CATEGORIFICATION,
KOSTANT’S PROBLEM AND
GENERALIZED VERMA MODULES

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1. Motivation — generalized Verma modules

\( g \) — semi-simple finite-dimensional complex Lie algebra.

\( g = n_- \oplus h \oplus n_+ \) — triangular decomposition.

\( p \supset h \oplus n_+ \) — parabolic subalgebra.

\( p = a \oplus n \)

\( n \) — nilpotent radical of \( p \)

\( a \) — Levi factor

\( V \) — simple \( a \)-module

\( nV = 0 \)

\( M(p, V) = U(g) \otimes_{U(p)} V \) — generalized Verma module

Question 1: What is the structure of \( M(p, V) \)?

Question 2: When is \( M(p, V) \) irreducible?
Discouragement: No classification of simple $\alpha$-modules.

Encouragement 1: Many partial cases are known, in particular, $\alpha = \mathfrak{h}$, $V$ finite-dimensional, $V$ weight dense with f.d. weight spaces, $V$ generic Gelfand-Zetlin, $V$ Whittaker. (Names: Verma, BGG, Jantzen, McDowell, Futorny, M., Milicic, Soergel, Khomenko, Mathieu, Britten, Lemire, others)

Encouragement 2: Annihilators of $V$ are classified via annihilators of simple highest weight modules.

Idea (following Milicic-Soergel’s study of the case when $V$ is a Whittaker module):

- Take a simple highest weight $\alpha$-module $V'$ with the same annihilator as $V$.
- Realize $M(\mathfrak{p}, V)$ and $M(\mathfrak{p}, V')$ as objects in some Coker-categories.
- Prove (using Harish-Chandra bimodules) that these categories are equivalent and that the equivalence sends $M(\mathfrak{p}, V)$ to $M(\mathfrak{p}, V')$.
- Deduce the structural properties of $M(\mathfrak{p}, V)$ from those of $M(\mathfrak{p}, V')$ and KL-type combinatorics.
Encouragement 1: Works for Whittaker and generic Gelfand-Zetlin modules.

Encouragement 2: The categories of Harish-Chandra bimodules which appear depend only on the annihilator of $V$.

Catch 1: Needs better understanding of the so-called Kostant’s problem for $V$ and some induced modules.

Catch 2: Answers the irreducibility question, but does not help to describe all subquotients of GVM as this description depends on more than the annihilator of $V$.

Example: The Verma module $M(s \cdot 0)$ over $\mathfrak{sl}_3$ is parabolically induced from a simple Verma $\mathfrak{sl}_3$-module, say $X$. The module $M(s \cdot 0)$ has simple subquotients

$$L(s \cdot 0), \ L(st \cdot 0), \ L(ts \cdot 0), \ L(sts \cdot 0).$$

Let $X'$ be a simple dense $\mathfrak{sl}_3$-module with the same annihilator as $X$. Then (Futorny) $M(p, X')$ has only three subquotients $N_1$, $N_2$ and $N_3$.

Mathieu’s functor can be used to associate $N_1$, $N_2$ and $N_3$ with $L(s \cdot 0)$, $L(st \cdot 0)$ and $L(sts \cdot 0)$ respectively.

$L(ts \cdot 0)$ is induced from a module with the annihilator, which is ‘‘strictly bigger’’ than that of $X$. 
2. **Kostant’s problem**

$M$ — $g$-module.

$L(M, M) = \text{Hom}_\mathbb{C}(M, M)^{\text{ad-fin}}$ — locally ad $U(g)$-finite $\mathbb{C}$-endomorphisms of $M$.

Kostant’s problem: For which (simple) $M$ is the natural injection

$$U(g)/\text{Ann}\,_{U(g)}(M) \hookrightarrow L(M, M)$$

surjective?

Answer is:

- not known in general, not even for simple highest weight modules

- known to be positive for Verma modules and for simple highest weight modules of the form $L(w_0^p w_0 \cdot \lambda)$, $\lambda$ is regular and dominant (Joseph, Gabber-Joseph).

- known to be negative for $L(st \cdot 0)$ in type $B_2$ (Joseph).
Theorem 1. (M.) Let $s$ be a set of simple roots for $p$. Then the answer to Kostant’s problem is positive for the simple highest weight module of the form $L(sw_0^p w_0 \cdot \lambda)$ where $\lambda$ is regular and dominant.

Example: For the regular block in type $B_2$ the answer to Kostant’s problem is thus positive for $L(0)$, $L(s \cdot 0)$, $L(t \cdot 0)$, $L(sts \cdot 0)$, $L(tst \cdot 0)$ and $L(tsts \cdot 0)$; and it is negative for $L(st \cdot 0)$ and $L(ts \cdot 0)$.

Theorem 2. (M.-Stroppel) Let $g = \mathfrak{sl}_n$. Then for simple highest weight modules of the form $L(x \cdot \lambda)$ where $\lambda$ is regular and dominant the answer to Kostant’s problem is a left cell invariant.
3. Why? Twisting FUNCTORS

$s$ — simple reflection corresponding to simple root $\alpha$

$X_\alpha$ — some non-zero element in $g_\alpha$

$U_\alpha$ — localization of $U(g)$ with respect to $X_\alpha$

$\Theta_\alpha$ — an automorphism of $g$ corresponding to $s$

Twisting functor (Arkhipov):

$$T_s : M \mapsto \Theta_\alpha(U_\alpha/U(g) \otimes g M).$$

Properties (Andersen-Stroppel, Khomenko-M.):

• $T_s$ commutes with projective functors.

• $R T_s$ is an autoequivalence of $D^b(O_0)$.

• $R T_s$’s satisfy braid relations and hence define an action of the braid group on $D^b(O_0)$.

• The action of $R T_s$’s on $D^b(O_0)$ categorifies the left regular representation of the Weyl group.

• $T_s M(x \cdot 0) \cong M(sx \cdot 0)$ if $sx > x$.

• $T_s$ is left adjoint to Joseph’s completion functor.
Kostant’s problem can be reduced to numerical calculations using:

- \( \text{Hom}_g(V, \mathcal{L}(M, M)) = \text{Hom}_g(M, M \otimes V^*), \ V \) — simple finite-dimensional.
- Annihilators of simple highest weight modules correspond bijectively to left cells.

Need: \( \dim \text{Hom}_g(L(x \cdot 0), L(x \cdot 0) \otimes V^*) \) is a left cell invariant.

Roughly speaking the left cell is a simple \( S_n \)-module, where \( S_n \) acts via twisting functors.

Twistings commute with projective functors \( - \otimes V^* \).

\( T_s L(x \cdot 0) \) is either 0 (if \( sx > s \)) or has simple top \( L(x \cdot 0) \) and semisimple radical consisting of \( L(sx \cdot 0) \) and some other modules \( L(y \cdot 0) \), where \( x \) and \( y \) are in the same left cell (multiplicity is given by KL-combinatorics).

Using the properties of (derived) twisting functors one can show that

\[ \dim \text{Hom}_g(L(x \cdot 0), L(x \cdot 0) \otimes V^*) \leq \dim \text{Hom}_g(L(y \cdot 0), L(y \cdot 0) \otimes V^*) \]

for any \( x, y \) in the same left cell.
4. Structure of generalized Verma modules

$V$ — simple $\alpha$-module

$\text{Coker}(V)$ — category of all modules $X$ which admit resolution $M_2 \rightarrow M_1 \rightarrow X \rightarrow 0$, where $M_2$ and $M_1$ are direct summands of some $E \otimes V$, $E$ finite-dimensional (Milicic-Soergel).

Need: $V$ — projective in $\text{Coker}(V)$

For $\mathfrak{sl}_n$ we can always substitute $V$ by some $\tilde{V}$, which will be projective in $\text{Coker}(\tilde{V})$ by Irving-Shelton.

Using “parabolic Harsh-Chandra homomorphism” (Drozd-Futorny-Ovsienko) we can assume that $M(\mathfrak{p}, \tilde{V})$ is projective in $\text{Coker}(M(\mathfrak{p}, \tilde{V}))$.

From the above results on Kostant’s problem it follows that Kostant’s problem has a positive answer for $M(\mathfrak{p}, \tilde{V})$.

Corollary: $\text{Coker}(M(\mathfrak{p}, \tilde{V}))$ is equivalent to a certain category of Harish-Chandra bimodules.
Blocks of \( \text{Coker}(M(\mathfrak{p}, \tilde{V})) \) are described by weakly properly stratified algebras in the sense of Cline-Parshall-Scott and Frisk.

This means that projectives in these categories are filtered by the so-called standard and proper standard modules, both having a clear categorical interpretation (and thus preserved by “nice” equivalences). Generalized Vermas correspond to proper standard modules.

Catch: Simple objects in these categories are not simple \( g \)-modules in general.

Example: \( g = \mathfrak{a} = \mathfrak{sl}_2, \; V = L(s \cdot 0) \).

The corresponding block of \( \text{Coker}(M(\mathfrak{p}, \tilde{V})) \) is equivalent to the category of modules over the algebra \( \mathbb{C}[x]/(x^2) \). It contains two indecomposable objects: the projective object \( P(s \cdot 0) \) and the simple object \( \hat{L}(s \cdot 0) \), which have the following Loewy filtrations:

\[
P(s \cdot 0) = \begin{cases} 
L(s \cdot 0) \\
L(0) \\
L(s \cdot 0)
\end{cases} , \quad \hat{L}(s \cdot 0) = \begin{cases} 
L(s \cdot 0) \\
L(0) \\
L(s \cdot 0)
\end{cases} ,
\]

There is no projective module in \( \text{Coker}(M(\mathfrak{p}, \tilde{V})) \) with simple top \( L(0) \).
This is very similar to the classical realization of $eAe$-modules inside $A$-modules for an Artin algebra $A$.

**Conclusion:** There is no hope to obtain a complete description of all composition factors of $M(p, V)$ in full generality using this approach.

On can only describe the rough structure of $M(p, V)$, that is multiplicities of those simples, for which there is a projective cover in $\text{Coker}(M(p, \tilde{V}))$.

Other simples correspond to “strictly bigger annihilators”.

**Theorem 3:** (M.-Stroppel) Let $L$ be the simple top of some projective in $\text{Coker}(M(p, \tilde{V}))$ then

$$[M(p, V) : L] = [M(p, L(\lambda)) : L(\mu)]$$

where $L(\lambda)$ is a simple highest weight module with the same annihilator as $V$ and the weight $\mu$ can be described explicitly (the right hand side is combinatorially understood).

**Corollary:** $M(p, V)$ is irreducible if and only if so is $M(p, L(\lambda))$. 