Twisted and shuffled filtrations on tilting modules

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Abstract

We prove that tilting modules in the category $\mathcal{O}_\lambda$ are filtered by different families of shuffled (or twisted) Verma modules.

Résumé

On prouve que les modules basculantes dans la catégorie $\mathcal{O}_\lambda$ ont des filtrations par des familles différentes de modules de Verma battres ou entorés.

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1 Introduction and the main result

Let $\mathfrak{g}$ be a semi-simple complex finite-dimensional Lie algebra. The Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for $\mathfrak{g}$, introduced in [BGG], contains several important and interesting families of $\mathfrak{g}$-modules, e.g. simple highest weight modules, Verma modules, projective modules, tilting modules, which appear naturally in that context. Considering the category of Harish-Chandra modules for $\mathfrak{g}$ (which are in fact $\mathfrak{g}\times\mathfrak{g}$-modules), the Bernstein-Gelfand – Joseph – Duflo equivalence of categories, see [Ja, Chapter 6], maps the principal series Harish-Chandra modules to the so-called *shuffled Verma modules* $M(x,y)$. Inside a fixed regular indecomposable block $\mathcal{O}_\lambda$, λ regular integral antidominant, of $\mathcal{O}$ shuffled Verma modules are indexed by pairs (x,y) of elements from the Weyl group W. Irving, in [I], gave an alternative construction of these modules in terms of the so-called *shuffling functors*, which are defined using the coherent translations θ_α, [Ja], through the α-wall. His construction is inductive and goes as follows. We start with setting $M(x,e)=M(x\cdot\lambda)$, the latter being the usual Verma module. If now $y\in W$ and $ys_\alpha>y$ for the simple reflection s_α, then the module $M(x,y)$ canonically embeds into $\theta_\alpha(M(x,y))$ and the quotient is exactly $M(x,ys_\alpha)$. Recently, in [AL] it was shown that the same family of modules can be obtained using *Arkhipov's twisting functor*, [Ar, AL], which also explains the alternative name *twisted Verma modules*, used in [AL].

Recall that, if $\mathcal{F}$ is a fixed family of modules, a module, M, is said to *have an $\mathcal{F}$-flag* (or *to be filtered by modules from $\mathcal{F}$*) if there is a filtration of M whose quotients belong to $\mathcal{F}$. Denote $\mathcal{F}_x=\{M(x,y)|y\in W\}$ resp. $\mathcal{F}^y=\{M(x,y)|x\in W\}$ and let w_0 be the longest element in W.

As it was known from [BGG], all projective modules in the category $\mathcal{O}$ are filtered by Verma modules. In [I, Theorem 4.1] it was shown that some projectives in $\mathcal{O}_\lambda$ are filtered by certain families of shuffled Verma modules. Moreover, roughly speaking, the bigger the indecomposable projective is, the more such filtrations it possess. Namely, the indecomposable projective cover $P(x\cdot\lambda)$, $x \in W$, of the simple module $L(x\cdot\lambda) \in \mathcal{O}_\lambda$ appears to have an $\mathcal{F}^y$-flag for all y, which can be written $y = s_1 \dots s_k$ with simple reflections s_i satisfying $xs_i > x$. In particular, the *big projective module* $P(\lambda)$ in $\mathcal{O}_\lambda$ has an $\mathcal{F}^y$-flag for all y.

If one writes $\{M(x,y)\}$ in a $W\times W$-array with respect to some total order extending the Bruhat order, the sets $\mathcal{F}_x$ and $\mathcal{F}^y$ represent rows resp. columns of the array. In particular, $\mathcal{F}^e$ and $\mathcal{F}_{w_0}$ represent Verma modules and $\mathcal{F}^{w_0}$ and $\mathcal{F}_e$ represent their duals. This is why the shuffled Verma modules are usually viewed as intermediate modules between Verma modules and their duals. This remark also stimulates to consider *tilting* modules in $\mathcal{O}_\lambda$, i.e. self-dual modules with a Verma flag, first constructed by [CI] (the term tilting module was introduced for $\mathcal{O}$ later, namely, after [R]). In particular, it is known that indecomposable tilting modules are indexed by Verma modules, namely, for each Verma module $M(x\cdot\lambda)$ there exists exactly one indecomposable tilting module $T(x\cdot\lambda)$, such that any Verma flag of $T(x\cdot\lambda)$ starts with $M(x\cdot\lambda)$.

By definition, all tilting modules have $\mathcal{F}_e$-, $\mathcal{F}_{w_0}$-, $\mathcal{F}^e$- and $\mathcal{F}^{w_0}$-flags. In particular, $P(\lambda)$ is an example of indecomposable tilting module. As we already mentioned, by Irving's result $P(\lambda)$ has an $\mathcal{F}^y$-flag for all y. Another example of tilting module in $\mathcal{O}_\lambda$ is the simple Verma module $M(\lambda)$, isomorphic to $M(x,x^{-1})$ for any $x \in W$ (see e.g. properties of $M(x,y)$ in [I]). Since $M(\lambda)$ occurs in each row and column of the $W\times W$ array $\{M(x,y)\}$, we get that $M(\lambda)$ has an $\mathcal{F}^y$- and an $\mathcal{F}_x$-flag for all x,y. The aim of this paper is to prove the following result, which is naturally motivated by the above discussion.

Theorem 1. *Any tilting module in $\mathcal{O}_\lambda$ has an $\mathcal{F}^y$- and an $\mathcal{F}_x$-flag for all $x,y \in W$.*

We also note that Soergel's equivalence of categories from [S1] extends this result to all regular anti-dominant λ, which is the classical case, considered in [I].

2 $\mathcal{F}^y$-flags on tilting modules

In this section we prove the first part of the main Theorem 1, namely, we will show that any tilting module in $\mathcal{O}_\lambda$ has an $\mathcal{F}^y$-flag for all $y \in W$. As we already mentioned, from [I] this follows for $P(\lambda) = T(w_0\cdot\lambda)$ and for $T(\lambda) = M(\lambda)$ the statement is obvious. For a simple root, α, let $s = s_\alpha$ be the corresponding reflection. Then we denote by $\mathcal{S}_s$ the corresponding shuffling functor, [I, Section 3] (we remark that in [I] this functor was denoted by C_s, $s = s_\alpha$, and we decided to use the other name to avoid confusions with Enright's completions, which are also usually denoted by C_s). Then for any $M \in \mathcal{O}_\lambda$ the module $\mathcal{S}_s(M)$ is the quotient of $\theta_\alpha(M)$ modulo the canonical image of M inside $\theta_\alpha(M)$. It is easy to see that this map is functorial. Shuffling functors produce the following connection between different $(\mathcal{F}^y)$'s, see [I, Corollary 3.2]:

Lemma 1. *Let α be a simple root and $y \in W$ such that $ys_\alpha > y$. If $M \in \mathcal{O}_\lambda$ has an $\mathcal{F}^y$-flag then $\mathcal{S}_{s_\alpha}(M)$ has an $\mathcal{F}^{ys_\alpha}$-flag.*

For $y \in W$ we denote by $\mathcal{O}_\lambda(y)$ the full subcategory of all modules from $\mathcal{O}_\lambda$ having an $\mathcal{F}^y$-flag. We start with the following observation:

Lemma 2. *Let $y \in W$ and $s = s_\alpha$ be a simple reflection such that $ys > y$. Then $\mathcal{S}_s : \mathcal{O}_\lambda(y) \to \mathcal{O}_\lambda(ys)$ is an equivalence of categories.*

Proof. Because of the exact sequence $0 \to M(x,y) \to \theta_s(M(x,y)) \to M(x,ys) \to 0$, [I, Theorem 2.1], where $x \in W$, the adjunction morphism $M(x,y) \to \theta_s(M(x,y))$ is injective and hence the image of $\mathcal{O}_\lambda(y)$ under $\mathcal{S}_s$ is contained in $\mathcal{O}_\lambda(ys)$ by Lemma 1. By [AL, Remark 1.2], there also exists a self-equivalence, $\tilde{\mathcal{S}}_s$ of the bounded derived category $\mathcal{D}^b(\mathcal{O}_\lambda)$ such that $\tilde{\mathcal{S}}_s(M) \simeq \mathcal{S}_s(M)$ for any $M \in \mathcal{O}_\lambda(y)$. In particular, $\mathcal{S}_s$ preserves the homomorphism rings between objects from $\mathcal{O}_\lambda(y)$ and thus the endomorphism ring of all objects from $\mathcal{O}_\lambda(y)$. Hence it sends indecomposables to indecomposables. Now it is sufficient to prove that any object in $\mathcal{O}_\lambda(ys)$ belongs to the image of $\mathcal{S}_s$. We will do it using induction in the length of $\mathcal{F}^{ys}$-filtration of $M \in \mathcal{O}_\lambda(ys)$. If this length is one, then $M \simeq M(x,ys)$ for some $x \in W$ and hence $M = \mathcal{S}_s(M(x,y))$. Now consider an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathcal{O}_\lambda(ys)$. Applying to this sequence the exact functor θ_s we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \theta_s(M_1) & \longrightarrow & \theta_s(M_2) & \longrightarrow & \theta_s(M_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0
\end{array}$$

where the columns are represented by natural morphisms $\theta_s(M) \to M$ (see I''_M in [GJ, Subsection 3.12]). As all $M_i \in \mathcal{O}_\lambda(ys)$, these morphisms are surjective by [GJ, Lemma 3.12]. Hence, by standard homological arguments and computing the character of K_i, $i = 1,2,3$, we can extend the diagram above to the following commutative diagram with exact columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \theta_s(M_1) & \longrightarrow & \theta_s(M_2) & \longrightarrow & \theta_s(M_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now as the two lower rows are exact the upper one is exact as well by 3×3-Lemma. From the inductive assumption we get that K_1 and K_3 have $\mathcal{F}^y$-flags and thus K_2 has an $\mathcal{F}^y$-flag as well. Moreover, by induction we also have $\theta_s(K_i) \simeq \theta_s(M_i)$, $i = 1, 3$, and that the morphisms $K_i \to \theta_s(M_i)$, $i = 1, 3$, are represented by the natural morphisms $K_i \to \theta_s(K_i)$. From this and [GJ, Subsection 3.12] it follows then that the natural morphism $K_2 \to \theta_s(K_2)$ is injective which then, together with $\theta_s^2 = \theta_s \oplus \theta_s$, guarantees that $\theta_s(K_2) \simeq \theta_s(M_2)$. Substituting now the map $K_2 \to \theta_s(M_2)$ with the natural morphism $K_2 \to \theta_s(K_2)$ we still get a commutative diagram and thus the composition of the natural morphisms $K_2 \to \theta_s(K_2)$ and $\theta_s(K_2) \simeq \theta_s(M_2) \to M_2$ must be zero. Therefore $M_2 \simeq \mathcal{S}_s(K_2)$, which completes the proof. □

We note that one can also use the following argument to prove the second part of Lemma 2: Having a module with $\mathcal{F}^{ys}$-filtration one uses shuffling functors to get a module, filtered by dual Verma modules, which then can be translated to a module with Verma flag by duality. Shuffling the latter we can get a module with $\mathcal{F}^{w_0 y}$-flag and applying the duality once more we get a module with $\mathcal{F}^y$-flag. It follows from [AL, Remark 1.2] that this procedure is inverse to $\mathcal{S}_s : \mathcal{O}_\lambda(y) \to \mathcal{O}_\lambda(ys)$.

Corollary 1. *Let $y \in W$ with the reduced decomposition $y = s_1 \dots s_k$. Then $\mathcal{S}_{s_k} \circ \dots \circ \mathcal{S}_{s_1} : \mathcal{O}_\lambda(e) \to \mathcal{O}_\lambda(y)$ is an equivalence of categories.*

Corollary 2. *The category $\mathcal{O}_\lambda(y)$ is closed under taking direct summands.*

Now the necessary statement (i.e. necessary part of Theorem 1) will follow from the following result.

Lemma 3. *Let $M \in \cap_{y \in W} \mathcal{O}_\lambda(y)$ and s be a simple reflection. Then $\theta_s(M) \in \cap_{y \in W} \mathcal{O}_\lambda(y)$.*

Proof. First we note that M is filtered by $\mathcal{F}^e$ and $\mathcal{F}^{w_0}$ and hence is a tilting module. In particular, it is self-dual. Hence $\theta_s(M)$ is self-dual as well. Let $y \in W$ be such that $ys > y$. Then the adjunction morphism $M \to \theta_s(M)$ is injective and thus its cokernel is filtered by $\mathcal{F}^{ys}$. As M is filtered by $\mathcal{F}^{ys}$ as well we get that $\theta_s(M)$ is filtered by $\mathcal{F}^{ys}$, in other words by $\mathcal{F}^w$ with $ws < w$. Now if we use the fact that M is self-dual and that the modules in $\mathcal{F}^{w_0 w}$ are exactly the duals to the modules in $\mathcal{F}^w$ (see [I]), we get that the module M is filtered by $\mathcal{F}^{w_0 w}$ with $ws < w$ and hence by $\mathcal{F}^t$ with $wt > t$. This completes the proof. □

Lemma 4. *Each indecomposable tilting module $T(w \cdot \lambda)$, $w \in W$, is a direct summand of some $M \in \cap_{y \in W} \mathcal{O}_\lambda(y)$.*

Proof. For $w = 0$ the module $T(\lambda)$ is a simple Verma module and hence belongs to all $\mathcal{O}_\lambda(y)$, $y \in W$. Now, by Lemma 3 the module $\theta_{s_1} \circ \dots \circ \theta_{s_k}(T(\lambda))$ also belongs to all $\mathcal{O}_\lambda(y)$, $y \in W$, for any sequence $s_1, \dots, s_k \in W$ of simple reflections. If we take $w = s_k \dots s_1$ to be a reduced decomposition of w, we can use [CI] and obtain that $T(w \cdot \lambda)$ is a direct summand of $\theta_{s_1} \circ \dots \circ \theta_{s_k}(T(\lambda))$. This completes the proof. □

Now the proof of the first statement of Theorem 1 is transparent. We use Lemma 4 and find some $M \in \cap_{y \in W} \mathcal{O}_\lambda(y)$ which has $T(w \cdot \lambda)$ as a direct summand. Now, by Corollary 2, all direct summands of M, in particular $T(w \cdot \lambda)$, belong to $\cap_{y \in W} \mathcal{O}_\lambda(y)$, which is the statement we needed.

3 $\mathcal{F}_x$-flags on tilting modules

In this section we prove the second part of Theorem 1, which appears to be a little bit easier than the first one. To produce different $\mathcal{F}_x$-flags on tilting modules we will use *Arkhipov's twisting functors* T_w, $w \in W$ (notation as in [AL], in [Ar] the author used Θ_w). According to [AL, Section 5], T_w sends $\mathcal{F}_{w_0}$, which consists of Verma modules, to $\mathcal{F}_{ww_0}$ for any $w \in W$. We again start with the simple tilting module.

Lemma 5. *The module $T(\lambda)$ has an $\mathcal{F}_x$-flag for any $x \in W$.*

Proof. Write $x = ww_0$ for uniquely defined $w \in W$ and choose Verma module $M(\mu) \in \mathcal{F}_{w_0}$ such that $T_w(M(\mu)) \simeq M(\lambda) = T(\lambda)$. This is possible since $T_w : \mathcal{F}_{w_0} \to \mathcal{F}_{ww_0}$ is bijective and $M(\lambda) \in \mathcal{F}_{ww_0}$. □

We have to note that the statement itself follows from the fact $M(\lambda) \in \mathcal{F}_{ww_0}$, however we will use the formula $T_w(M(\mu)) \simeq M(\lambda)$ in the arguments that follow.

Corollary 3. *For any finite-dimensional $\mathfrak{g}$-module F and any $x \in W$ the module $F \otimes T(\lambda)$ has an $\mathcal{F}_x$-flag.*

Proof. As above write $x = ww_0$. By [AL, Subsection 6.3], T_w commutes with $F\otimes\_$. Hence $F \otimes T(\lambda) \simeq F \otimes (T_w(M(\mu))) \simeq T_w(F \otimes M(\mu))$. As $M(\mu)$ is a Verma module, $F \otimes M(\mu)$ has a Verma flag, hence $\mathcal{F}_{w_0}$-flag. Then T_w will translate this flag to an $\mathcal{F}_x$-flag of $F \otimes T(\lambda)$. □

Lemma 6. *Let F and x be as in Corollary 3. Then each direct summand of $F \otimes T(\lambda)$ has an $\mathcal{F}_x$-flag.*

Proof. Here we use the fact ([Ar, AL]) that T_w extends to the functor LT_w on the bounded derived category $\mathcal{D}^b(\mathcal{O}_\lambda)$, moreover, LT_w is, in fact, an auto-equivalence on $\mathcal{D}^b(\mathcal{O}_\lambda)$. In particular, it preserves the endomorphism ring of each direct summand of $F \otimes M(\mu)$ as the latter are filtered by Verma modules, see [AL, Corollary 6.3]. Hence T_w sends indecomposable direct summands of $F \otimes M(\mu)$ to indecomposable direct summands of $F \otimes T(\lambda)$ and therefore transforms the Verma flags of the first ones to $\mathcal{F}_x$-flags of the last ones. This completes the proof. □

Now the proof of the second part of Theorem 1 is easily completed. By [CI] each indecomposable tilting module in $\mathcal{O}_\lambda$ occurs as a direct summand in some $F \otimes T(\lambda)$. Hence Lemma 6 implies the necessary statement.

We note that the problem to use analogous arguments in Section 2 was that the coherent translation θ_s does not commute with the functor $F\otimes\_$ in the general case.

4 Some corollaries and remarks

Corollary 4. *For a module, $M \in \mathcal{O}_\lambda$, the following conditions are equivalent:*

1. *M is a tilting module, i.e. is self-dual and filtered by Verma modules;*

2. *M has an $\mathcal{F}^y$-flag for all $y \in W$;*

3. *M has an $\mathcal{F}_x$-flag for all $x \in W$;*

4. *M has an $\mathcal{F}^y$-flag and an $\mathcal{F}_x$-flag for all $x, y \in W$.*

Proof. Immediate corollary of Theorem 1. □

Corollary 5. *Each category $\mathcal{O}_\lambda(y)$, $y \in W$, has almost split sequences.*

Proof. Follows from Corollary 1 and [R]. □

The next corollary is a famous result of Soergel, [S2]. In particular, Corollary 1 can be viewed as an extension of this result.

Corollary 6. *The categories $\mathcal{O}_\lambda(e)$ and $\mathcal{O}_\lambda(e)^{opp}$ are equivalent.*

Proof. As a special case of Corollary 1, the categories $\mathcal{O}_\lambda(e)$ and $\mathcal{O}_\lambda(w_0)$ are equivalent. But the last one is equivalent with $\mathcal{O}_\lambda(e)^{opp}$ by usual duality. □

Analogous results can be obtained via T_w for categories of modules, filtered by $\mathcal{F}_x$. In fact, in [S2] the functor T_{w_0} is used to prove the above statement.

We would like to finish with the remark that the homological characterization of $\mathcal{O}_\lambda(y)$ in the spirit of [R], where it was proved that $\mathcal{F}^e = {}^\perp\mathcal{F}^{w_0}$ and $\mathcal{F}^{w_0} = (\mathcal{F}^e)^\perp$ does not seem to be possible, e.g. as for any short exact sequence $0 \to K \to T_2 \to T_1 \to 0$ with tilting modules T_1 and T_2, the module K although filtered by $\mathcal{F}^e$ is not tilting in general (example: $T_1 = T(\lambda)$, $T_2 = \theta_\alpha(T(\lambda))$, then $K = M(s_\alpha \cdot \lambda)$ is not self-dual). In case of existence of any analogue of such homological characterization, from Theorem 1 and the long exact sequence it would follow that K is a tilting module as well.

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