

# Applications of the category of linear complexes of tilting modules associated with the category $\mathcal{O}$

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## Abstract

We use the category of linear complexes of tilting modules for the BGG category  $\mathcal{O}$ , associated with a semi-simple complex finite-dimensional Lie algebra  $\mathfrak{g}$ , to reprove in purely algebraic way several known results about  $\mathcal{O}$  obtained earlier by different authors using geometric methods. We also obtain several new results about the parabolic category  $\mathcal{O}(\mathfrak{p}, \Lambda)$ .

## 1 Introduction and preliminaries

Let  $\mathfrak{g}$  be a semi-simple complex finite-dimensional Lie algebra with a fixed triangular decomposition,  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ , and  $\pi$  be the corresponding basis of the root system of  $\mathfrak{g}$ . Let further  $\rho$  be the half of the sum of all positive roots and  $\mathbf{W}$  be the Weyl group of  $\mathfrak{g}$ . For  $w \in \mathbf{W}$  and  $\lambda \in \mathfrak{h}^*$  set  $w \cdot \lambda = w(\lambda + \rho) - \rho$ . Denote by  $w_0$  the longest element of  $\mathbf{W}$  and by  $\ell$  the length function on  $\mathbf{W}$ . We consider  $\mathbf{W}$  as a partially ordered set with respect to the Bruhat order  $<$  under the convention that the identity  $e$  is the smallest element.

Consider the principal block  $\mathcal{O}_0$  of the BGG category  $\mathcal{O}$  for  $\mathfrak{g}$ , associated with the triangular decomposition above (see [BGG, So1]). This category is a highest weight category in the sense of [CPS] and hence is equivalent to the module category of some finite-dimensional associative quasi-hereditary basic algebra, which we denote by  $\mathbf{A}$  (see [BGG, DR]). The Chevalley involution on  $\mathfrak{g}$  induces a natural involutive contravariant exact self-equivalence on  $\mathcal{O}$ , and hence on  $\mathbf{A}\text{-mod}$ , which we denote by  $\star$ . The isomorphism classes of simple modules in  $\mathcal{O}_0$  (and thus also in  $\mathbf{A}\text{-mod}$ ) are in a natural bijection with the elements of  $\mathbf{W}$  under the convention that  $e$  corresponds to the one-dimensional module in  $\mathcal{O}_0$ . For  $w \in \mathbf{W}$  we introduce the following notation:

- $L(w)$  is the simple  $\mathbf{A}$ -module, which corresponds to the simple highest weight module in  $\mathcal{O}$  with the highest weight  $w \cdot 0$  (see [Di, Chapter 7]);
- $\Delta(w)$  is the standard  $\mathbf{A}$ -module corresponding to  $L(w)$  (in  $\mathcal{O}$  the module  $\Delta(w)$  is the Verma module with the highest weight  $w \cdot 0$ , see [Di, Chapter 7]);
- $\nabla(w) = \Delta(w)^*$  is the corresponding costandard module for  $\mathbf{A}$  (the dual Verma module in  $\mathcal{O}$ );
- $P(w)$  is the indecomposable projective cover of  $L(w)$ ;
- $I(w) = P(w)^*$  is the indecomposable injective envelope of  $L(w)$ ;
- $T(w)$  is the indecomposable tilting  $\mathbf{A}$ -module, which corresponds to  $\Delta(w)$  (see [Ri]).

The principal result about the category  $\mathcal{O}$  is the so-called *Kazhdan-Lusztig* (or, simply, the KL-) Theorem, proved in [BB, BK], which describes the composition multiplicities of  $\Delta(w)$  in terms of the Kazhdan-Lusztig combinatorics of the Hecke algebra, see [KL]. This result and the geometric approach to  $\mathcal{O}$  have been used by Soergel in [So1] to show that  $\mathbf{A}$  is Koszul and even Koszul self-dual. This was further extended in [BGS], where it was shown that all associative algebras, associated with the blocks of  $\mathcal{O}$ , are Koszul and that the Koszul dual of the algebra of a singular block of  $\mathcal{O}$  is the algebra of the regular block of certain parabolic generalization of  $\mathcal{O}$ , introduced by Rocha-Caridi in [RC]. In [Ba] this result was even further extended to all parabolic-singular blocks. Both [BGS] and [Ba] use “heavy” geometric arguments.

The algebra  $\mathbf{A}$  can be given a nice combinatorial description via the coinvariant algebra  $\mathbf{C}$  associated with  $\mathbf{W}$ . In [So1] it is shown that  $\mathbf{C} \cong \text{End}_{\mathbf{A}}(P(w_0))$ ; that the module  $P(w_0)$ , viewed as a  $\mathbf{C}$ -module, can be described combinatorially; and that  $\mathbf{A} \cong \text{End}_{\mathbf{C}}(P(w_0)_{\mathbf{C}})$  (see also [KSX] for a completely different approach to the latter isomorphism). Furthermore, in [So1] it is shown that  $\text{End}_{\mathbf{A}}(P(w_0))$  is isomorphic to the quotient of the polynomial algebra over a homogeneous ideal, in particular, that it is  $\mathbb{Z}$ -graded (however the grading is not unique). If we let the generators of  $\text{End}_{\mathbf{A}}(P(w_0))$  to have degree 2, we obtain the grading, which coincides with the grading on  $\mathbf{C}$ , obtained via the interpretation of  $\mathbf{C}$  as the cohomology algebra of a certain flag manifold (see for example [Hi]). Further, the combinatorial construction of Soergel implies that  $P(w_0)$  is a  $\mathbb{Z}$ -graded  $\mathbf{C}$ -module, which makes  $\mathbf{A}$  into a  $\mathbb{Z}$ -graded algebra. In [BGS] it was shown that this grading is the Koszul grading on  $\mathbf{A}$  using some arguments from the “mixed” geometry.

The paper [St] initiates the study of the graded version of the category  $\mathcal{O}$  with respect to the “natural” grading described above. It is important that Soergel’s combinatorial description and hence the natural grading on  $\mathbf{A}$  do not depend on the KL-Theorem. In particular, in [St] the author obtains several results about the graded version of the category  $\mathcal{O}$ , which do not depend on the KL-Theorem. It seems that there is a hope that the graded approach might be a way to give an algebraic proof of the KL-Theorem. Because of Vogan’s formulation of the KL-Theorem, see [Vo] (see also Subsection 2.4 for the graded reformulation of this statement), and the graded description of the translation functors from [St], the KL-Theorem would follow if one would show that the natural grading on  $\mathbf{A}$  is positive in the sense that all non-zero components have non-negative degrees, and the radical of  $\mathbf{A}$  coincides with the sum of all components of positive degrees. However, so far it is not clear how to show this without using the KL-Theorem.

Let us now consider  $\mathbf{A}$  as a graded algebra with respect to the natural grading. Then it is easy to see that all simple, projective, injective, standard and costandard modules admit graded lifts. In [MO, Zh] it was shown that in this case all tilting modules admit graded lifts as well. In particular, the Ringel dual  $R(\mathbf{A})$  of  $\mathbf{A}$  (see [Ri]) is automatically graded. Fixing natural graded lifts of indecomposable tilting modules one can consider the category  $\mathcal{T}(\mathbf{A})$  of linear complexes of tilting modules for  $\mathbf{A}$ . In [MO] it was shown that  $\mathcal{T}(\mathbf{A})$  is equivalent to the category of locally finite-dimensional graded modules over the quadratic dual  $R(\mathbf{A})^!$  of  $R(\mathbf{A})$ . Using the Koszul self-duality of  $\mathbf{A}$  (see [So1]) and the Ringel self-duality of  $\mathbf{A}$  (see [So2]), one obtains that  $\mathcal{T}(\mathbf{A})$  is equivalent to the category of locally finite-dimensional graded  $\mathbf{A}$ -modules. In what follows we would like to derive several consequences from this fact.

In the present paper we use two “heavy” prerequisites. The first one is the KL-Theorem. Using the tilting module approach together with the KL-combinatorics (which follows from the KL-Theorem) we show in Section 2 that  $\mathbf{A}$  is Koszul and that the natural grading on  $\mathbf{A}$  is the Koszul one. However, we are not able to derive the Koszul self-duality for  $\mathbf{A}$  by our methods, hence we use this result of Soergel as the second “heavy” prerequisite. In Section 3 we give a brief synopsis about the category of linear complexes of tilting modules for a quasi-hereditary algebra, studied in [MO], and list some corollaries for the category  $\mathcal{O}_0$ . In Section 4 we show that the associative algebras, associated to all singular blocks of  $\mathcal{O}$  are Koszul. Already on this stage we need the Koszul self-duality for  $\mathbf{A}$ . We also use the machinery developed in [MO]. In Section 5 we recall the definition of the parabolic category  $\mathcal{O}_{\mathcal{S}}$  due to Rocha-Caridi, [RC], and show that all (regular and singular) blocks of this category are Koszul. This result happens to be a relatively easy

corollary from the corresponding result for the singular blocks of  $\mathcal{O}$ , which we prove earlier. It is possibly interesting to point out that the statement that the regular block of  $\mathcal{O}_S$  is Koszul does not require Soergel's Koszul self-duality result. In Section 6 we define a new properly stratified (in the sense of [Dl]) parabolic generalization of  $\mathcal{O}$  and  $\mathcal{O}_S$ . Roughly speaking, it is the translation of a singular block of  $\mathcal{O}$  out of the wall. We use this category and the corresponding category of linear complexes of tilting modules to reprove in Section 7 the parabolic-singular Koszul dualities from [BGS, Ba] in a purely algebraic way.

Apart from the KL-Theorem and Soergel's Koszul self-duality result, the main ingredients of this proof are:

- the way the composition of the Ringel and Koszul self-dualities on  $\mathbf{A}$  permute the primitive idempotents, see [So1, So2];
- the way the quadratic duality, defined via the linear complexes of tilting modules, works.

The simple modules in a singular block of  $\mathcal{O}$  are indexed by certain left cosets in  $\mathbf{W}$ . The simple modules in the corresponding parabolic category are indexed by the corresponding right cosets. The composition of the Ringel and Koszul self-dualities of  $\mathbf{A}$  switches the left and the right cosets in  $\mathbf{W}$  and, applied to a certain subcategory of  $\mathcal{I}(\mathbf{A})$ , gives the necessary duality. It is again perhaps interesting to note that for a finite-dimensional semi-simple Lie algebra  $\mathfrak{g}$  the Ringel self-duality of  $\mathcal{O}_0$  does not require the approach proposed in [So2], which is based on Arkhipov's twisting functor. Instead one may use a direct approach from [FKM], based on the translation functors, which substantially simplifies the argument.

The necessary preliminaries about the quadratic duality are collected in Section 3, where it is shown that this duality always switches the centralizer algebras, similar to those considered in Section 6, and the quotient algebras, similar to the blocks of  $\mathcal{O}_S$ .

Finally, we use  $\mathcal{I}(\mathbf{A})$  to study the properties of the parabolic generalization  $\mathcal{O}(\mathfrak{p}, \Lambda)$  of  $\mathcal{O}$ , considered in [FKM] (or, equivalently, using [BG], certain singular blocks in the category of Harish-Chandra bimodules for  $\mathfrak{g}$ ). The blocks of this category are equivalent to the module categories of certain properly stratified algebras, see [FKM]. Most of our results here are about the regular blocks. We show that the standard modules in the regular blocks have linear tilting coresolutions and that the costandard modules in the regular blocks have linear tilting resolutions. This can be proved in at least two different ways. The first way is analogous to that for  $\mathcal{O}_0$  and uses the properties of translation functors. The second way uses the existence of a

non-standard BGG-type resolutions for certain highest weight modules. We further compute the quadratic dual to the properly stratified algebra of the regular block of  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , which happens to be the parabolic quotient of  $\mathbf{A}$  similar to  $\mathcal{O}_S$ , but related to the left cosets instead of the right cosets. This is one more evidence for the strong asymmetry of  $\mathcal{O}$  with respect to the left and the right cosets of  $\mathbf{W}$  (the first one was obtained in [MS1]). However, this result gives a possibility to describe the lawyers of the tilting (co)resolutions of standard and costandard modules from  $\mathcal{O}(\mathfrak{p}, \Lambda)$  in terms of the Kazhdan-Lusztig combinatorics. We also derive several facts about the extensions between standard and proper standard modules in  $\mathcal{O}(\mathfrak{p}, \Lambda)$ .

## 2 Koszulity of the natural grading for the regular blocks

The aim of this section is to prove that the natural grading on  $\mathcal{O}_0$ , described in the introduction, is Koszul, in particular, that it is positive. Our approach is based on the results about the natural grading, obtained in [St], and on certain combinatorial consequences of the KL Theorem. In this section we do not use Soergel's Koszul self-duality of  $\mathbf{A}$ .

### 2.1 Kazhdan-Lusztig combinatorics

For  $w \in \mathbf{W}$  let  $\theta_w$  denote the indecomposable projective functor  $\theta_w : \mathcal{O}_0 \rightarrow \mathcal{O}_0$ , uniquely determined by the property  $\theta_w(\Delta(e)) \cong P(w)$ , see [BG, Theorem 3.3].  $\theta_w$  is exact and both left and right adjoint to  $\theta_{w^{-1}}$ . It is well-known, see for example [Ir, Corollary 5.2.4], that the KL Theorem implies the following combinatorial decomposition property for the indecomposable projective functors on  $\mathcal{O}_0$ : Let  $w \in \mathbf{W}$  be arbitrary and  $s \in \mathbf{W}$  be a simple reflection, then

$$\theta_s \theta_w \cong \begin{cases} \theta_w \oplus \theta_w, & ws < w \\ \theta_{ws} \oplus \bigoplus_{y < w, ys < y} \mu(y, w) \theta_y, & ws > w, \end{cases} \quad (2.1)$$

where  $\mu(y, w)$  is the *Kazhdan-Lusztig  $\mu$ -function* from [KL]. Evaluating (2.1) at  $\Delta(e)$  we obtain the following equivalent statement for  $w, s$  and  $\mu$  as above (see [Ir, Proposition 5.2.3]):

$$\theta_s P(w) \cong \begin{cases} P(w) \oplus P(w), & ws < w \\ P(ws) \oplus \bigoplus_{y < w, ys < y} \mu(y, w) P(y), & ws > w. \end{cases} \quad (2.2)$$

It is also well-known that  $\mu(y, w) = \dim \text{Ext}_{\mathbf{A}}(L(y), L(w))$  (this follows for example from the KL Theorem and [Vo, 3.7]).

## 2.2 Definition of the natural grading on $\mathcal{O}_0$

Recall that a quasi-hereditary algebra is called *standard Koszul*, [ADL], provided that it is positively graded, all standard modules admit linear projective resolutions (meaning that the  $l$ -th term of the resolution is generated in degree  $l$ ), and all costandard modules admit linear injective coresolutions (meaning that the  $l$ -th term of the coresolution is cogenerated in degree  $-l$ ). By [ADL, Theorem 1], every standard Koszul quasi-hereditary algebra is Koszul.

In all categories of graded modules and for all  $k \in \mathbb{Z}$  we denote by  $\langle k \rangle$  the functor of the shift of grading, which maps the degree  $l$  to the degree  $l - k$ . For categories of complexes and for all  $k \in \mathbb{Z}$  we denote by  $[k]$  the functor of the shift of the position in a complex, which maps the position  $l$  to the position  $l - k$ .

According to [So1], the  $\mathbf{C}$ -module  $P(w_0)$  admits the following decomposition:  $P(w_0)_{\mathbf{C}} = \bigoplus_{w \in \mathbf{W}} D(w)$ , with indecomposable modules  $D(w)$  constructed recursively as follows.  $D(e)$  is the simple  $\mathbf{C}$ -module, which we consider as the graded module, concentrated in degree 0. For a simple reflection,  $s \in \mathbf{W}$ , we denote by  $\mathbf{C}^s$  the subalgebra of  $s$ -invariants in  $\mathbf{C}$ . Let  $w \in \mathbf{W}$  and  $w = s_1 \dots s_k$  be a reduced decomposition of  $w$ . Then the module

$$\tilde{D}(w) = \mathbf{C} \otimes_{\mathbf{C}^{s_k}} \mathbf{C} \otimes_{\mathbf{C}^{s_{k-1}}} \cdots \otimes_{\mathbf{C}^{s_1}} D(e)\langle k \rangle$$

has one dimensional component of degree  $-k$ . The module  $D(w)$  is the indecomposable direct summand of  $\tilde{D}(w)$  such that  $D(w)_{-k} \neq 0$ . This fixes a grading on  $P(w_0)_{\mathbf{C}}$  and makes the algebra  $\mathbf{A} = \text{End}_{\mathbf{C}}(P(w_0)_{\mathbf{C}})$  into a graded algebra, see [So1]. We call this grading on  $\mathbf{A}$  *natural*. We denote by  $\mathbf{A}\text{-mod}$  and  $\mathbf{A}\text{-grmod}$  the categories of all finitely generated  $\mathbf{A}$ -modules and all finitely generated graded (with respect to the natural grading)  $\mathbf{A}$ -modules respectively. Remark that the morphisms in  $\mathbf{A}\text{-grmod}$  are homogeneous maps of degree 0. We set  $\text{ext}_{\mathbf{A}} = \text{Ext}_{\mathbf{A}\text{-grmod}}$ ,  $\text{hom}_{\mathbf{A}} = \text{Hom}_{\mathbf{A}\text{-grmod}}$ .

For standard graded lifts we will use the same symbol as for ungraded modules. We set  $L = \bigoplus_{w \in \mathbf{W}} L(w)$  and same for  $P, I, T, \Delta, \nabla$ . We concentrate  $L$  in degree 0 and fix a grading on  $P$  such that the natural surjection  $P \twoheadrightarrow L$  is a morphism in  $\mathbf{A}\text{-grmod}$ . Further, we fix a grading on  $I$  such that the natural inclusion  $L \hookrightarrow I$  is a morphism in  $\mathbf{A}\text{-grmod}$ . Then the natural maps  $P \twoheadrightarrow \Delta$  and  $\nabla \hookrightarrow I$  automatically induce gradings on  $\Delta$  and  $\nabla$ . Further, we fix a grading on  $T$  such that the natural inclusion  $\Delta \hookrightarrow T$  is a

morphism in  $\mathbf{A}\text{-grmod}$ . It follows automatically that the natural surjection  $T \rightarrow \nabla$  is a morphism in  $\mathbf{A}\text{-grmod}$ . We refer the reader to [MO, Section 5] for details.

By [St, Section 6] the duality  $\star$  lifts to a duality on  $\mathbf{A}\text{-grmod}$ , which we will denote by the same symbol. Note that  $\star$  acts on degrees via multiplication with  $-1$ . In [St, Subsection 3.2] it was shown that  $\theta_s$  admits a graded lift, that is it lifts to an exact and self-adjoint functor,  $\tilde{\theta}_s$ , on  $\mathbf{A}\text{-grmod}$ . It follows that  $\theta_w$  admits a graded lift,  $\tilde{\theta}_w$ , for every  $w \in \mathbf{W}$ .

### 2.3 Koszulity of the natural grading on $\mathcal{O}_0$

Here we are going to prove the following main result of the present section, first proved in [So1, Theorem 18] and [BGS, 4.5].

**Theorem 2.1.** *A is a standard Koszul quasi-hereditary algebra and the natural grading on  $\mathbf{A}$  is Koszul.*

The principal problem in proving this theorem is to show that that natural grading on  $\mathbf{A}$  is positive. Our strategy to prove this is the following: As a graded vector space we have  $\mathbf{A} = \text{hom}_{\mathbf{A}}(P, P)$ . We do not know whether  $\text{hom}_{\mathbf{A}}(P, P)$  is positively graded or not. The idea is to prove by induction on  $\ell(w)$  that  $P(w)$  is positively graded in the sense that it lives in non-negative degrees and its radical lives in positive degrees. For induction one would like to use (2.2), however, this is an ungraded statement. To solve the latter problem we use tilting modules. We show that  $\text{hom}_{\mathbf{A}}(P, P) = \text{hom}_{\mathbf{A}}(T, T)$  and obtain an ungraded tilting analogue for (2.2). The self-duality of tilting modules allows us to track the grading in this decomposition formula and to complete the induction in this way. We start with the following result, which is a graded version of [FKM, Proposition 4].

**Proposition 2.2.** (i) *Let  $w \in \mathbf{W}$  then there is an inclusion (of graded modules)  $T(w_0w)\langle -\ell(w_0) \rangle \hookrightarrow P(w)$ .*

(ii) *The restriction from  $P$  to  $T\langle -\ell(w_0) \rangle$  (the latter considered as a submodule of  $P$  via the inclusion constructed in (i)) induces an isomorphism,*

$$\mathbf{A} = \text{End}_{\mathbf{A}}(P) \cong \text{End}_{\mathbf{A}}(T) = R(\mathbf{A}),$$

*of graded algebras.*

*Proof.* We start with the ungraded version, proved in [FKM, Proposition 4]. Let  $w = s_1 \dots s_k$  be a reduced decomposition of  $w$ . Using the induction on  $\ell(w)$  one shows that applying  $\theta_{s_k} \dots \theta_{s_1}$  to the ungraded inclusion  $T(w_0) \hookrightarrow$

$P(e)$  induces the ungraded inclusion  $T(w_0w) \hookrightarrow P(w)$ . This proves the ungraded analogue of (i) and the ungraded analogue of (ii) follows from (i) using Enright's completion functor (or simply comparing the dimensions).

Obviously, the graded version of (ii) follows from the graded version of (i) and the ungraded version of (ii). Hence we are left to prove the graded version of (i). We start with

$$T(w_0)\langle -\mathfrak{l}(w_0) \rangle \hookrightarrow P(e). \quad (2.3)$$

Since  $T(w_0)$  is the simple socle of  $\Delta(x)$  for every  $x \in \mathbf{W}$  we certainly have  $T(w_0)\langle -k(x) \rangle \hookrightarrow \Delta(x)$  for some  $k(x)$ . In particular,  $k(w_0) = 0$  since  $T(w_0) \cong \Delta(w_0)$ . Using induction on  $\mathfrak{l}(x)$ , graded translation functors, and [St, Theorem 3.6] one shows that  $k(x) = \mathfrak{l}(w_0) - \mathfrak{l}(x)$ , in particular,  $k(e) = \mathfrak{l}(w_0)$ . Now (2.3) follows from the observation that  $P(e) \cong \Delta(e)$ .

The rest follows again by induction on  $\mathfrak{l}(w)$  applying  $\tilde{\theta}_{s_k} \dots \tilde{\theta}_{s_1}$  to (2.3) and using [St, Theorem 3.6].  $\square$

From (2.2) and the ungraded version of Proposition 2.2(i) we obtain the following tilting version of (2.2) (compare with [Ir, Proposition 5.2.6]): Let  $w \in \mathbf{W}$  be arbitrary and  $s \in \mathbf{W}$  be a simple reflection, then

$$\theta_s T(w) \cong \begin{cases} T(w) \oplus T(w), & ws > w \\ T(ws) \oplus \bigoplus_{y>w, ys>y} \mu(w, y) T(y), & ws < w. \end{cases} \quad (2.4)$$

We can now use (2.4) to prove the next statement, which is the key step in the proof of Theorem 2.1.

**Proposition 2.3.**  *$R(\mathbf{A})$  is a positively graded algebra.*

*Proof.* By definition (see [Ri, Section 5]) every indecomposable summand of  $T$  has both a standard and a costandard (graded) filtration, that is a filtration, whose subquotients are standard and costandard modules respectively. Moreover, every indecomposable summand is self-dual (with respect to  $\star$ ). Every (ungraded) morphism from  $T(x)$  to  $T(y)$ ,  $x, y \in \mathbf{W}$ , is a linear combination of morphisms, each of which is induced by a (unique up to a non-zero scalar) map from some subquotient of a standard filtration of  $T(x)$  to some subquotient of a costandard filtration of  $T(y)$ . This means that the statement of the proposition follows from the following lemma.

**Lemma 2.4.** *Let  $x \in \mathbf{W}$ . Then every subquotient of any standard filtration of  $T(x)$ , which is not isomorphic to  $\Delta(x)$ , has the form  $\Delta(y)\langle l \rangle$  with  $l > 0$ .*

*Proof.* We prove this by a downward induction on  $\mathfrak{l}(x)$  with the basis  $x = w_0$  being obvious. Let now  $x \in \mathbf{W}$  and  $s$  be a simple reflection such that  $\mathfrak{l}(x) > \mathfrak{l}(xs)$ . Consider the modules  $T(x)$  and  $\tilde{\theta}_s T(x)$ . Using [St, Theorem 3.6] and the inductive assumption we obtain that

- (a) every subquotient of any standard filtration of  $\tilde{\theta}_s T(x)$  has the form  $\Delta(y)\langle l \rangle$  with  $l \geq 0$ .

The question is when we can get  $l = 0$ ? First of all, again by [St, Theorem 3.6] we obtain that  $\Delta(xs)$  occurs as a subquotient of any standard filtration of  $\tilde{\theta}_s T(x)$ .

Fix now  $y \neq xs$  such that  $\Delta(y)$  occurs a subquotient of any standard filtration of  $\theta_s T(x)$ . Using [St, Theorem 3.6] and the inductive assumption, we get that every such occurrence comes from some occurrence of  $\Delta(y)\langle 1 \rangle$  as a subquotient of some standard filtration of  $T(x)$ . Moreover,  $ys > y$ . Let  $m_y$  denote the multiplicity of  $\Delta(y)\langle 1 \rangle$  in  $T(x)$ .

First we claim that  $m_y \leq \dim \operatorname{ext}_{\mathbf{A}}^1(L(y)\langle 1 \rangle, L(x))$ . Indeed, let  $f$  be a direct sum of all primitive idempotents of  $R(\mathbf{A})$ , which correspond to  $z \in \mathbf{W}$  such that  $\mathfrak{l}(z) \geq \mathfrak{l}(x)$ . By induction we can assume that the grading of  $fR(\mathbf{A})f$  is positive. Using Proposition 2.2 and the graded contravariant Ringel duality functor  $\operatorname{hom}_{\mathbf{A}}(-, \bigoplus_{l \in \mathbb{Z}} T\langle l \rangle)$ , the statement reduces to the analogous statement for projective modules. But for projective modules over positively graded quasi-hereditary algebras the corresponding inequality  $m_y \leq \dim \operatorname{ext}_{\mathbf{A}}^1(L(w_0x), L(w_0y)\langle -1 \rangle)$  is obvious.

From (2.4) and the interpretation of  $\mu(x, y)$ , given in Subsection 2.1, we have the following ungraded decomposition:

$$\theta_s T(x) \cong T(xs) \oplus \bigoplus_{y > x, ys > y} (\dim \operatorname{Ext}_{\mathbf{A}}^1(L(y), L(x))) T(y). \quad (2.5)$$

Let us recall that all tilting modules are self-dual (even as graded modules). This and (a) implies that (2.5) must be the case even in  $\mathbf{A}\text{-grmod}$  (otherwise any shifted direct summand must come with an isomorphic direct summand shifted in the opposite way, which would imply that there should occur some  $\Delta(u)\langle t \rangle$ ,  $t < 0$ , in any standard filtration of  $\tilde{\theta}_s T(x)$ , contradicting (a)). Further, since the  $s$ -translation of a standard filtration of  $T(x)$  gives rise to a standard filtration of  $\tilde{\theta}_s T(x)$  in a canonical way, it follows from [St, Theorem 3.6] that in  $\mathbf{A}\text{-grmod}$  the multiplicity of  $T(y)$  as a direct summand of  $\tilde{\theta}_s T(x)$  can not exceed  $m_y$ . Since

$$m_y \leq \dim \operatorname{ext}_{\mathbf{A}}^1(L(y)\langle 1 \rangle, L(x)) \leq \dim \operatorname{Ext}_{\mathbf{A}}^1(L(y), L(x)),$$

it follows that

$$m_y = \dim \operatorname{ext}_{\mathbf{A}}^1(L(y)\langle 1 \rangle, L(x)) = \dim \operatorname{Ext}_{\mathbf{A}}^1(L(y), L(x)), \quad (2.6)$$

which means that each occurrence of  $\Delta(y)$  as a subquotient of a standard filtration of  $\tilde{\theta}_s T(x)$  in fact comes from a direct summand of  $\tilde{\theta}_s T(x)$ , which is isomorphic to  $T(y)$ . This implies that every subquotient of any standard filtration of  $T(xs)$ , which is not isomorphic to  $\Delta(xs)$ , has the form  $\Delta(z)\langle l \rangle$  with  $l > 0$ , and completes the proof.  $\square$

$\square$

From the proof of Proposition 2.3 we obtain the following graded analogue of (2.4):

**Corollary 2.5.** *Let  $w \in \mathbf{W}$  and  $s$  be a simple reflection. Then*

$$\tilde{\theta}_s T(w) \cong \begin{cases} T(w)\langle 1 \rangle \oplus T(w)\langle -1 \rangle, & ws > w; \\ T(ws) \oplus \bigoplus_{y > w, ys > y} \mu(w, y)T(w), & ws < w. \end{cases}$$

*Proof.* The case  $ws > w$  follows directly from (2.4) and [St, Theorem 3.6]. The case  $ws < w$  was considered in the proof of Proposition 2.3.  $\square$

This allows us to obtain the following graded versions of the formulae from Subsection 2.1:

**Corollary 2.6.** *Let  $w \in \mathbf{W}$  and  $s$  be a simple reflection. Then we have*

$$\tilde{\theta}_s \tilde{\theta}_w \cong \begin{cases} \tilde{\theta}_w \langle 1 \rangle \oplus \tilde{\theta}_w \langle -1 \rangle, & ws < w \\ \tilde{\theta}_{ws} \oplus \bigoplus_{y < w, ys < y} \mu(y, w) \tilde{\theta}_y, & ws > w, \end{cases}$$

and

$$\tilde{\theta}_s P(w) \cong \begin{cases} P(w)\langle 1 \rangle \oplus P(w)\langle -1 \rangle, & ws < w \\ P(ws) \oplus \bigoplus_{y < w, ys < y} \mu(y, w)P(y), & ws > w. \end{cases}$$

*Proof.* This follows from Corollary 2.5, (2.1) and (2.2).  $\square$

Recall, that for a fixed graded module,  $M$ , over a graded algebra,  $A$ , the complex  $\mathcal{X}^\bullet$  is called *linear* (with respect to  $M$ ) provided that  $\mathcal{X}^i \in \text{add}(M\langle i \rangle)$  for all  $i \in \mathbb{Z}$ . Further, recall that the  $\mathbf{A}$ -module  $N$  is said to have a *linear tilting (co)resolution* if the tilting (co)resolution of  $N$  is a linear complex with respect to the graded lift of  $T$  fixed above.

**Proposition 2.7.** *The module  $\Delta$  admits a (finite) linear tilting coresolution and the module  $\nabla$  admits a (finite) linear tilting resolution.*

*Proof.* By duality it is enough to prove the statement for  $\Delta$ . It is further enough to show that  $\Delta(x)$  admits a linear tilting coresolution for every  $x \in \mathbf{W}$ . We show this by induction on  $l(w_0) - l(x)$ . The basis of the induction is obvious since  $\Delta(w_0)$  is a tilting module. Let  $x \in \mathbf{W}$  and  $s$  be a simple reflection such that  $xs < x$ . By induction we can assume that  $\Delta(x)$  has the linear tilting coresolution  $\mathcal{T}^\bullet(\Delta(x))$ . Applying  $\tilde{\theta}_s$  we obtain a tilting coresolution of  $\tilde{\theta}_s\Delta(x)$  and, since all tilting modules have costandard filtrations, the adjunction induces a morphism of complexes,  $\varphi^\bullet : \tilde{\theta}_s\mathcal{T}^\bullet(\Delta(x)) \rightarrow \mathcal{T}^\bullet(\Delta(x))\langle 1 \rangle$ , surjective on every component. Let  $\mathcal{Q}^\bullet$  denote the cone of  $\varphi^\bullet$  shifted by  $[-1]$ . Then  $\mathcal{Q}^\bullet$  is a tilting coresolution of  $\Delta(xs)$ .

We claim that, in the homotopy category, taking away all trivial direct summands of  $\mathcal{Q}^\bullet$ , that is direct summands of the form

$$\cdots \rightarrow 0 \rightarrow T(z) \xrightarrow{\text{id}} T(z) \rightarrow 0 \rightarrow \cdots,$$

the complex  $\mathcal{Q}^\bullet$  can be reduced to a linear complex,  $\overline{\mathcal{Q}}^\bullet$ , of tilting modules. Indeed, let us fix  $l \geq 0$  and let  $\mathcal{T}^l(\Delta(x)) \cong \bigoplus_{w \in \mathbf{W}} m_w T(w)\langle l \rangle$ . Then, by Corollary 2.5, we have

$$\tilde{\theta}_s\mathcal{T}^l(\Delta(x)) \cong \bigoplus_{ws > w} m_w T(w)\langle l+1 \rangle \oplus \bigoplus_{ws > w} m_w T(w)\langle l-1 \rangle \oplus X,$$

where  $X \in \text{add}(T\langle l \rangle)$ . It is easy to see that the morphism  $\varphi^\bullet$  induces an isomorphism between the corresponding direct summands  $T(w)\langle l+1 \rangle$  in  $\tilde{\theta}_s\mathcal{T}^l(\Delta(x))$  and  $\mathcal{T}^l(\Delta(x))\langle 1 \rangle$ , implying that  $\overline{\mathcal{Q}}^l \in \text{add}(T\langle l \rangle \oplus T\langle l-1 \rangle)$ .

On the other hand, let  $f^\bullet$  be the differential in  $\mathcal{T}^\bullet(\Delta(x))$ . Then each occurrence of  $T(w)\langle l \rangle$  in  $\mathcal{T}^l(\Delta(x))$  comes from some standard subquotient  $\Delta(w)\langle l \rangle$  in the cokernel of  $f^{l-2}$  and thus in  $\mathcal{T}^{l-1}(\Delta(x))$ . Hence, by Corollary 2.5, in the case  $ws > w$  it gives rise to an occurrence of  $T(w)\langle l-1 \rangle$  in  $\tilde{\theta}_s\mathcal{T}^{l-1}(\Delta(x))$ . Further, we obtain that  $\tilde{\theta}_s f^l$  induces a non-zero map and hence an isomorphism between the corresponding summands  $T(w)\langle l-1 \rangle$  in  $\tilde{\theta}_s\mathcal{T}^{l-1}(\Delta(x))$  and in  $\tilde{\theta}_s\mathcal{T}^l(\Delta(x))$ . This splits away and thus we obtain that  $\overline{\mathcal{Q}}^l \in \text{add}(T\langle l \rangle)$ , completing the proof.  $\square$

**Corollary 2.8.** *The module  $\Delta$  admits a (finite) linear projective resolution and the module  $\nabla$  admits a (finite) linear injective coresolution.*

*Proof.* This follows from Proposition 2.7 and the graded Ringel self-duality of  $\mathbf{A}$ .  $\square$

Now we are ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* From Proposition 2.3 we know that the induced grading on  $R(\mathbf{A})$  is positive. Hence the natural grading on  $\mathbf{A}$  is positive by Proposition 2.2(ii). By Corollary 2.8, every standard  $\mathbf{A}$ -module admits a linear projective resolution, and every costandard  $\mathbf{A}$ -module admits a linear injective coresolution. This means that  $\mathbf{A}$  is a standard Koszul quasi-hereditary algebra with respect to the underlying (natural) grading. The statement follows.  $\square$

## 2.4 The graded version of Vogan's conjecture

There are several known equivalent reformulations of the KL Conjecture. One of them is known as Vogan's conjecture, see [Vo], and can be formulated as follows: let  $w \in \mathbf{W}$  be arbitrary and  $s \in \mathbf{W}$  be a simple reflection such that  $sw < w$ , then the module  $\theta_s L(w)$  has Loewy length 3. Using the positivity of the natural grading on  $\mathbf{A}$  this can be reformulated as the following property of graded translation functors:

**Lemma 2.9.** *Let  $w \in \mathbf{W}$  be arbitrary and  $s \in \mathbf{W}$  be a simple reflection such that  $sw < w$ , then the module  $\tilde{\theta}_s L(w)$  is concentrated in degrees  $-1, 0$  and  $1$ , moreover, the degrees  $-1$  and  $1$  are one-dimensional and contain the simple top and the simple socle of  $\tilde{\theta}_s L(w)$  respectively.*

*Proof.* That the top of  $\tilde{\theta}_s L(w)$  is simple and lies in degree  $-1$ , and that the socle of  $\tilde{\theta}_s L(w)$  is simple and lies in degree  $1$  follows from [St, Section 8]. Positivity of the grading implies that the rest of  $\tilde{\theta}_s L(w)$  (which is exactly the Jantzen middle) must be concentrated in degree  $0$ .  $\square$

## 3 The category of linear complexes of tilting modules

### 3.1 Quadratic dual via linear complexes

Let  $A = \bigoplus_{i \geq 0} A_i$  be a basic positively graded algebra over some field  $\mathbb{k}$  with finite-dimensional graded components. Denote by  $A\text{-fgmod}$  the category of all graded  $A$ -modules with finite-dimensional graded components. Let us fix the natural gradings on simple and projective  $A$ -modules, which is induced from the positive grading on  $A$ . Let  $\mathcal{P}(A)$  denote the category, whose objects are all linear complexes of finitely generated projective  $A$ -modules, that is  $\mathcal{X}^\bullet \in \mathcal{P}(A)$  if and only if  $\mathcal{X}^l \in \text{add}({}_A A\langle l \rangle)$  for all  $l \in \mathbb{Z}$ , and morphism are all possible morphisms of complexes of graded modules. For an  $A$ -module,  $M$ , we denote by  $M^\bullet$  the complex satisfying  $M^0 = M$  and  $M^l = 0$  for all  $l \neq 0$ .

For a  $\mathbb{k}$ -vector space,  $V$ , we set  $V^* = \text{Hom}_{\mathbb{k}}(V, \mathbb{k})$ . For two  $\mathbb{k}$ -vector spaces,  $V$  and  $W$ , and for  $f \in \text{Hom}_{\mathbb{k}}(V, W)$  we denote by  $f^*$  the corresponding dual map from  $\text{Hom}_{\mathbb{k}}(W^*, V^*)$ . For finite-dimensional  $V$  and  $W$  we have an obvious canonical isomorphism,  $(V \otimes_{\mathbb{k}} W)^* \cong V^* \otimes_{\mathbb{k}} W^*$ .

Denote further by  $A^!$  the *quadratic dual* of  $A$ , defined as follows:  $A^! = A_0[A_1^*]/(\mu^*(A_2^*))$ , where  $\mu : A_1 \otimes_{A_0} A_1 \rightarrow A_2$  is the multiplication map. Note that  $A^!$  is quadratic by definition. If  $A$  is quadratic we have  $A \cong (A^!)^!$  canonically. Recall the following statement (see [MaSa, Theorem 2.4] or [MO, Theorem 7]):

**Theorem 3.1.** *There is an equivalence,  $F : A^! \text{-fgmod} \cong \mathcal{P}(A)$ .*

Let  $e \in A_0$  be an idempotent and let  $\mathcal{P}^e(A)$  denote the full subcategory of  $\mathcal{P}(A)$ , which consists of all  $\mathcal{X}^\bullet \in \mathcal{P}(A)$  such that  $\mathcal{X}^l \in \text{add}({}_A A e \langle l \rangle)$ . Note that there is a canonical bijection between the idempotents in  $A$  and  $A^!$  (since both  $A$  and  $A^!$  share the same semi-simple part  $A_0$  by definition). In particular, we can define  $B_e = A^!/(A^!(1-e)A^!)$ . In what follows we will need the following easy corollary from Theorem 3.1:

**Proposition 3.2.**  *$F$  induces the equivalence  $F^e : (eAe)^! \text{-fgmod} \cong \mathcal{P}^e(A)$ . In particular,  $(eAe)^! \cong B_e$  as graded algebras with respect to the induced gradings.*

*Proof.* The category of linear complexes of projective  $eAe$ -modules is isomorphic to  $\mathcal{P}^e(A)$  via the Auslander approximation functor, see [Au, Section 5]. This and Theorem 3.1 imply the first statement. Every projective object from  $\mathcal{P}^e(A)$  is by definition the maximal quotient of the corresponding projective object from  $\mathcal{P}(A)$ , which contains only simple subquotients of the form  $\text{add}({}_A A e^\bullet \langle l \rangle)$ . This gives the second statement and completes the proof.  $\square$

## 3.2 Application to the regular block of $\mathcal{O}$

Let  $\mathcal{T}(\mathbf{A})$  denote the category, whose objects are all bounded complexes  $\mathcal{X}^\bullet$  of graded  $\mathbf{A}$ -modules satisfying  $\mathcal{X}^l \in \text{add}(T \langle l \rangle)$  for all  $l \in \mathbb{Z}$  (such complexes are called *linear complexes of tilting modules*), and morphisms are all usual morphisms of complexes of graded modules, see [MO]. We also denote by  $\mathcal{P}(\mathbf{A})$  the category, whose objects are all bounded complexes  $\mathcal{X}^\bullet$  of graded  $\mathbf{A}$ -modules satisfying  $\mathcal{X}^l \in \text{add}(P \langle l \rangle)$  for all  $l \in \mathbb{Z}$ , and morphisms are all usual morphisms of complexes of graded modules (such complexes are called *linear complexes of projective modules*), see [MO].

**Theorem 3.3.** *(i) There is an equivalence,  $\mathcal{F} : \mathbf{A} \text{-fgmod} \cong \mathcal{T}(\mathbf{A})$ , which sends  $L(w)$  to  $T(w_0 w^{-1} w_0)^\bullet$  for every  $w \in \mathbf{W}$ .*

(ii) There is an equivalence,  $\mathcal{G} : \mathbf{A}\text{-fgmod} \cong \mathcal{P}(\mathbf{A})$ , which sends  $L(w)$  to  $P(w^{-1}w_0)^\bullet$  for every  $w \in \mathbf{W}$ .

*Proof.* By Theorem 3.1 there is an equivalence,  $\mathbf{A}^!\text{-fgmod} \cong \mathcal{P}(\mathbf{A})$ . Applying the graded version of the Ringel duality from [So2] gives an equivalence,  $R(\mathbf{A})^!\text{-fgmod} \cong \mathcal{S}(\mathbf{A})$ . Because of Proposition 2.2 we even obtain an equivalence,  $\mathbf{A}^!\text{-fgmod} \cong \mathcal{S}(\mathbf{A})$ . However,  $\mathbf{A}$  is Koszul and even Koszul self-dual by [So1, Theorem 18] and the natural grading on  $\mathbf{A}$  is the Koszul one by Theorem 2.1. Thus  $\mathbf{A}^! \cong \mathbf{A}$  by [BGS, 2.9], which proves the existence of both  $\mathcal{F}$  and  $\mathcal{G}$ . The correspondence on simple objects follows by a direct calculation from Proposition 2.2(i) and [So1, 3.5].  $\square$

## 4 Singular blocks of $\mathcal{O}$ and their Koszulity

Let  $\mathbf{G} \subset \mathbf{W}$  be a parabolic subgroup and  $w_0^{\mathbf{G}}$  be the longest element in  $\mathbf{G}$ . Denote by  $\mathbf{W}_{\mathbf{G}}$  the set of the longest coset representatives in  $\mathbf{W}/\mathbf{G}$ . Let  $\lambda \in \mathfrak{h}^*$  be a dominant integral (singular) weight with stabilizer  $\mathbf{G}$ . Denote by  $\mathcal{O}_\lambda$  the singular block of  $\mathcal{O}$ , which corresponds to  $\lambda$ , and by  $\mathbf{A}_{\mathbf{G}}$  the corresponding basic associative algebra. The bijection between the simple  $\mathbf{A}$ -modules and the elements of  $\mathbf{W}$ , described in the introduction, induces a bijection between the isomorphism classes of simple  $\mathbf{A}_{\mathbf{G}}$ -modules and the cosets from  $\mathbf{W}/\mathbf{G}$ , and, also, the elements of  $\mathbf{W}_{\mathbf{G}}$ . For  $\mathbf{A}_{\mathbf{G}}$ -modules we will use notation  $L_{\mathbf{G}}(w)$ ,  $w \in \mathbf{W}_{\mathbf{G}}$ , etc. Soergel's combinatorics from [So1] equips  $\mathbf{A}_{\mathbf{G}}$  with a natural grading in the following way. The algebra  $\text{End}_{\mathbf{A}_{\mathbf{G}}}(P_{\mathbf{G}}(w_0))$  is the subalgebra  $\mathbf{C}_{\mathbf{G}}$  of  $\mathbf{G}$ -invariants in  $\mathbf{C}$ , in particular, is graded. Moreover, the module  $P_{\mathbf{G}}(w_0)_{\mathbf{C}_{\mathbf{G}}}$  is a graded module. This induces a grading on  $\mathbf{A}_{\mathbf{G}} \cong \text{End}_{\mathbf{C}_{\mathbf{G}}}(P_{\mathbf{G}}(w_0)_{\mathbf{C}_{\mathbf{G}}})$ , which we call *natural*. Let  $\theta_{\mathbf{G}}^{\text{on}} : \mathcal{O}_0 \rightarrow \mathcal{O}_\lambda$  and  $\theta_{\mathbf{G}}^{\text{out}} : \mathcal{O}_\lambda \rightarrow \mathcal{O}_0$  denote the functors of translations onto and out of the  $\mathbf{G}$ -wall respectively. These functors are left and right adjoint to each other. Set  $\theta_{\mathbf{G}} = \theta_{\mathbf{G}}^{\text{out}}\theta_{\mathbf{G}}^{\text{on}}$ . Denote by  $\tilde{\theta}_{\mathbf{G}}^{\text{out}}$ ,  $\tilde{\theta}_{\mathbf{G}}^{\text{on}}$  and  $\tilde{\theta}_{\mathbf{G}}$  the corresponding graded lifts, existence of which follows from [St, Section 8]. Using [St, Theorem 8.4] one shows that  $(\tilde{\theta}_{\mathbf{G}}^{\text{out}}, \tilde{\theta}_{\mathbf{G}}^{\text{on}}\langle -\mathfrak{l}(w_0^{\mathbf{G}}) \rangle)$  and  $(\tilde{\theta}_{\mathbf{G}}^{\text{on}}, \tilde{\theta}_{\mathbf{G}}^{\text{out}}\langle \mathfrak{l}(w_0^{\mathbf{G}}) \rangle)$  are adjoint pairs of functors. In the same way as it is done in Section 2 we fix the graded lifts of simple, projective, injective, standard, costandard and tilting  $\mathbf{A}_{\mathbf{G}}$ -modules and will use for them analogous notation (for example  $L_{\mathbf{G}}(w)$  etc.). The main result of the present section is the following statement.

**Theorem 4.1.** *The algebra  $\mathbf{A}_{\mathbf{G}}$  is a standard Koszul quasi-hereditary algebra, and the natural grading on  $\mathbf{A}_{\mathbf{G}}$  is the Koszul one.*

Again, to prove this theorem we will need some preparation.

**Lemma 4.2.** *The natural grading on  $\mathbf{A}_{\mathbf{G}}$  is positive.*

*Proof.* Let  $x \in \mathbf{W}_{\mathbf{G}}$  and  $y \in \mathbf{W}$ . Then the adjointness of  $\theta_{\mathbf{G}}^{out}$  and  $\theta_{\mathbf{G}}^{on}$  gives

$$\mathrm{Hom}_{\mathbf{A}}(\theta_{\mathbf{G}}^{out} P_{\mathbf{G}}(x), L(y)) \cong \mathrm{Hom}_{\mathbf{A}_{\mathbf{G}}}(P_{\mathbf{G}}(x), \theta_{\mathbf{G}}^{on} L(y)). \quad (4.1)$$

However, since  $\theta_{\mathbf{G}}^{on}$  sends simple  $\mathbf{A}$ -modules to simple  $\mathbf{A}_{\mathbf{G}}$ -modules or zero, it follows that for a fixed  $y$  the right-hand side of (4.1) is either 0 for all  $x$ , or is 0 for all  $x$  except one, for which it is equal to  $\mathbb{C}$ . This implies that the projective module  $\theta_{\mathbf{G}}^{out} P_{\mathbf{G}}(x)$  is in fact indecomposable. This means that  $\tilde{\theta}_{\mathbf{G}}^{out}$  induces a homogeneous monomorphism from  $\mathbf{A}_{\mathbf{G}}$  to  $\mathbf{A}$  (note that the unit is not sent to unit under this homomorphism). In particular, it follows that the algebra  $\mathbf{A}_{\mathbf{G}}$  is a graded subalgebra of  $\mathbf{A}$  and hence Theorem 2.1 implies that  $\mathbf{A}_{\mathbf{G}}$  is positively graded.  $\square$

**Proposition 4.3.** *The module  $\tilde{\theta}_{\mathbf{G}}^{out} \Delta_{\mathbf{G}}(w) \langle \mathfrak{l}(w_0^{\mathbf{G}}) \rangle$ ,  $w \in \mathbf{W}_{\mathbf{G}}$ , admits a linear tilting coresolution in  $\mathbf{A}\text{-grmod}$ .*

*Proof.* Throughout the proof we fix  $w \in \mathbf{W}_{\mathbf{G}}$ . For  $i = 0, 1, \dots, \mathfrak{l}(w_0^{\mathbf{G}})$  let

$$\begin{aligned} W_i^w &= \{x \in w_0 \mathbf{G} w^{-1} w_0 : \mathfrak{l}(x) = \mathfrak{l}(w^{-1}) - \mathfrak{l}(w_0^{\mathbf{G}}) + i\}, \\ \overline{W}_i^w &= \{x \in w \mathbf{G} : \mathfrak{l}(x) = \mathfrak{l}(w) - \mathfrak{l}(w_0^{\mathbf{G}}) + i\} \end{aligned}$$

(in particular,  $W_0^w = \{w_0 w_0^{\mathbf{G}} w^{-1} w_0\}$  and  $\overline{W}_0^w = \{w w_0^{\mathbf{G}}\}$ ). For  $w \in \mathbf{W}$  we denote by  $\mathcal{T}^\bullet(\Delta(w))$  the linear tilting resolution of  $\Delta(w)$ .

For  $i = 0, 1, \dots, \mathfrak{l}(w_0^{\mathbf{G}})$  let  $X_i$  denote the (non-graded) trace in  $\theta_{\mathbf{G}}^{out} \Delta_{\mathbf{G}}(w)$  of all  $P(x)$  such that there is  $y \in \overline{W}_i^w$  satisfying  $y \geq x$ . We consider the trace for non-graded modules, however, since both  $\tilde{\theta}_{\mathbf{G}}^{out} \Delta_{\mathbf{G}}(w)$  and  $P(x)$  are graded, the trace itself will be a graded submodule of  $\tilde{\theta}_{\mathbf{G}}^{out} \Delta_{\mathbf{G}}(w)$ , see [MO, Lemma 4]. Set  $Y_i = (\theta_{\mathbf{G}}^{out} \Delta_{\mathbf{G}}(w)) / X_i$ . In particular, we have  $X_0 \cong \Delta(w w_0^{\mathbf{G}})$  and  $Y_{\mathfrak{l}(w_0^{\mathbf{G}})-1} \cong \Delta(w)$ .

**Lemma 4.4.** *For all  $x \in w \mathbf{G}$  and  $k > 0$  we have the following equality:  $\mathrm{Ext}_{\mathbf{A}}^k(\theta_{\mathbf{G}}^{out} \Delta_{\mathbf{G}}(w), \Delta(x)) = 0$ .*

*Proof.* The functors  $\theta_{\mathbf{G}}^{out}$ ,  $\theta_{\mathbf{G}}^{on}$  are exact, left and right adjoint to each other, and send projectives to projectives. Hence

$$\begin{aligned} \mathrm{Ext}_{\mathbf{A}}^k(\theta_{\mathbf{G}}^{out} \Delta_{\mathbf{G}}(w), \Delta(x)) &= \mathrm{Ext}_{\mathbf{A}}^k(\Delta_{\mathbf{G}}(w), \theta_{\mathbf{G}}^{on} \Delta(x)) = \\ &= \mathrm{Ext}_{\mathbf{A}}^k(\Delta_{\mathbf{G}}(w), \Delta_{\mathbf{G}}(w)) = 0 \end{aligned}$$

since Verma modules do not have self-extensions in  $\mathcal{O}$ .  $\square$

**Lemma 4.5.**  *$\mathrm{Ext}_{\mathbf{A}}^1(Y_i, \Delta(x)) = \mathbb{C}$  for  $i \in \{0, 1, \dots, \mathfrak{l}(w_0^{\mathbf{G}}) - 1\}$  and  $x \in \overline{W}_i^w$ .*

*Proof.* Applying  $\text{Hom}_{\mathbf{A}}(-, \Delta(x))$  to the short exact sequence

$$X_i \hookrightarrow \theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}(w) \rightarrow Y_i$$

we obtain the following fragment in the long exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathbf{A}}(Y_i, \Delta(x)) &\xrightarrow{\sim} \text{Hom}_{\mathbf{A}}(\theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}(w), \Delta(x)) \rightarrow \\ &\rightarrow \text{Hom}_{\mathbf{A}}(X_i, \Delta(x)) \rightarrow \text{Ext}_{\mathbf{A}}^1(Y_i, \Delta(x)) \rightarrow \text{Ext}_{\mathbf{A}}^1(\theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}(w), \Delta(x)). \end{aligned} \quad (4.2)$$

Since  $Y_i$  surjects, by the definition, onto  $\Delta(w)$ , which, in turn, is a submodule of  $\Delta(x)$  by [Di, 7.7.7], we get  $\text{Hom}_{\mathbf{A}}(Y_i, \Delta(x)) \neq 0$ . Using the adjointness of  $\theta_{\mathbf{G}}^{\text{out}}$  and  $\theta_{\mathbf{G}}^{\text{in}}$ , we obtain

$$\begin{aligned} \text{Hom}_{\mathbf{A}}(\theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}(w), \Delta(x)) &\cong \text{Hom}_{\mathbf{A}}(\Delta_{\mathbf{G}}(w), \theta_{\mathbf{G}}^{\text{in}} \Delta(x)) \cong \\ &\cong \text{Hom}_{\mathbf{A}}(\Delta_{\mathbf{G}}(w), \Delta_{\mathbf{G}}(w)) \cong \mathbb{C}, \end{aligned}$$

which establishes the isomorphism indicated in (4.2). Further, the only standard subquotient of  $X_i$ , which maps to  $\Delta(x)$  is  $\Delta(x)$  itself. This implies  $\text{Hom}_{\mathbf{A}}(X_i, \Delta(x)) \cong \mathbb{C}$ . Finally,  $\text{Ext}_{\mathbf{A}}^1(\theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}(w), \Delta(x)) = 0$  by Lemma 4.4. The exactness of (4.2) now gives the necessary statement.  $\square$

Lemma 4.5 implies that the modules  $Y_i$  can be constructed recursively starting from  $i = \mathfrak{l}(w_0^{\mathbf{G}})$  and descending to  $i = 0$ . On every step of this recursion one uses the universal extension procedure as follows: we start with  $Y_{\mathfrak{l}(w_0^{\mathbf{G}})} \cong \Delta(w)$  and proceed, involving the module  $\bigoplus_{x \in \overline{W}_i^w} \Delta(x)$  on the

step  $\mathfrak{l}(w_0^{\mathbf{G}}) - i$ . Note that non-isomorphic direct summands of the module  $\bigoplus_{x \in \overline{W}_i^w} \Delta(x)$  do not have self-extensions, see for example [RC, Section 9].

Let  $\mathfrak{a}$  be the semi-simple Lie subalgebra of  $\mathfrak{g}$ , associated with  $\mathbf{G}$ . Applying the parabolic induction to the classical BGG-resolution of a finite-dimensional  $\mathfrak{a}$ -module gives the following resolution,

$$0 \rightarrow \bigoplus_{x \in W_{\mathfrak{l}(w_0^{\mathbf{G}})}^w} \Delta(x) \rightarrow \cdots \rightarrow \bigoplus_{x \in W_1^w} \Delta(x) \rightarrow \Delta(w_0 w_0^{\mathbf{G}} w^{-1} w_0) \rightarrow 0, \quad (4.3)$$

of the generalized Verma module, which is the cokernel of the last non-zero map in (4.3). By [MO, Theorem 8], the sequence (4.3) induces, via the equivalence  $\mathcal{F}$  from Theorem 3.3, the following complex of elements from  $\mathcal{T}(\mathbf{A})$ :

$$\begin{aligned} 0 \rightarrow \bigoplus_{x \in \overline{W}_{\mathfrak{l}(w_0^{\mathbf{G}})}^w} \mathcal{T}^\bullet(\Delta(x))[-\mathfrak{l}(w_0^{\mathbf{G}})] &\rightarrow \cdots \rightarrow \bigoplus_{x \in \overline{W}_1^w} \mathcal{T}^\bullet(\Delta(x))[-1] \rightarrow \\ &\rightarrow \mathcal{T}^\bullet(\Delta(w w_0^{\mathbf{G}})) \rightarrow 0. \end{aligned} \quad (4.4)$$

The sequence (4.4) is exact in all terms except the last one. This allows us to take inductively the cone of all morphisms in (4.4) and, moreover, implies that on every step the complex of tilting modules we obtain is isomorphic to a linear complex of tilting modules. It is easy to see that the homology of the complex, obtained on every step, is concentrated in a single position (in particular, in position 0 on the last step). Let us denote this homology by  $Z_i$ ,  $i = \mathfrak{l}(w_0^{\mathbf{G}}), \mathfrak{l}(w_0^{\mathbf{G}}) - 1, \dots$

We show, by induction on  $i$ , that  $Z_i \cong Y_{i-1}$ . This is obvious for  $i = \mathfrak{l}(w_0^{\mathbf{G}})$ . Note that the restriction of all differentials in the BGG resolution to Verma modules are non-zero (see for example [RC, Section 10]). Let us now assume that  $Z_i \cong Y_{i-1}$ . In particular, we have  $\text{Ext}_{\mathbf{A}}^1(Z_i, \Delta(x)) = \mathbb{C}$  for every  $x \in \overline{W}_{i-1}^w$  by Lemma 4.5. The cone construction of  $Z_{i-1}$  now implies that  $Z_{i-1}$  is obtained from  $Z_i$  via the universal extension with  $\bigoplus_{x \in \overline{W}_{i-1}^w} \Delta(x)$ . By

the universality of this extension we obtain that  $Z_{i-1}$  must be isomorphic to  $Y_{i-2}$ .

In particular,  $Z_0 \cong \tilde{\theta}_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}(w) \langle \mathfrak{l}(w_0^{\mathbf{G}}) \rangle$  and this module has the linear tilting coresolution as prove above.  $\square$

**Lemma 4.6.** (i) *Let  $w \in \mathbf{W}_{\mathbf{G}}$ . Then the (ungraded) trace  $Tr_{\mathbf{G}}(w)$  of  $P_{\mathbf{G}}(w_0)$  in  $P_{\mathbf{G}}(w)$  is an indecomposable tilting  $\mathbf{A}_{\mathbf{G}}$ -module. Moreover, for  $w \neq w'$ ,  $w' \in \mathbf{W}_{\mathbf{G}}$ , we have  $Tr_{\mathbf{G}}(w) \not\cong Tr_{\mathbf{G}}(w')$ .*

(ii) *Restriction from  $P_{\mathbf{G}}$  to the trace of  $P_{\mathbf{G}}(w_0)$  in  $P_{\mathbf{G}}$  establishes the Ringel self-duality of  $\mathbf{A}_{\mathbf{G}}$ .*

*Proof.* From Proposition 2.2 it follows that  $T(w_0 w)$  is the trace of  $P(w_0)$  in  $P(w)$ .  $\theta_{\mathbf{G}}^{\text{on}}$  sends  $L(w_0)$  to  $L_{\mathbf{G}}(w_0)$ , which implies that the trace of  $P(w_0)$  in  $P(w)$  is mapped to the trace of  $\theta_{\mathbf{G}}^{\text{on}} P(w_0)$  in  $\theta_{\mathbf{G}}^{\text{on}} P(w)$ . This and the fact that  $\theta_{\mathbf{G}}^{\text{on}}$  sends tilting modules to tilting modules implies that  $Tr_{\mathbf{G}}(w)$  is a tilting module. On the other hand, from [KSX, 3.1] we have that every  $P_{\mathbf{G}}(w)$  has a two-step copresentation by modules from  $\text{add}(P_{\mathbf{G}}(w_0))$ . This implies that the Auslander  $P_{\mathbf{G}}(w_0)$ -coapproximation of  $Tr_{\mathbf{G}}(w)$ , see [Au, Section 5], is isomorphic to  $P_{\mathbf{G}}(w)$ . Using the same arguments as in [FKM, Theorem 4.1] we obtain that the restriction defines an isomorphism between the algebras  $\text{End}_{\mathbf{A}_{\mathbf{G}}}(P_{\mathbf{G}})$  and  $\text{End}_{\mathbf{A}_{\mathbf{G}}}(\bigoplus_{w \in \mathbf{W}_{\mathbf{G}}} Tr_{\mathbf{G}}(w))$ . Both (i) and (ii) follow.  $\square$

*Proof of Theorem 4.1.* Observe that  $\theta_{\mathbf{G}}^{\text{out}}$  sends tilting modules from  $\mathcal{O}_{\lambda}$  to tilting modules from  $\mathcal{O}_0$ , and projective modules from  $\mathcal{O}_{\lambda}$  to projective modules from  $\mathcal{O}_0$ . Adjointness with  $\theta_{\mathbf{G}}^{\text{on}}$  implies that  $\theta_{\mathbf{G}}^{\text{out}}$  sends indecomposable projective modules to indecomposable projective modules. This and Lemma 4.6 gives that  $\theta_{\mathbf{G}}^{\text{out}}$  even sends indecomposable tilting modules to

indecomposable tilting modules. In particular, for every  $w \in \mathbf{W}_{\mathbf{G}}$  the minimal tilting coresolution of  $\Delta_{\mathbf{G}}(w)$  is sent to a minimal tilting coresolution of  $\theta_{\mathbf{G}}^{out} \Delta_{\mathbf{G}}(w)$ . The latter one may, however, be chosen linear (up to a shift of grading) by Proposition 4.3. Applying  $\tilde{\theta}_{\mathbf{G}}^{out}$  we obtain that the original tilting coresolution of  $\Delta_{\mathbf{G}}(w)$  was linear. Using  $\star$  we also have that the tilting resolution of  $\nabla_{\mathbf{G}}(w)$  is linear as well. From [MO, Theorem 6] we now derive that  $\mathbf{A}_{\mathbf{G}}$  is standard Koszul with respect to the natural grading. The statement of the theorem follows.  $\square$

## 5 The parabolic category of Rocha-Caridi and its Koszulity

Let now  $\mathbf{G}$  and  $\mathbf{H}$  be two parabolic subgroups of  $\mathbf{W}$ , and  $\lambda$  and  $\mathcal{O}_{\lambda}$  be as in Section 4. Let further  $\mathbf{W}_{\mathbf{G}}^{\mathbf{H}}$  denote the set of all  $w \in \mathbf{W}$ , which are the longest coset representatives in  $\mathbf{W}/\mathbf{G}$  and the shortest coset representatives in  $\mathbf{H} \setminus \mathbf{W}$  at the same time. Denote by  $f_{\mathbf{G}}^{\mathbf{H}}$  the sum of all primitive idempotents of  $\mathbf{A}_{\mathbf{G}}$ , which correspond to  $w \in \mathbf{W}_{\mathbf{G}}^{\mathbf{H}}$ , and set  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}} = \mathbf{A}_{\mathbf{G}} / (\mathbf{A}_{\mathbf{G}}(1 - f_{\mathbf{G}}^{\mathbf{H}})\mathbf{A}_{\mathbf{G}})$ . Then the algebra  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  is the associative algebra of Rocha-Caridi's parabolic subcategory of  $\mathcal{O}_{\lambda}$ , associated with  $\mathbf{H}$ . The latter category is the full subcategory of  $\mathcal{O}_{\lambda}$ , which consists of all modules that are locally finite with respect to the parabolic subalgebra  $\mathfrak{p}$  of  $\mathfrak{g}$ , associated with  $\mathbf{H}$ . Let  $\mathfrak{a}$  be the semi-simple part of  $\mathfrak{p}$ . The natural grading on  $\mathbf{A}_{\mathbf{G}}$  induces a grading on  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  in an obvious way. We call the later grading again *natural*. The  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules will be denoted  $L_{\mathbf{G}}^{\mathbf{H}}(w)$  etc. The main result of this section is the following statement.

**Theorem 5.1.** *The algebra  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  is a standard Koszul quasi-hereditary algebra and the natural grading on  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  is the Koszul one.*

*Proof.* It is clear that the natural grading on  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  is positive because the natural grading on  $\mathbf{A}_{\mathbf{G}}$  is positive. Hence, to complete the proof it would suffice to show that standard  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules admit linear tilting coresolution and that costandard  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules admit linear tilting resolution and apply [MO, Theorem 6].

To show this we first forget about the grading for a moment. Let  $Z_{\mathbf{H}} : \mathbf{A}_{\mathbf{G}}\text{-mod} \rightarrow \mathbf{A}_{\mathbf{G}}^{\mathbf{H}}\text{-mod}$  denote Zuckerman's functor of taking the maximal subquotient, which belongs to  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}\text{-mod}$ . Let  $x \in \mathbf{W}_{\mathbf{G}}^{\mathbf{H}}$  and

$$0 \rightarrow P_k \rightarrow \cdots \rightarrow P_0 \rightarrow \Delta_{\mathbf{G}}(x) \rightarrow 0 \quad (5.1)$$

be a projective resolution of  $\Delta_{\mathbf{G}}(x) \in \mathbf{A}_{\mathbf{G}}\text{-mod}$ . Let us look at (5.1) inside  $\mathcal{O}_{\lambda}$  instead of  $\mathbf{A}_{\mathbf{G}}\text{-mod}$  keeping the notation. Since every element from  $\mathbf{W}_{\mathbf{G}}^{\mathbf{H}}$

corresponds to an  $\mathfrak{a}$ -dominant weight by definition, the corresponding Verma module  $\Delta_{\mathbf{G}}(x)$  is obtained using the parabolic induction from some dominant Verma  $\mathfrak{a}$ -modules. However, the dominant Verma module is projective in the category  $\mathcal{O}$  for the algebra  $\mathfrak{a}$  (we denote this category by  $\mathcal{O}(\mathfrak{a})$ ). Since  $U(\mathfrak{g})$ , considered as an  $\mathfrak{a}$ -module under the adjoint action, is a direct sum of finite-dimensional modules, the parabolic induction reduces (on the level of  $\mathfrak{a}$ -modules) to tensoring with finite-dimensional modules. Hence we obtain that  $\Delta_{\mathbf{G}}(x)$  (in particular,  $\Delta_{\mathbf{G}}(w_0^{\mathbf{G}})$ ) is a (possibly infinite) direct sum of projective modules from  $\mathcal{O}(\mathfrak{a})$ . Since all projective modules in  $\mathcal{O}_\lambda$  have Verma flags, it follows that all projective modules in  $\mathcal{O}_\lambda$  are (possibly infinite) direct sums of projective modules from  $\mathcal{O}(\mathfrak{a})$  as well.

This means that, up to  $\mathfrak{a}$ -homotopy, the sequence (5.1), considered as a sequence of  $\mathfrak{a}$ -modules, is a direct sum of trivial sequences of the form  $\cdots \rightarrow 0 \rightarrow X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow \dots$ . Since  $Z_{\mathbf{H}}$  commutes with the restriction to  $\mathfrak{a}$  (by definition), we obtain that the sequence

$$0 \rightarrow Z_{\mathbf{H}}P_k \rightarrow \cdots \rightarrow Z_{\mathbf{H}}P_0 \rightarrow Z_{\mathbf{H}}\Delta_{\mathbf{G}}(x) \rightarrow 0 \quad (5.2)$$

is exact. Since  $Z_{\mathbf{H}}$  sends projective  $\mathbf{A}_{\mathbf{G}}$ -modules to projective  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules or zero, (5.2) is in fact a projective resolution of the standard  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -module  $Z_{\mathbf{G}}\Delta_{\mathbf{G}}(x)$ .

Now let us go back to the graded algebras. From Theorem 4.1 we know that  $\mathbf{A}_{\mathbf{G}}$  is standard Koszul, in particular, all standard  $\mathbf{A}_{\mathbf{G}}$ -modules admit linear projective resolutions. This means that (5.1) can be chosen linear. Since  $Z_{\mathbf{H}}$  obviously respects the grading, we get that (5.2) is linear as well. Hence all standard  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules admit linear projective resolutions. Applying  $\star$  we get the dual statement for the costandard modules. This shows that  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  is standard Koszul, in particular, Koszul, by [ADL, Theorem 1], and completes the proof.  $\square$

**Remark 5.2.** *We remark that for the algebra  $\mathbf{A}_{\{e\}}^{\mathbf{H}}$  the proof of Theorem 5.1 does not use Section 4 and hence does not use Soergel's Koszul self-duality result.*

## 6 A new parabolic generalization of the category $\mathcal{O}$

In this section we will develop one auxiliary tool, which we will later use in Section 7. This is a new parabolic generalization of the category  $\mathcal{O}$ , which is not a highest weight category in general, but rather corresponds to a properly

stratified algebra. Roughly speaking, it is the translation of a singular block of  $\mathcal{O}$  (or some parabolic category) out of the wall.

Let  $\mathbf{G}$  and  $\mathbf{H}$  be two parabolic subgroups of  $\mathbf{W}$ . Let further  $f_{\mathbf{G}}^{\mathbf{H}}$  be the sum of all primitive idempotents of  $\mathbf{A}_{\{e\}}^{\mathbf{H}}$ , which correspond to  $w \in \mathbf{W}_{\mathbf{G}}^{\mathbf{H}}$ , and set  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}} = f_{\mathbf{G}}^{\mathbf{H}} \mathbf{A}_{\{e\}}^{\mathbf{H}} f_{\mathbf{G}}^{\mathbf{H}}$ . Let  $\mathbf{C}(\mathbf{G})$  denote the coinvariant algebra of  $\mathbf{G}$ , which we consider as a graded algebra with respect to the grading for which the generators have degree two.

**Theorem 6.1.** (i) *There is an isomorphism of graded algebras,*

$$\mathbf{B}_{\mathbf{G}}^{\mathbf{H}} \cong \mathbf{A}_{\mathbf{G}}^{\mathbf{H}} \otimes_{\mathbb{C}} \mathbf{C}(\mathbf{G});$$

- (ii) *the algebra  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$  is properly stratified (in general not quasi-hereditary) and has a duality;*
- (iii)  *$\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}\text{-mod}$  is equivalent to the full subcategory  $\mathcal{X}$  of  $\mathbf{A}_{\{e\}}^{\mathbf{H}}\text{-mod}$ , which consists of all  $X$  admitting a two-step projective presentation by modules from  $\text{add} \left( \mathbf{A}_{\{e\}}^{\mathbf{H}} f_{\mathbf{G}}^{\mathbf{H}} \right)$ ;*
- (iv)  *$\theta_{\mathbf{G}}^{\text{out}}$  sends standard  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules to standard  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ -modules;*
- (v) *as objects in  $\mathcal{X}$ , the proper costandard  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ -modules are  $\nabla_{\{e\}}^{\mathbf{H}}(ww_0^{\mathbf{G}})$ , where  $w \in \mathbf{W}_{\{e\}}^{\mathbf{H}}$  is a shortest representative of some coset from  $\mathbf{W}/\mathbf{G}$ ;*
- (vi) *as objects in  $\mathcal{X}$ , the tilting  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ -modules are  $T_{\{e\}}^{\mathbf{H}}(w)$ , where  $w \in \mathbf{W}_{\{e\}}^{\mathbf{H}}$  is a shortest representative of some coset from  $\mathbf{W}/\mathbf{G}$ .*

*Proof.* (iii) follows from [Au, Section 5]. The functor  $\theta_{\mathbf{G}}^{\text{out}}$  is exact and sends indecomposable projectives to indecomposable projectives (see the proof of Lemma 4.2). From the definition of  $f_{\mathbf{G}}^{\mathbf{H}}$  it follows that  $\theta_{\mathbf{G}}^{\text{out}}$  sends indecomposable  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -projectives to objects of the category  $\text{add} \left( \mathbf{A}_{\{e\}}^{\mathbf{H}} f_{\mathbf{G}}^{\mathbf{H}} \right)$ . This implies that  $\theta_{\mathbf{H}}^{\text{out}}$  maps  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}\text{-mod}$  to  $\mathcal{X}$ , in particular, the images of standard  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules belong to  $\mathcal{X}$ . Furthermore, every indecomposable projective  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ -module has a filtration, whose subquotients are the images of standard  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules, since  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  is quasi-hereditary and  $\theta_{\mathbf{G}}^{\text{out}}$  is exact. The existence of the duality for  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$  is proved in the same way as [MS2, Proposition 2.6]. This proves (ii) and (iv).

$\theta_{\mathbf{G}}^{\text{out}}$  sends tilting modules to tilting modules. Hence (vi) follows by tracking the highest weights.

Let  $w$  be as in (v). Then  $\nabla_{\mathbf{G}}^{\mathbf{H}}(w)$  is the costandard  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -module. It is easy to see that  $\theta_{\mathbf{G}}^{\text{out}} \nabla_{\mathbf{G}}^{\mathbf{H}}(w)$  surjects onto the dual Verma module  $\nabla_{\{e\}}^{\mathbf{H}}(ww_0^{\mathbf{G}})$ , and

that the kernel of this surjection is generated by a projective module from  $\mathcal{X}$ . Now (v) follows from the fact that  $\theta_{\mathbf{G}}^{on}$  sends  $\nabla_{\{e\}}^{\mathbf{H}}(ww_0^{\mathbf{G}})$  back to  $\nabla_{\mathbf{G}}^{\mathbf{H}}(w)$ .

We are left to prove (i). First we note that  $\theta_{\mathbf{G}}^{out}$  induces a monomorphism from  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  to  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ .

From [MS2, Theorem 6.1] we have that the center of  $\mathbf{A}$  surjects onto  $\text{End}_{\mathbf{A}}(\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0^{\mathbf{G}}))$  and that the later algebra is isomorphic to  $\mathbf{C}(\mathbf{G})$ . Note that  $\Delta_{\mathbf{G}}(w) \subset \Delta_{\mathbf{G}}(w_0^{\mathbf{G}})$  for all  $w \in \mathbf{W}_{\mathbf{G}}$ , which induces the inclusion  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w) \subset \theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0^{\mathbf{G}})$ .

**Lemma 6.2.** *For  $w \in \mathbf{W}_{\mathbf{G}}$  the restriction from  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0^{\mathbf{G}})$  to  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w)$  induces an isomorphism*

$$\text{End}_{\mathbf{A}}(\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0^{\mathbf{G}})) \cong \text{End}_{\mathbf{A}}(\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w)).$$

*Proof.* We start with the case  $w = w_0$ . We have  $L_{\mathbf{G}}(w_0) \cong \Delta_{\mathbf{G}}(w_0) \subset \Delta_{\mathbf{G}}(w_0^{\mathbf{G}})$  and  $[\Delta_{\mathbf{G}}(w_0^{\mathbf{G}}) : L_{\mathbf{G}}(w_0)] = 1$  by [Di, Section 7]. All top subquotients of the module  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0)$  are isomorphic to  $L(w_0)$ . Since  $L(w_0)$  is not a subquotient of any  $\theta_{\mathbf{G}}^{out}L_{\mathbf{G}}(w)$  other than  $\theta_{\mathbf{G}}^{out}L_{\mathbf{G}}(w_0)$ , it follows that  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0)$  is stable under all endomorphisms of  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0^{\mathbf{G}})$ . In particular, the restriction map for endomorphism rings is well-defined.

Using the same arguments as in Lemma 4.2 one shows that  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0)$  has simple top. The standard properties of the translation functors imply  $[\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0) : L(w_0)] = |\mathbf{G}|$ , which, together with (ii), implies the equality  $\dim \text{End}_{\mathbf{A}_{\mathbf{G}}}(\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0)) = |\mathbf{G}|$ . Since  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0^{\mathbf{G}})$  has a Verma flag, its socle consists of simple subquotients isomorphic to  $L(w_0)$ . The arguments of the previous paragraph imply  $[\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0^{\mathbf{G}})/\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0) : L(w_0)] = 0$ , which, in turn, implies that the restriction map for the endomorphism rings is injective. Since both these endomorphism rings have the same dimension, we derive that the restriction is in fact an isomorphism. This completes the proof in the case  $w = w_0$ .

In the general case it would suffice to show that the restriction from  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w)$  to  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w_0)$  induces an isomorphism between the endomorphism rings of these modules. However, the arguments, analogous to the ones above, show that  $\dim \text{End}_{\mathbf{A}_{\mathbf{G}}}(\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w)) = |\mathbf{G}|$  and that the restriction map is injective. The statement follows.  $\square$

**Lemma 6.3.** *Let  $w \in \mathbf{W}_{\mathbf{G}}$ . Then the canonical surjection  $\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w) \rightarrow \theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}^{\mathbf{H}}(w)$  induces an isomorphism,*

$$\text{End}_{\mathbf{A}}(\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}(w)) \cong \text{End}_{\mathbf{A}}(\theta_{\mathbf{G}}^{out}\Delta_{\mathbf{G}}^{\mathbf{H}}(w)).$$

*Proof.* The arguments, analogous to those in the proof of Lemma 6.2, show that both endomorphism rings have dimension  $|\mathbf{G}|$  and that the induced map is injective. The statement follows.  $\square$

From Lemma 6.2 and Lemma 6.3 it follows that

$$\text{End}_{\mathcal{X}}(\theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}^{\mathbf{H}}(w)) = \text{End}_{\mathbf{A}}(\theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}(w_0^{\mathbf{G}})) \cong \mathbf{C}(\mathbf{G}),$$

for every  $w \in \mathbf{W}_{\mathbf{G}}^{\mathbf{H}}$ , moreover, that the center of  $\mathbf{A}$  (which is  $\mathbf{C}$  by [So1]) surjects onto  $\text{End}_{\mathcal{X}}(\theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}^{\mathbf{H}}(w))$ .

Since  $\mathbf{C}$  is symmetric, the corresponding trace form defines a splitting  $\mathbf{C}(\mathbf{G}) \hookrightarrow \mathbf{C}$  of the epimorphism  $\mathbf{C} \rightarrow \mathbf{C}(\mathbf{G})$  constructed above. This allows us to consider  $\mathbf{C}(\mathbf{G})$  as a subalgebra of  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ , which is central and surjects onto all the endomorphism rings from the previous paragraph. The fact that  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$  is properly stratified implies that all standard  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ -modules are in fact free over  $\text{End}_{\mathcal{X}}(\theta_{\mathbf{G}}^{\text{out}} \Delta_{\mathbf{G}}^{\mathbf{H}}(w_0^{\mathbf{G}})) \cong \mathbf{C}(\mathbf{G})$  (with the free basis given by any basis in the corresponding proper standard module). This, in particular, implies that  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$  is a free  $\mathbf{C}(\mathbf{G})$ -module of rank  $\dim \mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ .

To complete the proof it is now enough to show that  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  and  $\mathbf{C}(\mathbf{G})$  generate  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ . We do this using the induction with respect to the partial order on  $\mathbf{W}_{\mathbf{G}}^{\mathbf{H}}$ , which makes  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$  stratified, see (ii). Let  $x, y \in \mathbf{W}_{\mathbf{G}}^{\mathbf{H}}$ . By induction we can consider a properly stratified quotient,  $\tilde{\mathbf{B}}$ , of  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ , for which  $x$  becomes a maximal element. Let also  $\tilde{\mathbf{A}}$  denote the corresponding quotient of  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ . In this case we have that the corresponding projective  $\tilde{\mathbf{B}}$ -module  $\tilde{P}(x)$  is standard. Then the properly stratified structure guarantees that the trace of  $\tilde{P}(x)$  in the projective module  $\tilde{P}(y)$  is a direct sum of say  $k$  copies of  $\tilde{P}(x)$ . This implies that  $\text{Hom}_{\tilde{\mathbf{B}}}(\tilde{P}(x), \tilde{P}(y))$  is a free  $\mathbf{C}(\mathbf{G})$ -module of rank  $k$ .

Let  $\tilde{P}^{\tilde{\mathbf{A}}}(x)$  and  $\tilde{P}^{\tilde{\mathbf{A}}}(y)$  denote the indecomposable projective  $\tilde{\mathbf{A}}$ -modules, which correspond to  $x$  and  $y$  respectively. Recall that  $\theta_{\mathbf{G}}^{\text{out}}$  sends indecomposable projective  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ -modules to indecomposable projective  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ -modules (realized as objects of  $\mathcal{X}$ ) preserving the standard filtration, and induces a bijection on the isomorphism classes of simple modules, compatible with the partial orders on these sets involved in the quasi-hereditary and properly stratified structures respectively. This implies that

$$\dim \text{Hom}_{\tilde{\mathbf{A}}}(\tilde{P}^{\tilde{\mathbf{A}}}(x), \tilde{P}^{\tilde{\mathbf{A}}}(y)) = k,$$

which shows that  $\text{Hom}_{\tilde{\mathbf{B}}}(\tilde{P}(x), \tilde{P}(y))$  is generated by  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  and  $\mathbf{C}(\mathbf{G})$ . For reversed  $x$  and  $y$  the necessary statement follows applying  $\star$ . This proves the ungraded version of (i) and the graded version follows from the remark that all the above arguments are compatible with the grading. This completes the proof.  $\square$

**Remark 6.4.** *It follows immediately from the proof of Theorem 6.1 that both standard and proper standard  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$ -modules admit graded lifts, in particular, it follows that  $\mathbf{B}_{\mathbf{G}}^{\mathbf{H}}$  is graded properly stratified in the sense of [MS1, Section 8].*

## 7 Beilinson-Ginzburg-Soergel's Theorem and Backelin's Theorem

**Theorem 7.1.** ([BGS, Theorem 1.1.3]) *Let  $\mathbf{G}$  be a parabolic subgroup of  $\mathbf{W}$ . Then the algebras  $\mathbf{A}_{\mathbf{G}}$  and  $\mathbf{A}_{\{e\}}^{\mathbf{G}}$  are Koszul dual to each other.*

*Proof.* Since the algebra  $\mathbf{C}(\mathbf{G})$  lives in even degrees only, from Theorem 6.1(i) it follows that

$$\left(\mathbf{B}_{\mathbf{G}}^{\{e\}}\right)^{\dagger} \cong \mathbf{A}_{\mathbf{G}}^{\dagger}. \quad (7.1)$$

Note again that  $\mathbf{A}^{\dagger} \cong \mathbf{A}$  by [So1, Theorem 18] and [BGS, Theorem 2.10.1]. Applying to the latter formula Proposition 3.2 and using the Ringel duality gives  $\left(\mathbf{B}_{\mathbf{G}}^{\{e\}}\right)^{\dagger} \cong \mathbf{A}_{\{e\}}^{\mathbf{G}}$ , which, using (7.1), implies  $\mathbf{A}_{\mathbf{G}}^{\dagger} \cong \mathbf{A}_{\{e\}}^{\mathbf{G}}$ . But from Theorem 4.1 and Theorem 5.1 we know that both  $\mathbf{A}_{\mathbf{G}}$  and  $\mathbf{A}_{\{e\}}^{\mathbf{G}}$  are Koszul with respect to the natural grading. The proof is now completed by applying [BGS, Theorem 2.10.1].  $\square$

**Theorem 7.2.** ([Ba, Theorem 1.1]) *Let  $\mathbf{G}$  and  $\mathbf{H}$  be parabolic subgroups of  $\mathbf{W}$ . Then the algebras  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  and  $\mathbf{A}_{w_0\mathbf{H}w_0}^{\mathbf{G}}$  are Koszul dual to each other.*

*Proof.* Since the algebra  $\mathbf{C}(w_0\mathbf{H}w_0)$  lives in even degrees only, from Theorem 6.1(i) it follows that

$$\left(\mathbf{B}_{w_0\mathbf{H}w_0}^{\mathbf{G}}\right)^{\dagger} \cong \left(\mathbf{A}_{w_0\mathbf{H}w_0}^{\mathbf{G}}\right)^{\dagger}. \quad (7.2)$$

By Theorem 7.1 and [BGS, Theorem 2.10.1] we have  $\left(\mathbf{A}_{\{e\}}^{\mathbf{G}}\right)^{\dagger} \cong \mathbf{A}_{\mathbf{G}}^{\{e\}}$ . Applying to the latter formula Proposition 3.2 and using the Ringel duality gives  $\left(\mathbf{B}_{w_0\mathbf{H}w_0}^{\mathbf{G}}\right)^{\dagger} \cong \mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ , which, using (7.2), implies  $\left(\mathbf{A}_{w_0\mathbf{H}w_0}^{\mathbf{G}}\right)^{\dagger} \cong \mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$ . But from Theorem 5.1 we know that both  $\mathbf{A}_{\mathbf{G}}^{\mathbf{H}}$  and  $\mathbf{A}_{w_0\mathbf{H}w_0}^{\mathbf{G}}$  are Koszul with respect to the natural grading. The proof is now completed by applying [BGS, Theorem 2.10.1].  $\square$

## 8 The category of linear complexes of tilting modules for the category $\mathcal{O}(\mathfrak{p}, \Lambda)$

Let  $\mathbf{G}$  and  $\mathbf{H}$  be two parabolic subgroups of  $\mathbf{W}$ . Let  $\mathbf{V}_{\mathbf{G}}^{\mathbf{H}}$  denote the set of all  $w \in \mathbf{W}_{\mathbf{G}}$  which are at the same time longest coset representatives for cosets from  $\mathbf{H} \setminus \mathbf{W}$ . Let  $q_{\mathbf{G}}^{\mathbf{H}}$  be the sum of all primitive idempotents in  $\mathbf{A}_{\mathbf{G}}$ , which correspond to  $w \in \mathbf{V}_{\mathbf{G}}^{\mathbf{H}}$ . Set  $\mathbf{C}_{\mathbf{G}}^{\mathbf{H}} = q_{\mathbf{G}}^{\mathbf{H}}\mathbf{A}_{\mathbf{G}}q_{\mathbf{G}}^{\mathbf{H}}$ . The algebra  $\mathbf{C}_{\mathbf{G}}^{\mathbf{H}}$  is the basic

associative algebra of the  $\mathbf{G}$ -singular block of the category  $\mathcal{O}(\mathfrak{p}, \Lambda)$ , studied in [FKM, MS2], where  $\mathfrak{p}$  is the parabolic subalgebra of  $\mathfrak{g}$ , associated with  $\mathbf{H}$ . The algebra  $\mathbf{C}_{\mathbf{G}}^{\mathbf{H}}$  is properly stratified and has a duality. Abusing notation we denote in this section the standard  $\mathbf{C}_{\mathbf{G}}^{\mathbf{H}}$ -modules by  $\Delta(w)$ ,  $w \in \mathbf{V}_{\mathbf{G}}^{\mathbf{H}}$ . The proper standard modules will be denoted by  $\overline{\Delta}(w)$ ,  $w \in \mathbf{V}_{\mathbf{G}}^{\mathbf{H}}$ .

Let now  $g_{\mathbf{H}}^{\mathbf{G}}$  be the sum of all primitive idempotents in  $\mathbf{A}_{\{e\}}^{\mathbf{G}}$ , which correspond to all  $w \in \mathbf{W}$  such that  $w$  is the shortest coset representative for a coset from  $\mathbf{G} \setminus \mathbf{W}$  and  $w$  is the shortest coset representative for a coset from  $\mathbf{W}/\mathbf{H}$  at the same time. Set  $\mathbf{D}_{\mathbf{H}}^{\mathbf{G}} = \mathbf{A}_{\{e\}}^{\mathbf{G}} / \left( \mathbf{A}_{\{e\}}^{\mathbf{G}} (1 - g_{\mathbf{H}}^{\mathbf{G}}) \mathbf{A}_{\{e\}}^{\mathbf{G}} \right)$ .

$\mathbf{C}_{\mathbf{G}}^{\mathbf{H}}$  inherits a  $\mathbb{Z}$ -grading from  $\mathbf{A}_{\mathbf{G}}$  and  $\mathbf{D}_{\mathbf{H}}^{\mathbf{G}}$  inherits a  $\mathbb{Z}$ -grading from  $\mathbf{A}_{\{e\}}^{\mathbf{G}}$ . We will call these gradings *natural*.

**Proposition 8.1.** *There is an isomorphism of algebras,  $(\mathbf{C}_{\mathbf{G}}^{\mathbf{H}})^{\dagger} \cong \mathbf{D}_{w_0 \mathbf{H} w_0}^{\mathbf{G}}$ , compatible with the natural gradings.*

*Proof.* This follows immediately from Proposition 3.2 and Theorem 7.1.  $\square$

In particular, we obtain that  $(\mathbf{C}_{\{e\}}^{\mathbf{H}})^{\dagger} \cong \mathbf{D}_{w_0 \mathbf{H} w_0}^{\{e\}}$  is an analogue of the algebra  $\mathbf{A}_{\{e\}}^{w_0 \mathbf{H} w_0}$  but with respect to the representatives in  $\mathbf{W}$  for cosets on the different side. The algebra  $\mathbf{C}_{\mathbf{G}}^{\mathbf{H}}$  has both tilting and cotilting modules, associated with the properly stratified structure. Via [MS2, Section 8] the natural grading on projective  $\mathbf{C}_{\mathbf{G}}^{\mathbf{H}}$ -modules induces natural graded lifts of simple, injective, standard, proper standard, costandard and proper costandard modules. Following [MO, Section 5], this allows us to fix a graded lift of the tilting module  $T(x)$  such that the natural inclusion  $\Delta(x) \hookrightarrow T(x)$  is a homogeneous map of degree 0. Further, we fix the graded lift of the cotilting module  $C(x)$  such that the natural projection  $C(x) \rightarrow \nabla(x)$  is a homogeneous map of degree 0. Note that  $T(x) \cong C(x)\langle -2l(w_0^{\mathbf{H}}) \rangle$ .

**Theorem 8.2.** (i) *All standard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules admit linear tilting coresolutions.*

(ii) *All costandard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules admit linear cotilting resolutions.*

*Proof.* Because of the duality it is enough to prove (i). In this case we have  $\Delta(w_0) = T(w_0)$  for which the statement is trivial. Now the statement follows by applying induction on the length of  $w$ , and using translation functors and the same arguments as in the proof of Proposition 2.7.  $\square$

**Remark 8.3.** *One can also prove Theorem 8.2 generalizing the arguments of Proposition 4.3. The main difference is that one will be forced to write an analogue of the parabolically induced BGG-resolution for cosets  $\mathbf{G}/\mathbf{H}$ .*

However, since this resolution consists only of modules with the scalar action of the center of  $\mathfrak{g}$ , the existence of such resolution follows directly from the proof of Proposition 4.3 using the functor  $\eta$  from [MS2, Section 3].

We believe that the statement of Theorem 8.2 remains valid for singular blocks (that is for all  $\mathbf{C}_{\mathbf{G}}^{\mathbf{H}}$ ) as well. However, the translation functor approach does not work appropriately for singular blocks, and a proper generalization of the corresponding arguments of Theorem 4.1 seems to be technically very complicated.

**Corollary 8.4.** (i) All standard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules admit linear projective resolutions.

(ii) All costandard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules admit linear injective coresolutions.

*Proof.* This follows from Theorem 8.2 and the Ringel self-duality of  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ , see [FKM, Theorem 3].  $\square$

**Corollary 8.5.** Let  $x, y \in \mathbf{V}_{\{e\}}^{\mathbf{H}}$  and  $x > y$ . Then

$$\mathrm{ext}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}^k(\overline{\Delta}(x)\langle m \rangle, \Delta(y)) \neq 0$$

implies  $m + 2\mathfrak{l}(w_0^{\mathbf{H}}) \leq k \leq \mathfrak{l}(x) - \mathfrak{l}(y)$ . In particular,

$$\mathrm{Ext}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}^k(\overline{\Delta}(x), \Delta(y)) = \mathrm{Ext}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}^k(\Delta(x), \Delta(y)) = 0$$

for all  $k \geq \mathfrak{l}(x) - \mathfrak{l}(y)$ .

*Proof.* Since all tilting  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules are cotilting at the same time, see for example [FKM, Section 6], the tilting coresolution  $\mathcal{T}^\bullet(\Delta(y))$  of  $\Delta(y)$  is cotilting (up to a shift of grading) at the same time. But this means that

$$\mathrm{ext}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}^k(\overline{\Delta}(x)\langle l \rangle, \Delta(y)) = \mathrm{Hom}(\overline{\Delta}(x)\langle l \rangle^\bullet, \mathcal{T}^\bullet(\Delta(y))),$$

where the last homomorphism space is taken in the homotopy category of the category  $\mathbf{C}_{\{e\}}^{\mathbf{H}}\text{-grmod}$ . Now the first statement follows from Theorem 8.2(i), Proposition 2.3, and the above remark that  $T(x) \cong C(x)\langle -2\mathfrak{l}(w_0^{\mathbf{G}}) \rangle$ . The second statement follows from the first one.  $\square$

**Corollary 8.6.** Let  $x, y \in \mathbf{V}_{\{e\}}^{\mathbf{H}}$  and  $x > y$ . Then

$$\mathrm{Ext}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}^{\mathfrak{l}(x) - \mathfrak{l}(y)}(\Delta(x), \Delta(y))$$

is a free  $\mathrm{End}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}(\Delta(x))$ -module of rank

$$\dim \mathrm{ext}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}^{\mathfrak{l}(x) - \mathfrak{l}(y)}(\overline{\Delta}(x)\langle \mathfrak{l}(x) - \mathfrak{l}(y) - 2\mathfrak{l}(w_0^{\mathbf{H}}) \rangle, \Delta(y)).$$

*Proof.* As in Corollary 8.5 we have that

$$\mathrm{Ext}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}^{\mathfrak{l}(x)-\mathfrak{l}(y)}(\Delta(x), \Delta(y)) = \mathrm{Hom}(\Delta(x)^{\bullet}, \mathcal{T}^{\bullet}(\Delta(y))),$$

where the last homspace is taken in the homotopy category of the category  $\mathbf{C}_{\{e\}}^{\mathbf{H}}\text{-mod}$ . Note that all (graded) standard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules have a graded filtration by proper standard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules, see [MS2, Section 8], and further observe that

$$\dim \mathrm{ext}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}^{\mathfrak{l}(x)-\mathfrak{l}(y)}(\overline{\Delta}(x)\langle \mathfrak{l}(x) - \mathfrak{l}(y) - 2\mathfrak{l}(w_0^{\mathbf{H}}) \rangle, \Delta(y))$$

is the number of occurrences of  $T(x)\langle \mathfrak{l}(x) - \mathfrak{l}(y) \rangle$  in  $\mathcal{T}^{\mathfrak{l}(x)-\mathfrak{l}(y)}(\Delta(y))$ . We have  $\mathrm{Hom}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}(\Delta(x), T(x)) \cong \mathrm{Hom}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}(\Delta(x), \Delta(x))$ , which is a free  $\mathrm{End}_{\mathbf{C}_{\{e\}}^{\mathbf{H}}}(\Delta(x))$ -module of rank one. The statement follows.  $\square$

Though  $\mathbf{D}_{\mathbf{G}}^{\mathbf{H}}$  is neither quasi-hereditary nor properly stratified in general (a counter example can be derived from [MS1, Remark 1.2]), the notion of a standard module is nevertheless well-defined for this algebra (and for any algebra with a fixed order on the set of the isomorphism classes of simple modules).

**Corollary 8.7.** *Under the equivalence provided by Proposition 8.1, the linear tilting coresolutions of standard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules are standard  $\mathbf{D}_{w_0^{\mathbf{H}}\mathbf{H}w_0}^{\{e\}}$ -modules; and (appropriately shifted) linear cotilting resolutions of costandard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules are costandard  $\mathbf{D}_{w_0^{\mathbf{H}}\mathbf{H}w_0}^{\{e\}}$ -modules.*

*Proof.* The proof is analogous to that of [MO, Proposition 5].  $\square$

**Remark 8.8.** *All standard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules are quotients of Verma modules in  $\mathcal{O}$  and hence have central character. Applying the equivalence  $\eta$  from [MS2, Section 3], we obtain that the multiplicities of simple modules in the composition series of standard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules coincide with the multiplicities of the corresponding (under  $\eta$ ) simple  $\mathbf{A}_{\{e\}}^{\mathbf{H}}$ -modules in the composition series of the corresponding (again under  $\eta$ ) standard  $\mathbf{A}_{\{e\}}^{\mathbf{H}}$ -modules. Moreover, everything is compatible with the grading. Therefore, the components of the linear tilting coresolution of a standard  $\mathbf{C}_{\{e\}}^{\mathbf{H}}$ -modules can be computed using the Kazhdan-Lusztig combinatorics (see also [CC]).*

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