *(b)* (A_n, A_{n-1}, A_n), (B_n, B_{n-1}, B_n), (C_n, C_{n-1}, C_n), *or* (G_2, A_1, G_2)*;*
- *(c)* (A_1, e, e)*;*
- *(d)* (A_n, A_{n-1}, A_{n-1})*;*
- *(e)* (A_n, A_{n-1}, A_{n-2}), *where* A_{n-2} *is obtained from* A_n *by taking away the first and the last roots;*
- *(f)* (B_2, A_1, A_1) *or* (C_2, A_1, A_1) *and* $\mathbf{G} = \mathbf{H}$*;*
- *(g)* (B_n, B_{n-1}, B_{n-1}) *or* (C_n, C_{n-1}, C_{n-1}), *where* $n \geq 3$*;*
- *(h)* (A_2, A_1, e)*.*

(2) The category ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ *has tame representation type if and only if the Coxeter type of the associated triple is*

- *(a)* $(A_3, A_1 \times A_1, A_3)$, (A_2, e, A_2), (B_2, e, B_2), (G_2, e, G_2), (B_3, A_2, B_3), (C_3, A_2, C_3), *or* (D_n, D_{n-1}, D_n) *where* $n \geq 4$*;*
- *(b)* (B_2, A_1, A_1) *or* (C_2, A_1, A_1) *and* $\mathbf{G} \neq \mathbf{H}$*;*
- *(c)* $(A_n, A_{n-1}, A_1 \times A_{n-2})$, $n > 2$*;*
- *(d)* (A_n, A_{n-1}, A_{n-2}), $n > 2$, *where* A_{n-2} *is included into* A_{n-1} *and contains either the first or the last root of* A_n*;*
- *(e)* (A_n, A_{n-1}, A_{n-2}), $n > 2$, *where* A_{n-2} *is not included into* A_{n-1}*;*
- *(f)* (A_3, A_2, e), (B_2, A_1, e), (C_2, A_1, e)*.*

(3) In all other cases the category ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ *has wild representation type.*

According to [FKM, KM], the category ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ is equivalent to the $\mathbf{H}$-singular block of a certain parabolic generalization, $\mathcal{O}(\mathfrak{p}, \Lambda)$, of the BGG category $\mathcal{O}$, where $\mathfrak{p}$ is the parabolic subalgebra of $\mathfrak{g}$ associated to $\mathbf{G}$. In particular, in the case $\mathbf{H} = \{e\}$ (i.e. μ regular) Theorem 1.1 gives the classification of the representation type of the blocks of the category $\mathcal{O}$ obtained in [FNP] (see also [BKM] for the argument closer to the one we use here). In the case $\mathbf{H} = \mathbf{W}$ (i.e. μ the most singular) Theorem 1.1 reduces to the classification of the representation type for the algebra $\mathtt{C}(\mathbf{W}, \mathbf{G})$ of $\mathbf{G}$-invariants in the coinvariant algebra associated to $\mathbf{W}$. This result was obtained in [GP] and, in fact, our argument in the present paper is based upon it. Another important ingredient in the proof of Theorem 1.1, the latter being presented in Section 3, is the classification of the representation type of all centralizer subalgebras in the Auslander algebra $\mathtt{A}_n$ of $\Bbbk[x]/(x^n)$. This classification is given in Section 2. Two series of centralizer subalgebras, namely those considered in Lemma 2.7 and Lemma 2.8, seems to be rather interesting and non-trivial. The paper is finished with an extension of Theorem 1.1 to the case of a semi-simple $\mathfrak{g}$ in Section 4, where one more interesting tame algebra arises.

We would like to finish the introduction with a remark that just recently a first step towards the classification of the representation type of the blocks of Rocha-Caridi's parabolic analogue $\mathcal{O}_S$ of $\mathcal{O}$ was made in [BN]. The next step would be to complete this classification and then to classify the representation type of the "mixed" version of $\mathcal{O}_S$ and $\mathcal{O}(\mathfrak{p}, \Lambda)$. As the results of [BN] and of the present paper suggest, this might give some interesting tame algebras in a natural way.

2 Representation type of the centralizer subalgebras in the Auslander algebra of $\Bbbk[x]/(x^n)$

Let $\Bbbk$ be an algebraically closed field. Let further $n > 1$ be a positive integer and $\mathtt{A}_n$ be the algebra given by the following quiver with relations:

$$\begin{tikzcd}
1 \arrow[r, bend left, "a_1"] & 2 \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_2"] & \cdots \arrow[l, bend left, "b_1"] \arrow[r, bend left, "a_{n-1}"] & n \arrow[l, bend left, "b_{n-1}"]
\end{tikzcd}
\qquad
\begin{array}{l}
a_i b_i = b_{i+1} a_{i+1}, \quad i = 1, \dots, n-2, \\
a_{n-1} b_{n-1} = 0.
\end{array}$$

The algebra $\mathtt{A}_n$ is the Auslander algebra of $\Bbbk[x]/(x^n)$. For $X \subset \{2, 3, \dots, n\}$ let e_X denote the direct sum of all primitive idempotents of $\mathtt{A}_n$, which corrrespond to the vertexes from $\{1\} \cup X$. Set $\mathtt{A}_n^X = e_X \mathtt{A}_n e_X$. The main result of this section is the following:

Theorem 2.1. *(i) The algebra $\mathtt{A}_n^X$ has finite representation type if and only if $X \subset \{2, n\}$.*

(ii) The algebra $\mathtt{A}_n^X$ has tame representation type if and only if $X = \{3\}$, $\{2, 3\}$, $\{n-1\}$, $\{n-1, n\}$ or $n = 4$ and $X = \{2, 3, 4\}$.

(iii) The algebra $\mathtt{A}_n^X$ is wild in all other cases.

To prove Theorem 2.1 we will need the following lemmas:

Lemma 2.2. *The algebra $\mathtt{A}_n^{\{m\}}$ has infinite representation type for $m \in \{3, \dots, n-1\}$ and $n \geq 4$.*

Proof. The algebra $\mathtt{A}_n^{\{m\}}$ is given by the following quiver with relations:

$$\begin{tikzcd}
1 \arrow[loop left, "x"] \arrow[r, bend left, "a"] & m \arrow[l, bend left, "b"] \arrow[loop right, "y"]
\end{tikzcd}
\qquad
\begin{array}{ll}
ax = ya, & xb = by, \\
ab = y^{m+1}, & ba = x^{m+1}, \\
y^{n-m+1} = 0, &
\end{array}
\tag{1}$$

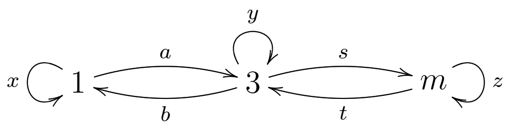
where $x = b_1 a_1$, $y = b_m a_m$, $a = a_{m-1} \dots a_1$, $b = b_1 \dots b_{m-1}$. Modulo the square of the radical $\mathtt{A}_n^{\{m\}}$ gives rise to the following diagram of infinite type:

$$\begin{tikzcd}
1 \arrow[d, no head] \arrow[dr, no head] & m . \arrow[d, no head] \arrow[dl, no head] \\
1 & m
\end{tikzcd}$$

Hence $\mathtt{A}_n^{\{m\}}$ has infinite representation type as well. $\square$

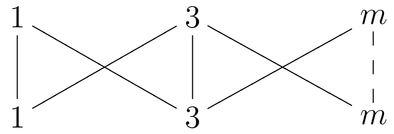
Lemma 2.3. *The algebra* $\mathtt{A}_n^X$ *is wild for* $X = \{3, m\}$*, where* $m > 4$.

Proof. In this case the algebra $\mathtt{A}_n^X$ is given by the following quiver with relations:



$$\begin{aligned} &ax = ya, && xb = by, \\ &sy = zs, && yt = tz, \\ &ab = y^2, && ba = x^2, \\ &st = z^{m-3}, && ts = y^{m-3}, \\ &z^{n-m+1} = 0, \end{aligned}$$

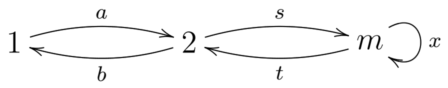
where $x = b_1a_1$, $y = b_3a_3$, $z = b_ma_m$, $a = a_2a_1$, $b = b_1b_2$, $s = a_{m-1}\dots a_3$, $b = b_3\dots b_{m-1}$. Note that $z = 0$ if $m = n$. Modulo the square of the radical $\mathtt{A}_n^X$ gives rise to the following wild diagram:



(here the dashed line indicates that it disappears in the case $m = n$). Hence $\mathtt{A}_n^X$ is wild as well. $\square$

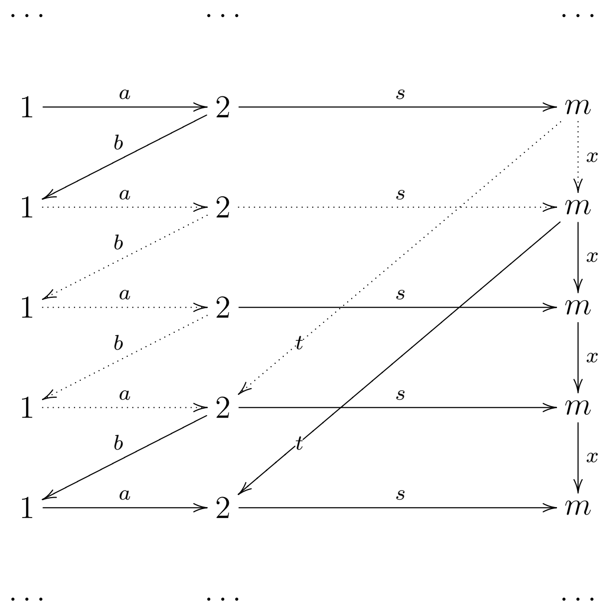
Lemma 2.4. *The algebra* $\mathtt{A}_n^X$ *is wild for* $X = \{2, n-1\}$ *and* $n \geq 5$.

Proof. Set $m = n - 1$. The algebra $\mathtt{A}_n^X$ is given by the following quiver with relations:



$$\begin{aligned} &sab = xs, && abt = tx, \\ &st = 0, && ts = (ab)^{n-3}, \\ &x^2 = 0, \end{aligned}$$

where $a = a_1$, $b = b_1$, $s = a_{n-2}\dots a_2$, $t = b_2\dots b_{n-2}$, $x = b_{n-1}a_{n-1}$. The universal covering of $\mathtt{A}_n^X$ has the wild fragment indicated by the dotted arrows in the following picture:



Hence $\mathtt{A}_n^X$ is wild as well. ☐

Lemma 2.5. *The algebra* $\mathtt{A}_5^{\{3,4\}}$ *is wild.*

Proof. The algebra $\mathtt{A}_5^{\{3,4\}}$ is given by the following quiver with relations:

$$\begin{tikzcd}
1 \arrow[loop left, "x"] \arrow[r, bend left, "a"] & 3 \arrow[l, bend left, "b"] \arrow[r, bend left, "s"] & 4 \arrow[l, bend left, "t"]
\end{tikzcd}
\qquad
\begin{array}{ll}
ax = tsa, & xb = bts, \\
ba = x^2, & ab = (ts)^2, \\
(st)^2 = 0, &
\end{array}$$

where $a = a_2a_1$, $b = b_1b_2$, $s = a_3$, $t = b_3$, $x = b_1a_1$. The universal covering of $\mathtt{A}_5^{\{3,4\}}$ has the wild fragment indicated by the dotted arrows in the following picture:

$$\begin{tikzcd}
\cdots & \cdots & \cdots \\
1 \arrow[r, "a"] \arrow[d, "x"'] & 3 \arrow[r, dotted, "s"] \arrow[ddl, dotted, "b" near end] & 4 \arrow[dl, dotted, "t"] \\
1 \arrow[r, dotted, "a"] \arrow[d, dotted, "x"'] & 3 \arrow[r, dotted, "s"] & 4 \arrow[dl, "t"] \\
1 \arrow[r, "a"] & 3 \arrow[r, "s"] & 4 \\
\cdots & \cdots & \cdots
\end{tikzcd}$$

Hence $\mathtt{A}_5^{\{3,4\}}$ is wild as well. ☐

Lemma 2.6. *The algebra* $\mathtt{A}_n^{\{m\}}$ *is wild for* $m \in \{4, \dots, n-2\}$ *and* $n \geq 6$.

Proof. The algebra $\mathtt{A}_n^{\{m\}}$ is given by (1). We consider its quotient $\mathtt{B}$ given by additional relations $x^3 = y^3 = ab = ba = 0$ (which is possible because of our restrictions on m and n). Then the universal covering of $\mathtt{B}$ exists and has the following fragment,

$$\begin{tikzcd}
m \arrow[dr, "y"] & & 1 \arrow[dl, "a"'] \arrow[dr, "x"] \arrow[dd, dashed] & & & & & & \\
& m \arrow[dr, "y"] & & 1 \arrow[dl, "a"'] \arrow[dr, "x"] & & m \arrow[dl, "b"'] \arrow[dr, "y"] & & 1 \arrow[dl, "a"'] \arrow[dr, "x"] & \\
& & m & & 1 & & m & & 1
\end{tikzcd}\quad ,$$

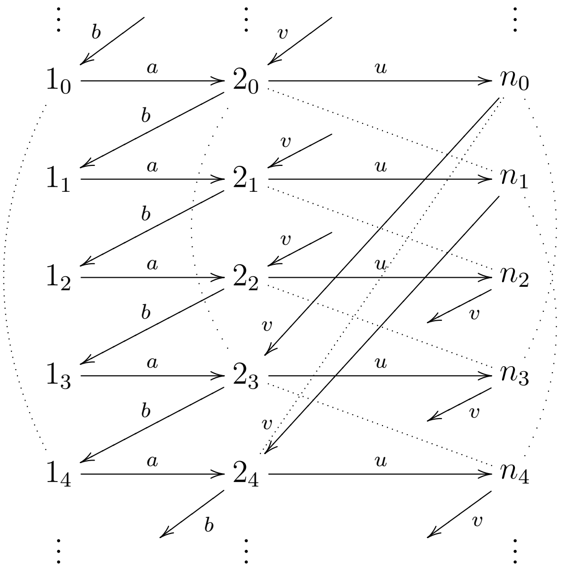
which is wild by [Un]. This implies that $\mathtt{B}$ and hence $\mathtt{A}_n^{\{m\}}$ is wild. ☐

Lemma 2.7. *The algebra* $\mathtt{A}_n^{\{2,n\}}$, $n \geq 4$, *is of finite representation type.*

Proof. The algebra $\mathtt{A}_n^{\{2,n\}}$, $n \geq 4$, is given by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[r, bend left, "u"] & n \arrow[l, bend left, "v"] \end{tikzcd} \qquad uv = uab = abv = 0,\ vu = (ab)^{n-2}, \tag{2}$$

where $a = a_1$, $b = b_1$, $u = a_{n-1} \dots a_2$, $v = b_2 \dots b_{n-1}$. Note that these relations imply $(ab)^{n-1} = (ba)^n = 0$. The projective $\mathtt{A}_n^{\{2,n\}}$-module $P(1)$ is injective, so we can replace $\mathtt{A}_n^{\{2,n\}}$ by $\mathtt{A}' = \mathtt{A}_n^{\{2,n\}}/\mathrm{soc}(P(1)) = \mathtt{A}_n^{\{2,n\}}/((ba)^{n-1})$, which has the same indecomposable modules except $P(1)$, see [DK, Lemma 9.2.2]. So from now on we consider the algebra $\mathtt{A}'$, i.e. add the relation $(ba)^{n-1} = 0$ to (2). The algebra $\mathtt{A}'$ has a simply connected covering $\widetilde{\mathtt{A}}$, see [BoGa], given by the following quiver with relations:



In this picture we show the case $n = 5$; actually the arrow starting at n_k ends at 2_{n-2+k}. We do not label the arrows a, b, u, v. They satisfy the same relations as in $\mathtt{A}'$, which are shown by the dotted lines. Consider the full subcategory $\mathtt{B}_m \subseteq \widetilde{\mathtt{A}}$ with the set of objects $\mathbf{S} = \{1_k, m \leq k \leq m+n-1;\ 2_k, m \leq k \leq m+n-2;\ n_m\}$. Let M be an $\widetilde{\mathtt{A}}$-module, N_m be its restriction onto $\mathtt{B}_m$, $N_m = \bigoplus_{i=1}^{s} K_i$, where K_i are indecomposable $\mathtt{B}_m$-modules. It is well known that every K_i is completely determined by the subset of objects $\mathbf{S}_i = \{x \mid K_i(x) \neq 0\}$ and if $1_m \in \mathbf{S}_i$, then $1_{m+n-1} \notin \mathbf{S}_i$. Moreover, all $K_i(x)$ with $x \in \mathbf{S}_i$ are one-dimensional and all arrows between these objects correspond to the identity maps. Since $uab = abv = 0$, K_i splits out of the whole module M whenever $\mathbf{S}_i \supseteq \{2_m, 2_{m+n-2}\}$. Suppose that, for every integer m, N_m does not contain such direct summands. It implies that $M(vu) = 0$.

Therefore M can be considered as a module over $\overline{\mathtt{A}}$, where $\overline{\mathtt{A}}$ is given by the following quiver

$$\begin{tikzcd}
 & \cdots & n' \arrow[d, "v"] & & n' \arrow[d, "v"] & \cdots & n' \arrow[d, "v"] & \cdots & \\
\cdots & 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[d, "u"] & 1 \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[d, "u"] & \cdots \arrow[r, "a"] & 2 \arrow[r, "b"] \arrow[d, "u"] & 1 & \cdots \\
 & \cdots & n & & n & \cdots & n & \cdots &
\end{tikzcd}$$

with relations $uv = uab = abv = (ab)^{n-2} = 0$. One easily checks that any indecomposable representation of $\overline{\mathtt{A}}$ is at most of dimension $2n-5$. Hence, $\overline{\mathtt{A}}$ is representation (locally) finite, i.e. for every object $x \in \overline{\mathtt{A}}$ there are only finitely many indecomposable representations M with $M(x) \neq 0$. By [BoGa], so is $\mathtt{A}_n^{\{2,n\}}$ as well, which completes the proof. $\square$

Lemma 2.8. *The algebra* $\mathtt{A}_n^{\{n-1,n\}}$ *is tame.*

Proof. For $q = n-1$ the algebra $\mathtt{A}_n^{\{n-1,n\}}$ is given by the following quiver with relations

$$\begin{tikzcd}
1 \arrow[loop left, "c"] \arrow[r, bend left, "u"] & q \arrow[l, bend left, "v"] \arrow[r, bend left, "a"] & n \arrow[l, bend left, "b"]
\end{tikzcd} \qquad c^n = ab = uv = 0,\ vu = c^{n-2},\ cv = vba,\ uc = bau,$$

where $c = a_1b_1$, $a = a_{n-1}$, $b = b_{n-1}$, $u = a_{n-2}\dots a_1$, $v = b_1\dots b_{n-2}$. The projective module $P(1)$ is also injective, hence, we again can replace $\mathtt{A}$ by $\mathtt{A}' = \mathtt{A}/\mathrm{soc}(P(1)) = \mathtt{A}/(c^{n-1})$. Let M be an $\mathtt{A}'$-module. Choose a basis in $M(1)$ so that the matrix $C = M(c)$ is in the Jordan normal form, or, further,

$$M(c) = \bigoplus_{i=1}^{n-1} J_i \otimes I_{m_i},$$

where J_i is the nilpotent Jordan block of size $i \times i$ and I_m is the identity matrix of size $m \times m$. Thus

$$J_i \otimes I_m = \begin{pmatrix} 0 & I_m & 0 & \dots & 0 & 0 \\ 0 & 0 & I_m & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & I_m \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{i \times i}$$

Choose bases in $M(n-1)$ and $M(n)$ so that the matrices $A = M(a)$ and $B = M(b)$ are in the form

$$A = \begin{pmatrix} 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where the vertical (horizontal) stripes of A are of the same size as the horizontal (respectively, vertical) stripes of B; we do not precise these sizes here. Set $r = n(n-1)/2$; it is the number of the horizontal and vertical stripes in A. Then $M(u)$ and $M(v)$ can be considered as block matrices: $M(u) = U = (U_k^{ij})_{5\times r}$ and $M(v) = V = (V_{ij}^k)_{r\times 5}$, where $k = 1,\dots,5$ correspond to the k-th horizontal stripe of B; $i = 1,\dots,n-1,\ j = 1,\dots,i$, and the stripe (ij) corresponds to the j-th horizontal stripe of the matrix $J_i \times I_{m_i}$ in the decomposition of C. The conditions $uc = bau$ and $cv = vba$ imply that the only nonzero blocks U_k^{ij} and V_{ij}^k can be

$$\begin{array}{ll} U_k^{ii} & \text{and } U_1^{i,i-1} = U_5^{ii} \\ V_{i1}^k & \text{and } V_{i2}^5 = V_{i1}^1, \end{array}$$

moreover, $U_5^{11} = V_{11}^1 = 0$. Changing bases in the spaces $M(x)$ so that the matrices A, B, C remain of the same form, we can replace U and V respectively by $T^{-1}US$ and $S^{-1}VT$, where S, T are invertible matrices of the appropriate sizes such that $SA = AS$ and $TU = UQ$, $QV = VT$ for an invertible matrix Q. We also consider S and T as block matrices: $S = (S_{st}^{ij})_{r\times r}$ and $T = (T_l^k)_{5\times 5}$ respectively to the division of A, B, C. Then the conditions above can be rewritten as follows:

- S_{st}^{ij} can only be nonzero if $i - j < s - t$ or $i - j = s - t,\ s \le i$;
- $S_{st}^{ij} = S_{st'}^{ij'}$ if $t - j = t' - j'$;
- T is block triangular: $T_l^k = 0$ if $k < l$, and $T_1^1 = T_5^5$;
- all diagonal blocks S_{ij}^{ij} and T_k^k are invertible.

Especially, for the vertical stripes U^{ii} and for the horizontal stripes U_k of the matrix U the following transformations are allowed:

1. Replace U^{ii} by $U^{ii}Z$.
2. Replace U_k by ZU_k, where $k = 2, 3, 4$.
3. Replace U_1 and U_5 respectively by ZU_1 and ZU_5.
4. Replace U^{ii} by $U^{ii} + U^{jj}Z$, where $j < i$.
5. Replace U_k by $U_k + U_lZ$, where $k < l$.

Here Z denotes an arbitrary matrix of the appropriate size, moreover, in cases 1–3 it must be invertible. One can easily see that, using these transformations, one can subdivide all blocks U_k^{ii} into subblocks so that each stripe contains at most one nonzero block, which is an identity matrix. Note that the sizes of the horizontal substripes of U_1 and U_5 must be the same. Let Λ^{ii} and Λ_k be respectively the sets of the vertical and the horizontal stripes of these subdivisions. Note that all stripes U^{ij} must be subdivided respectively to the subdivision of U^{ii} and recall that $U_1^{i,i-1} = U_5^{ii}$. Especially, there is a one-to-one correspondence $\lambda \mapsto \lambda'$ between Λ_5 and Λ_1.

We make the respective subdivision of the blocks of the matrix V too. The condition $UV = 0$ implies that, whenever the λ-th vertical stripe of U is nonzero ($\lambda \in \Lambda^{ii}$), the λ-th horizontal stripe of V is zero. The conditions $VU = C^{n-1}$ can be rewritten as

$$V_{ij}U^{st} = \begin{cases} I & \text{if } (i,j,s,t) = (n-1,1,n-1,n-1), \\ 0 & \text{otherwise.} \end{cases}$$

It implies that there are no zero vertical stripes in the new subdivision of $U^{n-1,n-1}$. Moreover, if $\lambda \in \Lambda^{ii}$, $\mu \in \Lambda_k$, and the block V^{λ}_{μ} is nonzero, then the μ-th vertical stripe of U is zero if $i \neq n-1$; if $i = n-1$ this stripe contains exactly one non-zero block, namely, $U^{\mu}_{\lambda} = I$. We denote by $\overline{\Lambda}^{ii}$ and $\overline{\Lambda}_k$ the set of those stripes from Λ^{ii} and Λ_k, which are not completely defines by these rules. Let $\lambda \in \Lambda_5$, λ' be the corresponding element of Λ_1. If the blocks U^{μ}_{λ} and $U^{\mu'}_{\lambda'}$ are both nonzero, write $\mu \sim \mu'$. Note that there is at most one element μ' such that it holds, and $\mu' \neq \mu$.

One can verify that the sets $\overline{\Lambda}^{ii}$ and $\overline{\Lambda}_k$ can be linearly ordered so that, applying the transformations of types 1–5 form above, we can replace a stripe V^{λ} by $V^{\lambda} + V^{\lambda'}Z$ with $\lambda' < \lambda$ and a stripe V_{μ} by $V_{\mu} + ZV_{\mu'}$, where $\lambda' < \lambda$, $\mu' < \mu$ for any matrix Z (of the appropriate size). We can also replace V^{λ} by $V^{\lambda}Z$, where Z is invertible, and replace simultaneously V_{μ} and $V_{\mu'}$, where $\mu' \sim \mu$, by ZV_{μ} and $ZV_{\mu'}$ (if μ' does not exist, just replace V_{μ} by ZV_{μ}) with invertible Z. Therefore, we obtain a special sort of the matrix problems considered in [Bo], which is known to be tame. Hence, the algebra $\mathtt{A}_n^{\{n-1,n\}}$ is tame as well. □

Proof of Theorem 2.1. Lemma 2.7 and Lemma 2.2 imply Theorem 2.1(i). The statement of Theorem 2.1(iii) follows from Theorem 2.1(i) and Theorem 2.1(ii) by [Dr1]. Hence we have to prove Theorem 2.1(ii) only.

It is known, see for example [DR], that $\mathtt{A}_n$ has finite representation type for $n \leq 3$, is tame for $n = 4$, and is wild for all other n. This, in particular, proves Theorem 2.1(ii) for $n \leq 4$.

If $n \geq 6$ then from Lemma 2.6 it follows that if $\mathtt{A}_n^X$ is tame then $X \subset \{2,3,n-1,n\}$. From Theorem 2.1(i) we know that $X \not\subset \{2,n\}$. From Lemma 2.3 it follows that $\{3,n-1\} \not\subset X$ and $\{3,n\} \not\subset X$. From Lemma 2.4 it follows that $\{2,n-1\} \not\subset X$. This leaves us the cases $X = \{n-1,n\}$, $\{n-1\}$, $\{2,3\}$ and $\{3\}$. In the first two cases $\mathtt{A}_n^X$ is tame by Lemma 2.8. The algebra $\mathtt{A}_n^{\{2,3\}}$ is given by the following quiver with relations:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[r, bend left, "s"] & 3 \arrow[l, bend left, "t"] \end{tikzcd} \qquad \begin{array}{l} ab = ts, \\ (st)^{n-2} = 0, \end{array}$$

where $a = a_1$, $b = b_1$, $s = a_2$, $t = b_2$. It is tame as a quotient of the classical tame problem from [NR]. Hence $\mathtt{A}_n^{\{3\}}$ is tame as well.

For $n = 5$ the proof is analogous using also Lemma 2.5. This completes the proof. □

3 Proof of Theorem 1.1

We briefly recall the structure of ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ following [So, FKM]. By [BeGe, Theorem 5.9], the category ${}^{\infty}_{\lambda}\mathcal{H}^1_0$ is equivalent to the block $\mathcal{O}_\lambda$ of the BGG category $\mathcal{O}$, [BGG]. The simple modules in $\mathcal{O}_\lambda$ are in the natural bijection with the cosets $\mathbf{W}/\mathbf{G}$. For $w \in \mathbf{W}$ let $L(w)$ denote the corresponding simple module in $\mathcal{O}_\lambda$, and $P(w)$ be the projective cover of $L(w)$. Then [So] implies that for the longest element $w_0 \in \mathbf{W}$ one has $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0)) \cong \mathtt{C}(\mathbf{W},\mathbf{G})$. The group $\mathbf{H}$ acts on $\mathbf{W}\cdot\lambda$. Let $P(\lambda,\mathbf{H})$ denote the direct sum of indecomposable projective modules that correspond to the longest elements in all orbits of this action. The category ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ is equivalent, by [KM], to the module category over $\mathtt{B}(\mathbf{G},\mathbf{H}) = \mathrm{End}_{\mathcal{O}_\lambda}(P(\lambda,\mathbf{H}))$. From [So] it follows that $\mathtt{B}(\mathbf{G},\mathbf{H})$ depends on $\mathbf{G}$ rather than on λ.

Note that $P(w_0)$ is always a direct summand of $P(\lambda,\mathbf{H})$. Hence $\mathtt{C}(\mathbf{W},\mathbf{G})$ is a centralizer subalgebra of $\mathtt{B}(\mathbf{G},\mathbf{H})$. In particular, for ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ to be of finite representation type, $\mathtt{C}(\mathbf{W},\mathbf{G})$ must be of finite representation type as well. According to [GP, Theorem 7.2], $\mathtt{C}(\mathbf{W},\mathbf{G})$ is of finite representation type in the following cases:

(I) $\mathbf{W}=\mathbf{G}$;

(II) $\mathbf{W}$ is of type A_n and $\mathbf{G}$ is of type A_{n-1};

(III) $\mathbf{W}$ is of type B_n and $\mathbf{G}$ is of type B_{n-1};

(IV) $\mathbf{W}$ is of type C_n and $\mathbf{G}$ is of type C_{n-1};

(V) $\mathbf{W}$ is of type G_2 and $\mathbf{G}$ is of type A_1.

Moreover, in all this cases $\mathtt{C}(\mathbf{W},\mathbf{G}) \cong \mathbb{C}[x]/(x^r)$, where $r = [\mathbf{W}:\mathbf{G}]$. The last observation and [FKM, Theorem 1] imply that in all the above cases the category ${}^{\infty}_{\lambda}\mathcal{H}^1_0$ is equivalent to $\mathtt{A}_r-$mod.

The case (I) gives Theorem 1.1(1a). In the cases (II), (III), (IV), and (V) from Theorem 2.1(i) we have the following possibilities for $\mathtt{B}(\mathbf{G},\mathbf{H})$:

$\mathtt{B}(\mathbf{G},\mathbf{H})$ **has one simple module.** This implies $\mathbf{W}=\mathbf{H}$ and gives Theorem 1.1(1b).

$\mathtt{B}(\mathbf{G},\mathbf{H})$ **has two simple modules.** These simples correspond either to the dominant and anti-dominant weights in $\mathcal{O}_\lambda$ or to the anti-dominant weight and its neighbor. By a direct calculation we get the following: the case $r=2$ gives Theorem 1.1(1c), and the case $r>2$ gives Theorem 1.1(1d).

$\mathtt{B}(\mathbf{G},\mathbf{H})$ **has three simple modules.** These simples correspond to the following weights in $\mathcal{O}_\lambda$: the anti-dominant one, its neighbor, and the dominant one. By a direct calculation we get the following: the case $r=3$ gives Theorem 1.1(1h), and the case $r>3$ gives Theorem 1.1(1e), Theorem 1.1(1f), and Theorem 1.1(1g). This proves Theorem 1.1(1).

Let us now proceed with the tame case, that is with Theorem 1.1(2). If $\mathtt{C}(\mathbf{W},\mathbf{G})$ is of finite representation type, that is in the cases (I)–(V), Theorem 2.1(ii) give us the following possibilities for $\mathtt{B}(\mathbf{G},\mathbf{H})$:

$\mathtt{B}(\mathbf{G},\mathbf{H})$ has two simple modules. These simples correspond to the following weights in $\mathcal{O}_\lambda$: either the anti-dominant one and the neighbor of its neighbor, or the anti-dominant one and the neighbor of the dominant one. By a direct calculation we get that these cases lead to Theorem 1.1(2b) and Theorem 1.1(2c).

$\mathtt{B}(\mathbf{G},\mathbf{H})$ has three simple modules. These simples correspond to the following weights in $\mathcal{O}_\lambda$: either the anti-dominant one, its neighbor, and the neighbor of its neighbor, or the anti-dominant, its neighbor and the dominant one. By a direct calculation we get that these cases lead to Theorem 1.1(2d) and Theorem 1.1(2e).

$\mathtt{B}(\mathbf{G},\mathbf{H})$ has four simple modules. In this case $r = 4$ and a direct calculation gives Theorem 1.1(2f).

The rest (that is Theorem 1.1(2a)) should correspond to the case when $\mathtt{C}(\mathbf{W},\mathbf{G})$ is tame. According to [GP, Theorem 7.2], $\mathtt{C}(\mathbf{W},\mathbf{G})$ is tame in the following cases:

(VI) $\mathbf{W}$ has rank 2 and $\mathbf{G} = \{e\}$;

(VII) $\mathbf{W}$ is of type A_3 and $\mathbf{G}$ is of type $A_1 \times A_1$;

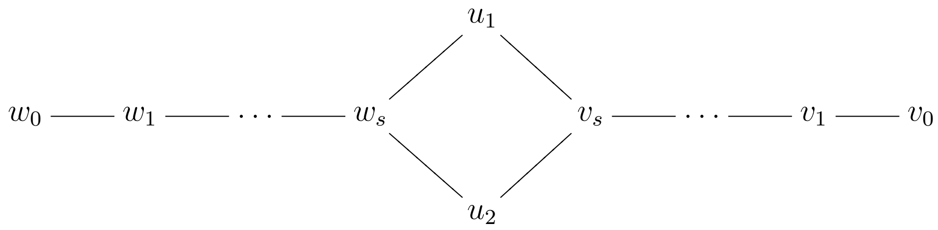
(VIII) $\mathbf{W}$ is of type B_3 and $\mathbf{G}$ is of type A_2;

(IX) $\mathbf{W}$ is of type C_3 and $\mathbf{G}$ is of type A_2;

(X) $\mathbf{W}$ is of type D_n and $\mathbf{G}$ is of type D_{n-1}.

For $\mathbf{W} = \mathbf{H}$ the cases (VI), (VII), (VIII), (IX), and (X) give exactly Theorem 1.1(2a). Let us now show that the rest is wild.

If $\mathbf{W} \neq \mathbf{H}$ then ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ contains at least two non-isomorphic indecomposable projective modules, one of which is $P(w_0)$ and the other one is some $P(w)$. We consider first the cases (VII), (VIII), (IX), and (X). In all these cases the restriction of the Bruhat order to $\mathbf{W}/\mathbf{G}$ gives the following poset:



(3)

From [GP, Theorem 7.3] it follows that in all these cases the algebra $\mathtt{C}(\mathbf{W},\mathbf{G})$ has two generators.

We consider the centralizer subalgebra $\mathtt{D}(w) = \mathrm{End}_{\mathcal{O}_\lambda}(P(w_0) \oplus P(w))$. This algebra is positively graded, see for example [St1].

Lemma 3.1. *Let A be a basic associative algebra, e be an idempotent of A and f be a primitive direct summand of e. Assume that there exists two non-isomorphic A-modules M and N satisfying the following properties:*

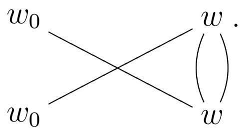
(1) both M and N have simple top and simple socles isomorphic to the simple A-module $L^A(f)$, corresponding to f;

(2) $e\,\mathrm{rad}(M)/\mathrm{soc}(M) = e\,\mathrm{rad}(N)/\mathrm{soc}(N) = 0$.

Then $\dim \mathrm{Ext}^1_{eAe}(L^{eAe}(f), L^{eAe}(f)) > 1$.

Proof. Recall from [Au, Chapter 5] that $eAe-$mod is equivalent to the full subcategory $\mathcal{M}$ of $A-$mod, consisting of all Ae approximations of modules from $A-$mod. Let M' and N' be the Ae-approximations of M and N respectively. Then the eAe-modules eM' and eN' have, because of (1) and (2), length two with both composition subqoutients isomorphic to $L^{eAe}(f)$. Again by [Au, Chapter 5], any eAe-isomorphism of eM' and eN' induces an A-isomorphism of M' and N'. The latter induces an isomorphism of the maximal images of both M' and N' in the injective module $I(f)$. However, these images are isomorphic to M and N respectively because of (1). This implies the statement. □

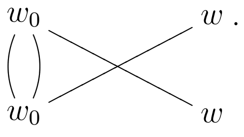
Assume first that the multiplicity of $L(w)$ in $P(v_0)$ is greater than 1 (in particular, $w \neq v_0$). This means that there exists $x > w$, which is not a neighbor of w, and such that $\mathrm{Ext}^1_{\mathcal{O}_\lambda}(L(w), \Delta(x)) \neq 0$. Fix a non-split extension from the last space. The highest occurrence of $L(w)$ in the radical filtration of $\Delta(x)$ induces, via this extension, a module, M, with simple top and socle isomorphic to $L(w)$ and such that $\mathrm{rad}(M)/\mathrm{soc}(M)$ does not contain neither $L(w)$ nor $L(w_0)$, but contains $L(x)$. Let now $u > w$ be a neighbor. Then the same arguments as above give module N with simple top and socle isomorphic to $L(w)$ and such that $\mathrm{rad}(N)/\mathrm{soc}(N)$ does not contain neither $L(w)$ nor $L(w_0)$ nor $L(x)$. Consider the quotient $Q(w)$ of $\mathtt{D}(w)$ modulo the radical square. The unique occurrence of $L(w_0)$ in the composition series of $\Delta(w)$ gives a morphism, $\alpha : P(w_0) \to P(w)$, which does not belong to the square of the radical. Further, the unique occurrence of $\Delta(w)$ in the Verma flag of $P(w_0)$ gives a morphism, $\beta : P(w) \to P(w_0)$, which does not belong to the square of the radical. By Lemma 3.1 we also obtain that $Q(w)$ contains at least two loops at the point w. Factoring extra arrows out, $Q(w)$ gives rise to the following wild configuration:



(4)

Hence $\mathtt{D}(w)$ and thus ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ is wild in this case.

Assume now that $w \in \{v_0, v_1, \dots, v_s\}$. In this case $Q(w)$ obviously contains α, β as above. Moreover $Q(w)$ also contains two loops at the point w_0 which correspond to two generators of $\mathtt{C}(\mathbf{W}, \mathbf{G})$. Passing, if necessary, to a quotient of $Q(w)$, we obtain the following wild configuration:



(5)

This implies that $\mathtt{D}(w)$ and hence ${}^{\infty}_{\lambda}\mathcal{H}^{1}_{\mu}$ is wild in this case as well.

Now we can assume that the multiplicity of $L(w)$ in $P(v_0)$ is 1 and restrict ourselves to the case when $w \in \{w_1, \dots, w_s, u_1, u_2\}$. Then from [Ba, Proposition 2.12] it follows that $P(w)$ has simple socle, in particular, $P(w)$ is a submodule of $P(w_0)$. Injectivity of $P(w_0)$ thus gives a surjection from $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0)) \cong \mathtt{C}(\mathbf{W}, \mathbf{G})$ to $\mathrm{End}_{\mathcal{O}_\lambda}(P(w))$. Note that $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0))$ is the center of $\mathcal{O}_\lambda$ by [So]. We still have α and β as above, which do not belong to the square of the radical. Further, using the embedding $P(w) \hookrightarrow P(w_0)$ one also obtains that α generates $\mathrm{Hom}_{\mathcal{O}_\lambda}(P(w_0), P(w))$ as $\mathtt{C}(\mathbf{W}, \mathbf{G})$-module and β generates $\mathrm{Hom}_{\mathcal{O}_\lambda}(P(w), P(w_0))$ as $\mathtt{C}(\mathbf{W}, \mathbf{G})$-module.

Let $w = w_i$, $i \neq 0, 1, s$. Then from $P(w)$ we have a unique indecomposable extension M of $\Delta(w)$ and $\Delta(w_{i+1})$. The positivity of the grading on $\mathcal{O}_\lambda$ implies that M has a quotient, N, of Loewy length 3 with simple socle $L(w)$. Moreover, since all occurrences of $L(w_0)$ in M are in degree at least 2, it follows that $\mathrm{soc}(N)/\mathrm{rad}(N)$ contains neither $L(w_0)$ nor $L(w)$. Since $i + 1 \leq s$, we also obtain that $\mathrm{soc}(N)/\mathrm{rad}(N)$ contains neither $L(u_1)$ nor $L(u_2)$. Let Q be the (unique) quotient of $P(w)$ with the Verma flag, consisting of $\Delta(x)$, $x = w_i, \dots, w_s, u_1, u_2$. Then the cokernel of the composition of α with the canonical projection $P(w) \twoheadrightarrow Q$ has $L(w_0)$ with composition multiplicity 1, occurring in the socle; and has $L(w)$ with multiplicity 2. The last gives rise to a module, M', with simple top and simple socle isomorphic to $L(w)$ and such that $\mathrm{soc}(M')/\mathrm{rad}(M')$ contains $L(u_1)$ and $L(u_2)$. From Lemma 3.1 it now follows that the quotient of $\mathtt{D}(w)$ modulo the square of the radical gives rise to the wild configuration (4). Hence $\mathtt{D}(w)$ is wild in this case.

Now we assume that $w = w_s$ and $s > 1$. In this case the unique indecomposable extension of $\Delta(w_s)$ and $\Delta(u_1)$ has a quotient, M, with simple top and simple socle isomorphic to $L(w_s)$, moreover, $\mathrm{soc}(M)/\mathrm{rad}(M)$ does not contain neither $L(w_0)$ nor $L(w)$ nor $L(u_2)$ but contains $L(u_1)$. The same arguments applied to the indecomposable extension of $\Delta(w_s)$ and $\Delta(u_2)$ give a module, N, with simple top and simple socle isomorphic to $L(w_s)$ such that $\mathrm{soc}(N)/\mathrm{rad}(N)$ does not contain neither $L(w_0)$ nor $L(w)$ nor $L(u_1)$ but contains $L(u_2)$. From Lemma 3.1 it follows that the quotient of $\mathtt{D}(w)$ modulo the square of the radical gives rise to the wild configuration (4). Hence $\mathtt{D}(w)$ is wild in this case.

Let now $w = u_1$ (the case $w = u_2$ is analogous). In this case $\mathtt{D}(w)$ has the following quiver:

$$\begin{tikzcd}
w_0 \arrow[loop left, "x"] \arrow[r, bend left, "\alpha"] & w \arrow[l, bend left, "\beta"] \arrow[loop right, "y"]
\end{tikzcd} .$$

Note that α is surjective as a homomorphism from $\mathrm{End}_{\mathtt{D}(w)}(P(w_0))$ to $\mathrm{End}_{\mathtt{D}(w)}(P(w))$ since $P(w)$ has simple socle (the latter follows from [Ba, Proposition 2.12] or can be proved analogously to [St2, Theorem 7.1]). This and the fact that $\mathrm{End}_{\mathcal{O}_\lambda}(P(w_0))$ is central implies relations $\alpha x = y\alpha$ and $\beta y = x\beta$. Using [GP, 7.12-7.16] one also easily gets the following additional relations: $y^{s+2} = 0$, $\alpha\beta = cy^{s+1}$ for some $0 \neq c \in \mathbb{C}$, $x\beta\alpha = \beta\alpha x = 0$ and $(\beta\alpha)^2 = x^{2s+3}$. In the case in the universal covering of $\mathtt{D}(w)$ has the following fragment (shown for $s=1$):

$$\begin{tikzcd}
\mathbf{w_0} \arrow[r, "\alpha"] \arrow[d, "x"'] \arrow[dr, dashed] & w \arrow[d, "y"] \arrow[ddl, "\beta"] \\
w_0 \arrow[r, "\alpha"] \arrow[d, "x"'] & w \\
w_0 &
\end{tikzcd} \tag{6}$$

(here the dashed arrow indicates the commutativity of the corresponding square). Evaluating the Tits form of this fragment at the point $(1,2,2,2,2)$, where 1 is placed in the bold vertex, we obtain $-1<0$ implying that the fragment (6) is wild. Hence $\mathtt{D}(w)$ is wild as well.

Finally, let us assume that $w=w_1$. We will show that this can not happen (under the assumption that the multiplicity of $L(w)$ in $P(v_0)$ is 1). Indeed, in this case we would have that the quiver of $\mathtt{D}(w)$ would look as follows:

$$\begin{tikzcd}
w_0 \arrow[r, bend left, "\alpha"] & w \arrow[l, bend left, "\beta"] \arrow[loop right, "\gamma"]
\end{tikzcd} .$$

According to [GP, 7.12-7.16], γ can be chosen such that the endomorphism algebra of $P(w_0)$ is given by the generators $\beta\alpha$ and $\beta\gamma\alpha$ with the relations

$$(\beta\alpha)^n = (\beta\gamma\alpha)^2 \quad \text{for some} \quad n\geq 4, \tag{7}$$

$$\beta\alpha\beta\gamma\alpha = \beta\gamma\alpha\beta\alpha = 0. \tag{8}$$

However, β is an inclusion and α is surjective as a homomorphism from $\mathrm{End}_{\mathtt{D}(w)}(P(w_0))$ to $\mathrm{End}_{\mathtt{D}(w)}(P(w))$ (see above). Hence from (8) we obtain $\alpha\beta\gamma=\gamma\alpha\beta=0$, which implies $(\beta\alpha)^n=0$ using (7). This contradicts [GP, 7.12-7.16] mentioned above and completes the cases (VII), (VIII), (IX), and (X).

Now let us consider the case (VI). If $\mathbf{H}\neq\mathbf{W}$, then ${}^{\infty}_{\lambda}\mathcal{H}^1_\mu$ necessarily contains an indecomposable projective module, which corresponds to some w such that $\mathfrak{l}(w_0)-\mathfrak{l}(w)=2$. Translating $L(w_0)$ with respect to two simple roots gives rise to two non-isomorphic modules with simple top and simple socle isomorphic to $L(w_0)$, which has $L(w_0)$ with multiplicity 2 and do not have any occurrence of $L(w)$. Hence from Lemma 3.1 it follows that the quotient of the corresponding $\mathtt{D}(w)$ modulo the square of the radical gives rise to the wild configuration (5). Hence $\mathtt{D}(w)$ is wild in this case. This proves Theorem 1.1(2).

To complete the proof we just note that Theorem 1.1(3) follows from Theorem 1.1(1), Theorem 1.1(2) by [Dr1].

4 The case of a semi-simple algebra $\mathfrak{g}$

Theorem 1.1 is formulated for a simple algebra $\mathfrak{g}$. However, in the case of a semi-simple algebra the result is almost the same. In a standard way it reduces to the description of the representation types of the tensor products of algebras, described in Theorem 1.1.

Theorem 4.1. *Let $k > 1$ be a positive integer, and $\mathtt{X}_i$, $i = 1, \dots, k$, be basic algebras associated to non-semi-simple categories from the list of Theorem 1.1. Then the algebra $\mathtt{X}_1 \otimes \cdots \otimes \mathtt{X}_k$ is never of finite representation type, and it is of tame representation type only in the following two cases:*

(1) $k = 2$ and both $\mathtt{X}_1$ and $\mathtt{X}_2$ have Coxeter type (A_1, e, A_1);

(2) $k = 2$, one of $\mathtt{X}_1$ and $\mathtt{X}_2$ has Coxeter type (A_1, e, A_1), and the other one has Coxeter type (A_1, e, e).

Proof. The algebra in (1) is isomorphic to $\mathbb{C}[x, y]/(x^2, y^2)$ and hence is tame with well-known representations. Let us thus consider the algebra $\mathtt{X}$ of the case (2). This algebra is given by the following quiver with relations

$$\begin{tikzcd} 1 \arrow[loop left, "x"] \arrow[r, bend left, "u"] & 2 \arrow[l, bend left, "v"] \arrow[loop right, "y"] \end{tikzcd} \qquad x^2 = y^2 = uv = 0,\ ux = yu,\ xv = vy. \tag{9}$$

Lemma 4.2. *The algebra of* (9) *is tame.*

Proof. This algebra is tame by [Be], however, since the last paper is not easily available and does not contain a complete argument, we prove the tameness of $\mathtt{X}$. Consider the subalgebra $\mathtt{X}' \subset \mathtt{X}$ generated by x, y, u. Its indecomposable representations are

$$\begin{tikzcd} e_8 \arrow[d] & e_9 \arrow[d] \arrow[dr] & f_{10} \arrow[d] & e_{10} \arrow[d] \arrow[dr] & & e_{11} \arrow[r] \arrow[d] & f_8 \arrow[d] \\ e_1 & e_2 & f_3 & e_3 & f_6 & e_4 \arrow[r] & f_1 \\ e_5 & e_6 \arrow[dr] & f_9 \arrow[d] & e_7 \arrow[r] & f_5 & f_{11} \arrow[d] & f_7 \\ & & f_2 & & & f_4 & \end{tikzcd} \tag{10}$$

Here the elements e_i form a basis of the space corresponding to the vertex 1, the elements f_j form a basis of the space corresponding to the vertex 2, the vertical arrows show the action of x and y, and the arrows going from left to right show the action of u. Let M be an $\mathtt{X}$-module. Decompose it as $\mathtt{X}'$-module. Then the matrix V describing the action of v divides into the blocks V_{ij}, $i, j = 1, 2, \dots, 11$, corresponding to the basic elements e_i and f_j from above. Moreover, since $uv = 0$, the blocks V_{ij} can only be nonzero if $i \in \{1, 2, 3, 5, 8\}$; since $xv = vy$, $V_{ij} = 0$ if $i > 4, j < 5$ or $i > 7, j < 8$, and $V_{ij} = V_{i+7,j+7}$ for $i, j \in \{1, 2, 3, 4\}$. If M' is another $\mathtt{X}$-module, $V' = (V'_{ij})$ is the corresponding block matrix, a homomorphism $M \to M'$ is given by a pair of matrices S, T, where $S : M(1) \to M(1)$, $T : M(2) \to M(2)$. Divide them into blocks corresponding to the division of V: $S = (S_{ij})$, $T = (T_{ij})$, $i, j = 1, 2, \dots, 11$. One can easily check that such block matrices define a homomorphism $M \to M'$ if and only if the following conditions hold:

- S and T are block triangular, i.e. $S_{ij} = 0$ and $T_{ij} = 0$ if $i > j$.

- $S_{ij} = S_{i+7,j+7}$ and $T_{ij} = T_{i+7,j+7}$ for $i, j \in \{1, 2, 3, 4\}$.
- $S_{ii} = T_{jj}$ if in the list (10) there is an arrow $e_i \to f_j$.
- $S_{ij} = T_{kl}$ if in the list (10) there are arrows $e_i \to f_k$ and $e_j \to f_l$.
- $S_{ij} = 0$ if $(i, j) \in \{(4, 5), (4, 6), (6, 8), (7, 8), (7, 9)\}$.
- $T_{ij} = 0$ if $(i, j) \in \{(3, 5), (4, 5), (4, 6), (7, 8)\}$.

Certainly, S, T define an isomorphism if and only if all diagonal blocks are invertible. In particular, we can replace the part $V_1 = (V_{11}\ V_{12}\ V_{13}\ V_{14})$ by $S_1^{-1} V_1 T_1$, where S_1 is any invertible matrix and $T_1 = (T_{ij}),\ i, j \in \{1, 2, 3, 4\}$ is any invertible block triangular matrix. So we can suppose that V_1 is of the form

$$\left(\begin{array}{cc|cc|cc|cc}
0 & I^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I^{(2)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I^{(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(4)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),$$

where the vertical lines show the division of V_1 into blocks, $I^{(k)}$ denote identity matrices (of arbitrary sizes). Denote the parts of the blocks V_{1j} to the right of $I^{(k)}$ by $V_{1k,j}$ and those to the right of the zero part of V_1 by V_{5j}. Using automorphisms, we can make zero all $V_{11,j}$ and $V_{12,j}$, as well as the blocks $V_{13,j}$ and $V_{14,j}$ for $j > 6$. Note that $V_{1j} = V_{8,j+7}$, and we can also make zero all parts of the blocks $V_{1,j+7}$ over the parts $I^{(j)}$ of the blocks $V_{8,j+7}$. Subdivide the blocks of S and T corresponding to this subdivision of V_1. Note that, since $S_{22} = T_{99} = T_{33}$, we must also subdivide the blocks S_{2j} into $S_{20,j}$ and $S_{21,j}$ respective to the zero and nonzero parts of V_{13}. Then the extra conditions for the new blocks are:

$$S_{21,20} = 0 \quad \text{and} \quad S_{1k,1l} = 0 \quad \text{if} \quad k > l.$$

Therefore, we get a matrix problem considered in [Bo]. It is described by the semichain

$$\begin{tikzcd}
f_5 \arrow[r] & f_6 \arrow[r] \arrow[dr] & f_7 \arrow[dr] & & & \\
& & f_8 \arrow[r] & f_9 \arrow[r] & f_{10} \arrow[r] & f_{11}
\end{tikzcd}$$

for the columns, the chain

$$e_5 \to e_3 \to e_{21} \to e_{20} \to e_{15} \to e_{14} \to e_{13}$$

for the rows, and the unique equivalence $e_3 \sim f_6$. This matrix problem is tame, hence, the algebra `X` is tame as well. $\square$

If $k>2$ then each of $\mathtt{X}_1$, $\mathtt{X}_2$, and $\mathtt{X}_3$ has projective module with non-trivial endomorphism ring and thus $\mathtt{X}_1\otimes\mathtt{X}_2\otimes\mathtt{X}_3$ contains a centralizer subalgebra, which surjects onto $\mathbb{C}[x,y,z]/(x,y,z)^2$. The later algebra is wild by [Dr2] and hence $\mathtt{X}$ is wild.

If $k=2$ but none of the conditions (1), (2) is satisfied, then one of the algebras $\mathtt{X}_1$ and $\mathtt{X}_2$ has a projective module, whose endomorphism algebra surjects onto $\mathbb{C}[x]/(x^3)$, and the other one has a projective module, whose endomorphism algebra surjects onto $\mathbb{C}[y]/(y^3)$. Hence there is a centralizer subalgebra in $\mathtt{X}$, which surjects onto $\mathbb{C}[x,y]/(x^3,y^2)$, the later being wild by [Dr2]. This shows that $\mathtt{X}_1\otimes\mathtt{X}_2$ is wild as well and completes the proof. $\square$

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