REGULAR STRONGLY TYPICAL BLOCKS OF $\mathcal{O}^{\mathfrak{q}}$

ANDERS FRISK AND VOLODYMYR MAZORCHUK

Abstract. We use the technique of Harish-Chandra bimodules to prove that regular strongly typical blocks of the category $\mathcal{O}$ for the queer Lie superalgebra $\mathfrak{q}_n$ are equivalent to the corresponding blocks of the category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_n$.

1. Introduction and the main result

For $n \in \mathbb{N}$ let $\mathfrak{q}_n$ denote the queer Lie superalgebra and $\mathcal{O}^{\mathfrak{q}}$ denote the category $\mathcal{O}$ for $\mathfrak{q}_n$. The category $\mathcal{O}^{\mathfrak{q}}$ decomposes into a direct sum of blocks, which can be *typical* or *atypical*. Atypical blocks are very complicated and may contain infinitely many simple objects. Typical blocks are much easier and are always equivalent to module categories over finite-dimensional algebras. In [Fr] it was shown that the finite-dimensional algebras describing typical blocks are always properly stratified in the sense of [Dl]. Among all typical blocks one separates *strongly typical* ones, which are described ([Fr]) by quasi-hereditary algebras in the sense of [CPS, DR].

A very general conjecture about the combinatorial structure of the category $\mathcal{O}^{\mathfrak{q}}$ is given in [Br, 4.8]. In the special case of regular strongly typical blocks this conjecture says that multiplicities of simple highest weight supermodules in Verma supermodules for $\mathfrak{q}_n$ are given by Kazhdan-Lusztig combinatorics. For this special case a much stronger conjecture was formulated in [Fr, 3.9], namely that strongly typical blocks of $\mathcal{O}^{\mathfrak{q}}$ are equivalent to the corresponding blocks of the category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_n$. A strong evidence for this conjecture, established in [Fr], was a similarity between the quasi-hereditary structures in both cases. Moreover, [Fr, 3.9] contains also an explicit conjecture for the structure of all typical blocks. The aim of this paper is to prove the conjecture from [Fr, 3.9] (and hence the conjecture from [Br, 4.8] as well) for *regular* strongly typical blocks.

There is a natural restriction functor from $\mathcal{O}^{\mathfrak{q}}$ to the category $\mathcal{O}$ for the Lie algebra $\mathfrak{gl}_n$. However, unlike the case of most of the other Lie superalgebras, this restriction functor does not induce an equivalence in a straightforward way. The problem is that the highest weights of Verma supermodules over $\mathcal{O}^{\mathfrak{q}}$ are not one-dimensional (because the Cartan subsuperalgebra of $\mathfrak{q}$ is not commutative). Subsequently, under restriction Verma supermodules are not mapped to the corresponding

Date: January 15, 2009.

Verma modules but rather to direct sums of Verma modules (see Proposition 2). This suggests that the naive restriction functor is a direct sum of several copies of some "smaller" functor, which defines the desired equivalence of categories. This is exactly what we prove in this paper.

The main idea of the proof is to realize the induction functor (the left adjoint to the restriction) as a tensor product with some Harish-Chandra bimodule. This requires several definitions and some technical work as we are forced to go beyond the original categories and instead work with the so-called thick version of the category $\mathcal{O}$. Unfortunately, along the way we use some properties of Harish-Chandra bimodules, which require an additional assumption of regularity of the blocks we work with. The main result of the paper is the equivalence of blocks of categories $\mathcal{O}$ for $\mathfrak{q}_n$ and $\mathfrak{gl}_n$, see Theorem 1. This extends earlier results of Penkov, Serganova and Gorelik (see [PS, Go2]) to the case of the algebra $\mathfrak{q}_n$ and verifies conjectures from [Fr, 3.9] and [Br, 4.8] in our setup. At the same time Theorem 1 is a refinement (in the case of $\mathfrak{q}_n$) of the main result of [Pe].

The paper is organized as follows: In Section 2 we give all necessary definitions and formulate our main result. In Section 3 we collect auxiliary technical results about Harish-Chandra bimodules. The main result is proved in Section 4. We conclude the paper with some remarks in Section 5.

Acknowledgments. For the second author the research was partially supported by the Swedish Research Council. We would like to thank the referees for several corrections and very useful suggestions, which led to improvements in the original version of the paper.

2. Definitions, preliminaries and formulation of the main result

For all undefined notions we refere the reader to [Fr]. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all positive and nonnegative integers, respectively, and fix $n \in \mathbb{N}$. Set $\mathbb{N}_n = \{1, \dots, n\}$. For an algebraically closed field $\Bbbk$ of characteristic zero let $\mathfrak{g} = \mathfrak{gl}_n$ denote the general linear Lie algebra of $n \times n$ matrices over $\Bbbk$. Let $\mathfrak{q} = \mathfrak{q}_n$ denote the *queer Lie superalgebra* over $\Bbbk$, which consists of block matrices of the form

$$\mathtt{M}(A, B) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad A, B \in \mathfrak{gl}_n.$$

The even and the odd spaces $\mathfrak{q}_{\overline{0}}$ and $\mathfrak{q}_{\overline{1}}$ consist of the matrices $\mathtt{M}(A, 0)$ and $\mathtt{M}(0, B)$, respectively, and we have $\mathfrak{q} = \mathfrak{q}_{\overline{0}} \oplus \mathfrak{q}_{\overline{1}}$. For a homogeneous element $X \in \mathfrak{q}$ we denote by $\overline{X} \in \mathbb{Z}/2\mathbb{Z}$ the degree of X. Then the Lie superbracket in $\mathfrak{q}$ is given by $[X, Y] = XY - (-1)^{\overline{X}\overline{Y}} YX$, where $X, Y \in \mathfrak{q}$ are homogeneous.

For $i, j \in \mathbb{N}_n$ let $E_{ij} \in \mathfrak{gl}_n$ denote the corresponding matrix unit. We have the *Cartan subsuperalgebra* $\mathfrak{h}$ of $\mathfrak{q}$, which is the linear span of $H_i = \mathtt{M}(E_{ii}, 0)$ and $\mathtt{M}(0, E_{ii})$, $i \in \mathbb{N}_n$. The superalgebra $\mathfrak{h}$ inherits from $\mathfrak{q}$ the decomposition $\mathfrak{h} = \mathfrak{h}_{\overline{0}} \oplus \mathfrak{h}_{\overline{1}}$.

Let $\{\varepsilon_i : i \in \mathbb{N}_n\}$, denote the basis of $\mathfrak{h}_{\overline{0}}^*$, which is dual to the basis $\{H_i : i \in \mathbb{N}_n\}$ of $\mathfrak{h}_{\overline{0}}$. Then $\Phi = \{\varepsilon_i - \varepsilon_j : i, j \in \mathbb{N}_n, i \neq j\}$ is the *root system* of $\mathfrak{q}$ with the corresponding Weyl group $W \cong \mathbf{S}_n$. We also have the standard set $\Phi^+ = \{\varepsilon_i - \varepsilon_j : i, j \in \mathbb{N}_n, i < j\}$ of *positive* roots. Set $\overline{\Phi}^+ = \{\varepsilon_i + \varepsilon_j : i, j \in \mathbb{N}_n, i < j\}$. The group W acts on $\mathfrak{h}_{\overline{0}}^*$ in the usual way and via the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$, where ρ is the half of the sum of all positive roots. Let $(\cdot, \cdot)$ denote the standard nondegenerate W-invariant bilinear form, given by $(\varepsilon_i, \varepsilon_j) = \delta_{i,j}$.

For $\alpha \in \Phi$ set $\mathfrak{q}^\alpha = \{v \in \mathfrak{q} : [H, v] = \alpha(H)v \text{ for all } H \in \mathfrak{h}_{\overline{0}}\}$. Then $\mathfrak{q}^\alpha = \mathfrak{q}^\alpha_{\overline{0}} \oplus \mathfrak{q}^\alpha_{\overline{1}}$, where both components are one-dimensional, and we further have the decomposition

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{q}^\alpha.$$

This induces the natural triangular decomposition $\mathfrak{q} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ with respect to our choice Φ^+ of positive roots.

Elements in $\mathfrak{h}_{\overline{0}}^*$ are called *weights* and are written $\lambda = (\lambda_1, \dots, \lambda_n)$ with respect to the basis $\{\varepsilon_i\}$. For $\lambda, \mu \in \mathfrak{h}_{\overline{0}}^*$ we write $\lambda \leq \mu$ provided that $\mu - \lambda \in \mathbb{N}_0\Phi^+$. For $\lambda \in \mathfrak{h}_{\overline{0}}^*$ we denote by W^λ the *integral Weyl group* of λ (the set of all $w \in W$ such that $w\lambda \in \lambda + \mathbb{Z}\Phi^+$). A weight λ is called

- *integral* provided that $\lambda_i \in \mathbb{Z}$ for any $i \in \mathbb{N}_n$;
- *dominant* provided that $w\lambda \leq \lambda$ for any $w \in W^\lambda$;
- *regular* provided that $w\lambda \neq \lambda$ for any $w \in W^\lambda$;
- *typical* provided that $(\alpha, \lambda) \neq 0$ for all $\alpha \in \overline{\Phi}^+$;
- *strongly typical* provided that it is typical and $\lambda_i \neq 0$ for all i.

Strongly typical weights were originally defined as the highest weights for which the corresponding highest weight modules are not annihilated by the ghost element T, introduced in [Go1]. The above definition follows from the explicit description of T for the superalgebra $\mathfrak{q}_n$, see for example [Go3, Theorem 10.4].

The algebra $\mathfrak{q}_{\overline{0}}$ is identified with the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ in the obvious way, and $\mathfrak{g}$ inherits from $\mathfrak{q}$ the triangular decomposition $\mathfrak{g} = \mathtt{n}_- \oplus \mathtt{h} \oplus \mathtt{n}_+$. A $\mathfrak{q}$-supermodule M is called a *weight* supermodule if it is a weight (that is $\mathtt{h}$-diagonalizable) module with respect to $\mathfrak{g}$.

We consider the category $\mathfrak{SM}$ of all $\mathfrak{q}$-supermodules, where all morphisms are homogeneous maps of degree zero. This is an abelian category with usual kernels and cokernels. Let $\Pi : \mathfrak{SM} \to \mathfrak{SM}$ denote the autoequivalence, which changes the parity. Let further $\mathfrak{M}$ denote the category of all $\mathfrak{g}$-modules, which is also abelian with usual kernels and cokernels. Let $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}} : \mathfrak{S}\mathfrak{M} \to \mathfrak{M}$ denote the functor of restriction from $\mathfrak{q}$ to $\mathfrak{g}$, which sends a $\mathfrak{q}$-supermodule $M = M_{\overline{0}} \oplus M_{\overline{1}}$ to the $\mathfrak{g}$-module $M_{\overline{0}}$. We denote by $\mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{q}} : \mathfrak{M} \to \mathfrak{S}\mathfrak{M}$ the left adjoint of $\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}}$.

Let $\mathcal{O}^{\mathfrak{q}}$ and $\mathcal{O}^{\mathfrak{g}}$ denote the BGG categories $\mathcal{O}$ for $\mathfrak{q}$ and $\mathfrak{g}$, respectively (see [BGG]). These are full subcategories in the respective categories of finitely generated (super)module, which consist of weight (super)modules, which are locally $U(\mathfrak{n}_+)$- and $U(\mathfrak{n}_+)$-finite, respectively.

Let $U(\mathfrak{q})$ and $U(\mathfrak{g})$ denote the *universal enveloping (super)algebra* of $\mathfrak{q}$ and $\mathfrak{g}$, respectively. Let, further, $Z(\mathfrak{q})$ and $Z(\mathfrak{g})$ denote the *(super)center* of $U(\mathfrak{q})$ and $U(\mathfrak{g})$, respectively. The action of the (super)center gives rise to the following *block decomposition* of $\mathcal{O}^{\mathfrak{q}}$ and $\mathcal{O}^{\mathfrak{g}}$, indexed by *central characters*:

$$\mathcal{O}^{\mathfrak{q}} = \oplus_{\chi}\mathcal{O}^{\mathfrak{q}}_{\chi}, \qquad \mathcal{O}^{\mathfrak{g}} = \oplus_{\hat{\chi}}\mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}.$$

For any χ we have the inclusion functor $\mathrm{incl}_{\chi} : \mathcal{O}_{\chi} \to \mathcal{O}$ and the projection functor $\mathrm{proj}_{\chi} : \mathcal{O} \to \mathcal{O}_{\chi}$, which are both left and right adjoint to each other. Similarly for $\hat{\chi}$.

Throughout the paper we fix a regular dominant strongly typical weight λ for $\mathfrak{q}$. Denote by $\hat{\lambda}$ the corresponding $\mathfrak{g}$-weight (note that we formally have $\lambda = \hat{\lambda}$ as elements in $\mathfrak{h}_{\overline{0}}^{*}$, however, it is convenient to use different notation to specify the algebra we work with). Let $\chi = \chi_{\lambda}$ and $\hat{\chi}$ be the central characters for $\mathfrak{q}$ and $\mathfrak{g}$, which correspond to λ and $\hat{\lambda}$, respectively. We also denote by $\mathfrak{m}_{\chi}$ the kernel of χ and by $\mathtt{m}_{\hat{\chi}}$ the kernel of $\hat{\chi}$. If λ is integral, then the block $\mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}$ in indecomposable. If, in addition, n is odd, the block $\mathcal{O}^{\mathfrak{q}}_{\chi}$ is indecomposable as well. If n is even, then we have a decomposition

$$\mathcal{O}^{\mathfrak{q}}_{\chi} \cong \tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi} \oplus \Pi\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$$

such that $\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$ is indecomposable for integral λ. To make our notation independent of the parity of n we will denote by $\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$ the whole of $\mathcal{O}^{\mathfrak{q}}_{\chi}$ for odd n. Abusing notation we will denote the inclusion and projection functors between $\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$ and $\mathcal{O}^{\mathfrak{q}}$ in the same way as above.

We have the following pair of functors:

$$\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi} \underset{\mathrm{F}:=\mathrm{proj}_{\chi}\circ\mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{q}}\circ\mathrm{incl}_{\hat{\chi}}}{\overset{\mathrm{G}:=\mathrm{proj}_{\hat{\chi}}\circ\mathrm{Res}_{\mathfrak{g}}^{\mathfrak{q}}\circ\mathrm{incl}_{\chi}}{\rightleftarrows}} \mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}$$

From the definitions we have that (F, G) is an adjoint pair of functors. The main result of this paper is the following theorem:

Theorem 1. *There is a direct summand* F_1 *of* F *and a direct summand* G_1 *of* G *such that the functors* F_1 *and* G_1 *are mutually inverse equivalences of categories.*

Before we proceed it is necessary to say why the original functors F and G are not equivalences of categories. Both $\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$ and $\mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}$ are equivalent to categories of modules over finite-dimensional quasi-hereditary algebras (see [BGG] for $\mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}$ and [Fr] for $\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$). Moreover, simple objects in both categories are naturally indexed by elements from W. Quasi-hereditary structure on both categories comes with the collection of standard modules.

Standard modules in $\mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}$ are the usual Verma modules $M(\hat{\mu})$, $\hat{\mu} \in W \cdot \hat{\lambda}$ (observe that here we have the dot action of W). In $\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$ standard modules are *Verma supermodules*. They are constructed as follows: For $\mu \in \mathfrak{h}_{\overline{0}}^*$ let $V(\mu)$ be a simple $\mathfrak{h}$-supermodule of weight μ. The supermodule $V(\mu)$ is unique if n is odd and satisfies $\Pi(V(\mu)) \cong V(\mu)$. If n is even then there are exactly two simple $\mathfrak{h}_{\overline{0}}$-supermodules of weight μ, namely $\Pi(V(\mu))$ and $V(\mu)$ (we denote by $V(\mu)$ the supermodule, which will give rise to the Verma supermodule in $\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$). We have $V(\mu) = V(\mu)_{\overline{0}} \oplus V(\mu)_{\overline{1}}$ and

$$\dim_{\Bbbk}(V(\mu)_{\overline{0}}) = \dim_{\Bbbk}(V(\mu)_{\overline{1}}) = 2^{\lfloor (n-1)/2 \rfloor} =: \mathbf{k}.$$

Letting $\mathfrak{n}_+$ act trivially on $V(\mu)$ and inducing the obtained module up to $U(\mathfrak{q})$ we obtain the *Verma supermodule* $\Delta(V(\mu))$ (note that these supermodules where called *Weyl modules* in [Go3]). The weight μ is a highest weight of $\Delta(V(\mu))$ and has both even and odd dimension $\mathbf{k}$. The standard modules in $\tilde{\mathcal{O}}^{\mathfrak{q}}_{\chi}$ are $\Delta(V(\mu))$, $\mu \in W\lambda$ (observe that here we have the usual action of W).

Proposition 2. *For every $w \in W$ we have*

$$\mathrm{G}\Delta(V(w\lambda)) \cong \underbrace{M(w\cdot\hat{\lambda}) \oplus M(w\cdot\hat{\lambda}) \oplus \cdots \oplus M(w\cdot\hat{\lambda})}_{\mathbf{k}\ \textit{summands}} =: \mathbf{k}M(w\cdot\hat{\lambda}),$$

$$\mathrm{F}M(w\cdot\hat{\lambda}) \cong \underbrace{\Delta(V(w\lambda)) \oplus \cdots \oplus \Delta(V(w\lambda))}_{\mathbf{k}\ \textit{summands}} =: \mathbf{k}\Delta(V(w\lambda)).$$

Proof. We will need the following combinatorial lemma:

Lemma 3. *Let P denote the multiset $\{\sum_{\alpha\in I}\alpha : I \subset \Phi^+\}$.*

(i) For any $w \in W$ we have

$$(\{w\cdot\lambda\} + P) \cap W\lambda = \{w\lambda\}, \quad (\{w\lambda\} - P) \cap W\cdot\lambda = \{w\cdot\lambda\}.$$

(ii) For any $w \in W$ there is a unique element $x \in P$ such that we have $w\cdot\lambda + x = w\lambda$.

Proof. The second equality of (i) follows from the first one, so we will prove only the first equality. Fix $w \in W$. Then we have

$$\rho - w\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha - \frac{1}{2}\sum_{\beta\in w\Phi^+}\beta = \sum_{\alpha\in\Phi^+, w^{-1}\alpha\notin\Phi^+}\alpha = \sum_{\alpha\in\Phi^+\cap w\Phi^-}\alpha$$

(here $\Phi^- = -\Phi^+$), which is an element of P. This yields $w\lambda - w\cdot\lambda \in P$ and hence $w\lambda \in \{w\cdot\lambda\} + P$.

For any $\alpha \in \Phi^+$ we either have $\alpha \in w\Phi^+$ or $\alpha \in w\Phi^-$. This implies that $\rho + wP = w\rho + P$ (bijection of multisets) and hence

$$w\cdot\lambda + P = w\lambda + w\rho - \rho + P = w\lambda + wP.$$

Using the latter equality we obtain

$$\begin{aligned}(w\cdot\lambda + P)\cap W\lambda &= (w\lambda + wP)\cap W\lambda\\ &= w(\lambda+P)\cap W\lambda\\ &= w((\lambda+P)\cap W\lambda).\end{aligned}$$

We have $\lambda \leq \lambda+\mu$ for every $\mu \in P$. On the other hand, for a dominant regular λ and $w \in W$ the inequality $\lambda \leq w\lambda$ forces w to be the identity element. Hence the multiset $(\lambda + P)\cap W\lambda$ consists of exactly one element. The claim of the lemma follows. □

Fix $w \in W$. Consider the module $N = U(\mathfrak{g}) \otimes_{U(\mathsf{g})} M(w\cdot\hat{\lambda})$. By the PBW Theorem for Lie superalgebras ([Ro]), the algebra $U(\mathfrak{g})$ is a free $U(\mathsf{g})$-module with basis $\bigwedge \mathfrak{g}_{\overline{1}}$. In particular, it follows that the module N is free over $U(\mathfrak{n}_-)$ of finite rank. Hence N has a filtration whose subquotients are Verma supermodules. As every Verma supermodule is free over $U(\mathfrak{n}_-)$ by definition, we obtain that the highest weight elements of subquotients in any Verma filtration of N are images of the elements from the space $\bigwedge(\mathfrak{n}_+)_{\overline{1}} \otimes v$, where v is a highest weight vector of $M(w\cdot\hat{\lambda})$. Hence the corresponding highest weights belong to $w\cdot\hat{\lambda} + P$. By Lemma 3(i), the only weight from $W\lambda$, which intersects $w\cdot\hat{\lambda} + P$ is $w\lambda = w\hat{\lambda}$. This yields that $\mathrm{F}M(w\cdot\hat{\lambda})$ is isomorphic to the direct sum of several copies of $\Delta(V(w\lambda))$, say k copies.

Similarly, the restriction of $\Delta(V(w\lambda))$ to $U(\mathsf{g})$ is a $U(\mathfrak{n}_-)$-free module of finite rank, and hence has a Verma filtration. The highest weight elements of subquotients of this Verma filtration are images of elements from $\bigwedge(\mathfrak{n}_-)_{\overline{1}}\otimes v$, where v is a highest weight vector of $\Delta(V(w\lambda))$. Hence the corresponding highest weights have form $w\lambda - P$. By Lemma 3(i), the only weight from $W\cdot\hat{\lambda}$, which intersects $w\lambda - P$ is $w\cdot\hat{\lambda} = w\cdot\lambda$. This yields that $\mathrm{G}\Delta(V(w\lambda))$ is isomorphic to the direct sum of several copies of $M(w\cdot\hat{\lambda})$, say m copies.

By adjunction we have

$$\mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{F}M(w\cdot\hat{\lambda}), \Delta(V(w\lambda))) = \mathrm{Hom}_{U(\mathsf{g})}(M(w\cdot\hat{\lambda}), \mathrm{G}\Delta(V(w\lambda))),$$

which yields $k = m$ as Verma (super)modules have trivial endomorphism ring.

Finally, the multiplicity m equals the even multiplicity of the weight $w\cdot\hat{\lambda} = w\cdot\lambda$ in the space $\bigwedge(\mathfrak{n}_-)_{\overline{1}}\otimes V(w\lambda)$. By Lemma 3(ii), the multiplicity of the weight $w\cdot\lambda - w\lambda$ in $\bigwedge(\mathfrak{n}_-)_{\overline{1}}$ equals 1. Since $\dim_{\Bbbk}(V(w\lambda)_{\overline{0}}) = \dim_{\Bbbk}(V(w\lambda)_{\overline{1}}) = \mathbf{k}$, it follows that $m = \mathbf{k}$ and the proof is complete. □

From Proposition 2 it follows that in order to prove Theorem 1 we have to decompose the functors F and G into a direct sum of **k** copies of isomorphic functors. For this we will need the technique of Harish-Chandra bimodules.

3. Harish-Chandra $U(\mathfrak{q})-U(\mathfrak{g})$-bimodules

This section is inspired by and based on [Go2, 3.1.2]. Let M, N be $U(\mathfrak{g})$-modules. Then the space $\mathrm{Hom}_{\mathbb{C}}(M,N)$ has the natural structure of a $U(\mathfrak{g})$-bimodule and contains the subbimodule $\mathcal{L}(M,N)$, consisting of all elements, the adjoint action of $\mathfrak{g}$ on which is locally finite (see [Ja, Kapitel 6]). Similarly one defines $\mathcal{L}(M,N)$ in the case M and N are $U(\mathfrak{q})$-modules and in the case M is a $U(\mathfrak{g})$-module and N is a $U(\mathfrak{q})$-module. As $U(\mathfrak{q})$ is a finite extension of $U(\mathfrak{g})$ ([Ro]), in all cases we can impose the condition that the adjoint action of the Lie algebra $\mathfrak{g}$ on elements from $\mathcal{L}(M,N)$ is locally finite.

Fix $r \in \mathbb{N}$. Let $\mathtt{m}$ denote the maximal ideal of $U(\mathtt{h})$, generated by the elements $H - \hat{\lambda}(H)$, $H \in \mathtt{h}$. Consider the $U(\mathtt{h})$-module $U(\mathtt{h})/\mathtt{m}^r$. Let $\mathtt{n}_+$ act on $U(\mathtt{h})/\mathtt{m}^r$ trivially and construct the *thick Verma module* $M^r(\hat{\lambda})$ as follows (see [So]):

$$M^r(\hat{\lambda}) = U(\mathfrak{g}) \bigotimes_{U(\mathtt{h}\oplus\mathtt{n}_+)} U(\mathtt{h})/\mathtt{m}^r.$$

Since $\hat{\lambda}$ is regular, by [So] we have

(1) $\mathrm{Ann}_{U(\mathfrak{g})}(M^r(\hat{\lambda})) = U(\mathfrak{g})\mathtt{m}^r_{\hat{\chi}}, \quad \mathcal{L}(M^r(\hat{\lambda}), M^r(\hat{\lambda})) \cong U(\mathfrak{g})/U(\mathfrak{g})\mathtt{m}^r_{\hat{\chi}}.$

Our main result in this section is the following statement, which generalizes [Go2, 3.1.2] to our setup:

Proposition 4. *There is an isomorphism of $U(\mathfrak{q})-U(\mathfrak{g})$-bimodules as follows:*

$$U(\mathfrak{q})/U(\mathfrak{q})\mathtt{m}^r_{\hat{\chi}} \cong \mathcal{L}(M^r(\hat{\lambda}), \mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{q}} M^r(\hat{\lambda})).$$

Proof. Consider the homomorphism φ of $U(\mathfrak{q})-U(\mathfrak{g})$-bimodules, defined as follows: $\varphi : U(\mathfrak{q}) \to \mathcal{L}(M^r(\hat{\lambda}), \mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{q}} M^r(\hat{\lambda}))$, where $\varphi(u)(m) = u \otimes m$. By (1) we have $\mathrm{Ann}_{U(\mathfrak{g})}(M^r(\hat{\lambda})) = U(\mathfrak{g})\mathtt{m}^r_{\hat{\chi}}$ and hence the map φ induces a $U(\mathfrak{q})-U(\mathfrak{g})$-bimodule homomorphism

$$\overline{\varphi} : U(\mathfrak{q})/U(\mathfrak{q})\mathtt{m}^r_{\hat{\chi}} \to \mathcal{L}(M^r(\hat{\lambda}), \mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{q}} M^r(\hat{\lambda})).$$

Since $U(\mathfrak{q})$ is free over $U(\mathfrak{g})$, we conclude that $\overline{\varphi}$ is injective. Let us prove that $\overline{\varphi}$ is surjective.

By the PBW theorem we have $U(\mathfrak{q}) \cong \bigwedge \mathfrak{q}_{\overline{1}} \otimes U(\mathfrak{g})$. By Kostant separation theorem (see [Ko]), there is a submodule H of the adjoint $\mathfrak{g}$-module $U(\mathfrak{g})$ such that $U(\mathfrak{g}) \cong H \otimes Z(\mathfrak{g})$. This gives us the following isomorphism of adjoint $\mathfrak{g}$-modules:

$$U(\mathfrak{q})/U(\mathfrak{q})\mathtt{m}^r_{\hat{\chi}} \cong \bigwedge \mathfrak{q}_{\overline{1}} \otimes H \otimes Z(\mathfrak{g})/\mathtt{m}^r_{\hat{\chi}}.$$

On the other hand, since $\bigwedge \mathfrak{q}_{\overline{1}}$ is finite-dimensional, we also have the following isomorphism of adjoint $\mathfrak{g}$-modules:

$$\begin{aligned}
\mathcal{L}(M^r(\hat{\lambda}), \mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{q}} M^r(\hat{\lambda})) &\overset{\mathrm{PBW}}{=} \mathcal{L}(M^r(\hat{\lambda}), \bigwedge \mathfrak{q}_{\overline{1}} \otimes M^r(\hat{\lambda})) \\
\text{(by [Ja, 6.8])} &= \bigwedge \mathfrak{q}_{\overline{1}} \otimes \mathcal{L}(M^r(\hat{\lambda}), M^r(\hat{\lambda})) \\
\text{(by (1))} &= \bigwedge \mathfrak{q}_{\overline{1}} \otimes U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{\hat{\chi}}^r \\
&= \bigwedge \mathfrak{q}_{\overline{1}} \otimes H \otimes Z(\mathfrak{g})/\mathfrak{m}_{\hat{\chi}}^r.
\end{aligned}$$

The claim of the proposition follows. □

4. Proof of the main result

We will need the following lemma:

Lemma 5. *There exists $r \in \mathbb{N}$ such that $\mathfrak{m}_{\hat{\chi}}^r M = 0$ for any $M \in \mathcal{O}_{\hat{\chi}}^{\mathfrak{g}}$.*

Proof. The category $\mathcal{O}_{\hat{\chi}}^{\mathfrak{g}}$ is equivalent to the category of modules over some finite-dimensional algebra ([BGG]). Hence it has a projective generator Q of finite length, say r, and every object in $\mathcal{O}_{\hat{\chi}}^{\mathfrak{g}}$ is a quotient of a direct sum of some copies of Q. As $\mathfrak{m}_{\hat{\chi}}$ annihilates all $M(w \cdot \hat{\lambda})$, we have that $\mathfrak{m}_{\hat{\chi}}$ annihilates all simple objects and hence that $\mathfrak{m}_{\hat{\chi}}^r$ annihilates Q. The claim follows. □

Now we have to define thick Verma supermodule $\Delta(V^r(\lambda))$. Let $V^r(\lambda)$ denote the indecomposable projective cover of the simple object $V(\lambda)$ in the category $\mathfrak{F}_{\lambda}^r$ of all finite-dimensional $\mathfrak{h}$-supermodules, annihilated by $\mathfrak{m}^r$. Let $\mathfrak{n}_+$ act on $V^r(\lambda)$ trivially and define the *thick Verma supermodule* $\Delta(V^r(\lambda))$ as follows:

$$\Delta(V^r(\lambda)) = U(\mathfrak{q}) \bigotimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} V^r(\lambda).$$

Let $\hat{\mathfrak{F}}_{\lambda}^r$ denote the category of all finite-dimensional $\mathtt{h}$-modules, annihilated by $\mathfrak{m}^r$. Then $U(\mathtt{h})/\mathfrak{m}^r$ is the unique (up to isomorphism) indecomposable projective module in $\hat{\mathfrak{F}}_{\lambda}^r$.

Lemma 6. *We have* $\mathrm{proj}_{\chi} \circ \mathrm{Ind}_{\mathfrak{g}}^{\mathfrak{q}} M^r(\hat{\lambda}) \cong \mathbf{k}\Delta(V^r(\lambda))$*,*

Proof. By adjunction it is enough to show that the kernel of $\mathfrak{m}_{\hat{\chi}}^r$ on the $U(\mathfrak{g})$-module $\Delta(V^r(\lambda))_{\overline{0}}$ is isomorphic to $\mathbf{k}M^r(\hat{\lambda})$.

Lemma 7. *We have* $\mathrm{Res}_{\mathtt{h}}^{\mathfrak{h}} V^r(\lambda)_{\overline{0}} \cong \mathbf{k}\, U(\mathtt{h})/\mathfrak{m}^r$*.*

Proof. The restriction functor is left adjoint to the coinduction functor $\mathrm{Coind}_{\mathtt{h}}^{\mathfrak{h}} \cong \Pi^n \mathrm{Ind}_{\mathtt{h}}^{\mathfrak{h}}$, the latter being exact ([Fr, Proposition 22]). Hence

$\mathrm{Res}^{\mathfrak{h}}_{\mathtt{h}}$ maps $V^r(\lambda)$ to a projective module in $\hat{\mathfrak{F}}^r_\lambda$, that is $\mathrm{Res}^{\mathfrak{h}}_{\mathtt{h}} V^r(\lambda)_{\overline{0}} \cong k\, U(\mathtt{h})/\mathtt{m}^r$. Let $\Bbbk_\lambda$ denote the simple object in $\hat{\mathfrak{F}}^r_\lambda$. We have

$$\begin{aligned} k &= \dim \mathrm{Hom}_{\hat{\mathfrak{F}}^r_\lambda}(\mathrm{Res}^{\mathfrak{h}}_{\mathtt{h}} V^r(\lambda)_{\overline{0}}, \Bbbk_\lambda) \\ \text{(by adjunction)} &= \dim \mathrm{Hom}_{\mathfrak{F}^r_\lambda}(V^r(\lambda), \mathrm{Coind}^{\mathfrak{h}}_{\mathtt{h}} \Bbbk_\lambda) \\ \text{(by [Fr, Proposition 22])} &= \dim \mathrm{Hom}_{\mathfrak{F}^r_\lambda}(V^r(\lambda), \Pi^n \mathrm{Ind}^{\mathfrak{h}}_{\mathtt{h}} \Bbbk_\lambda) \\ \text{(by projectivity)} &= \mathrm{length}(\Pi^n \mathrm{Ind}^{\mathfrak{h}}_{\mathtt{h}} \Bbbk_\lambda) \\ \text{(by PBW)} &= \mathbf{k}. \end{aligned}$$

□

The claim of Lemma 6 follows from Lemma 7 by the same arguments as in the proof of Proposition 2. □

Proof of Theorem 1. Choose r as given by Lemma 5. Then the functor F is a direct summand (defined by the projection on $\tilde{\mathcal{O}}^{\mathfrak{q}}_\chi$) of the functor

$$U(\mathfrak{q})/U(\mathfrak{q})\mathtt{m}^r_{\hat{\chi}} \bigotimes_{U(\mathfrak{g})} {}_{-}.$$

By Proposition 4, the latter functor is isomorphic to the functor

$$\mathcal{L}(M^r(\hat{\lambda}), \mathrm{Ind}^{\mathfrak{q}}_{\mathfrak{g}} M^r(\hat{\lambda})) \bigotimes_{U(\mathfrak{g})} {}_{-}.$$

By Lemma 6 we have that the module $\mathrm{proj}_\chi \circ \mathrm{Ind}^{\mathfrak{q}}_{\mathfrak{g}} M^r(\hat{\lambda})$ decomposes into a direct sum of $\mathbf{k}$ copies of $\Delta(V^r(\lambda))$. Hence the additivity of the functor $\mathcal{L}(X, {}_{-})$ implies that the functor F decomposes into a direct sum of $\mathbf{k}$ copies of some functor F_1. By adjunction, the adjoint G decomposes into a direct sum of $\mathbf{k}$ copies of some functor G_1 such that $(\mathrm{F}_1, \mathrm{G}_1)$ forms an adjoint pair.

From Proposition 2 we have

$$\mathrm{F}_1 M(w \cdot \hat{\lambda}) \cong \Delta(V(w\lambda)), \quad \mathrm{G}_1 \Delta(V(w\lambda)) \cong M(w \cdot \hat{\lambda})$$

for all $w \in W$. As all Verma (super)modules are not annihilated by F_1 and G_1, respectively, it follows that the adjunction morphisms $\mathrm{Id}_{\mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}} \to \mathrm{G}_1\mathrm{F}_1$ and $\mathrm{F}_1\mathrm{G}_1 \to \mathrm{Id}_{\tilde{\mathcal{O}}^{\mathfrak{q}}_\chi}$ are nonzero. As the endomorphism ring of a Verma (super)module is trivial, it follows that these adjunction morphisms are isomorphisms.

As any simple object in both $\mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}$ and $\tilde{\mathcal{O}}^{\mathfrak{q}}_\chi$ is a unique quotient of some Verma (super)module, we conclude that the adjunction morphisms are isomorphisms, when evaluated on all simple objects. As any object in $\mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}$ and $\tilde{\mathcal{O}}^{\mathfrak{q}}_\chi$ has finite length, a standard induction on the length implies that the adjunction morphisms are isomorphisms of functors. This completes the proof. □

5. Concluding remarks

Remark 8. For the superalgebra $\mathfrak{pq}(n) = \mathfrak{q}(n)/(\mathtt{M}(\mathrm{Id}, 0))$, where Id is the identity matrix, the notion of a strongly typical weight coincides with that for $\mathfrak{q}(n)$, see [Go3, 10.4 and (35)]). The main result of the paper transfers to $\mathfrak{pq}(n)$ in a straightforward way for example using the following argument: By the definition, the element $\mathtt{M}(\mathrm{Id}, 0)$ acts as the scalar $\chi(\mathtt{M}(\mathrm{Id}, 0))$ on the block $\mathcal{O}^{\mathfrak{q}}_{\chi}$. When $\chi(\mathtt{M}(\mathrm{Id}, 0)) = 0$, we obtain that the block $\mathcal{O}^{\mathfrak{q}}_{\chi}$ coincides with the corresponding block for $\mathfrak{pq}(n)$.

Remark 9. For two other Q-type superalgebras, namely $\mathfrak{sq}(n) = \{\mathtt{M}(A, B) : \mathrm{tr}(B) = 0\}$ and $\mathfrak{psq}(n) = \mathfrak{sq}(n)/(\mathtt{M}(\mathrm{Id}, 0))$ the notion of a strongly typical weight is more complicated and is not given by linear conditions for $n > 2$ (see [Go3, 10.4 and (35)]). More precisely, for these two superalgebras a weight λ is *strongly typical* provided that

$$\sum_{i=1}^{n} \lambda_1 \cdots \lambda_{i-1}\lambda_{i+1} \cdots \lambda_n \neq 0. \tag{2}$$

From [Go3, A.3] it follows that for such strongly typical weights the corresponding simple $U(\mathfrak{h})$-supermodules (which are determined by simple supermodules over some Clifford algebra) are projective in the category of weight supermodules. Using this and following the proof of [Fr, Theorem 12] one shows that for strongly typical weights the corresponding blocks of the category $\mathcal{O}$ are described by quasi-hereditary algebras. Now one easily checks that the main result of the paper and all proofs transfer mutatis mutandis to both $\mathfrak{sq}(n)$ and $\mathfrak{psq}(n)$ with respect to the definition of strongly typical weights, given by (2).

Remark 10. To extend the main result to singular weights one has to develop the theory of Harish-Chandra bimodules for superalgebras in the similar way as done for Lie algebras in [BG].

Remark 11. Blocks which are typical but not strongly typical are described by properly stratified rather than by quasi-hereditary algebras (see [Fr]). Hence the results of this paper do not extend to such blocks. In [Fr, 3.9] it is conjectured that such blocks are tensor products of strongly typical blocks with $\Bbbk[x]/(x^2)$-mod.

Remark 12. As one of the referees pointed out, a natural question is whether the result can be extended to larger categories than $\mathcal{O}$.

References

[BG] J. Bernstein, S. Gelfand, Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. Compositio Math. **41** (1980), no. 2, 245–285.

[BGG] I. Bernstrein, I. M. Gelfand, S. I. Gelfand, A certain category of $\mathfrak{g}$-modules. Funkcional. Anal. i Prilozen. **10** (1976), no. 2, 1–8.

[Br] J. Brundan, Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra $\mathfrak{q}(n)$. Adv. Math. **182** (2004), no. 1, 28–77.

[CPS] E. Cline, B. Parshall and L. Scott, Finite dimensional algebras and highest weight categories, J. Reine Angew. Math. **391** (1988), 85–99.

[Dl] V. Dlab, Properly stratified algebras. C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 3, 191–196.

[DR] V. Dlab, C. Ringel, Quasi-hereditary algebras. Illinois J. Math. **33** (1989), no. 2, 280–291.

[Fr] A. Frisk, Typical blocks of the category $\mathcal{O}$ for the queer Lie superalgebra, J. Algebra Appl. **6** (2007), no. 5, 731–778.

[Go1] M. Gorelik, On the ghost centre of Lie superalgebras. Ann. Inst. Fourier (Grenoble) **50** (2000), no. 6, 1745–1764 (2001).

[Go2] M. Gorelik, Strongly typical representations of the basic classical Lie superalgebras. J. Amer. Math. Soc. **15** (2002), no. 1, 167–184.

[Go3] M. Gorelik, Shapovalov determinants of Q-type Lie superalgebras. IMRP Int. Math. Res. Pap. 2006, Art. ID 96895, 71 pp.

[Ja] J. C. Jantzen, Einhüllende Algebren halbeinfacher Lie-Algebren. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **3**. Springer-Verlag, Berlin, 1983.

[Ko] B. Kostant, Lie group representations on polynomial rings. Amer. J. Math. **85** (1963), 327–404.

[Pe] I. Penkov, Generic representations of classical Lie superalgebras and their localization. Monatsh. Math. **118** (1994), no. 3-4, 267–313

[PS] I. Penkov, V. Serganova, Representations of classical Lie superalgebras of type I. Indag. Math. (N.S.) **3** (1992), no. 4, 419–466.

[Ro] L. Ross, Representations of graded Lie algebras. Trans. Amer. Math. Soc. **120** (1965) 17–23.

[So] S. Soergel, Équivalences de certaines catégories de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), no. 15, 725–728.

Department of Mathematics, Uppsala University, Box 480, SE-75106, Uppsala, SWEDEN

frisken@math.uu.se
mazor@math.uu.se