SCHUR–WEYL DUALITIES FOR SYMMETRIC INVERSE SEMIGROUPS

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1. CLASSICAL SCHUR-WEYL DUALITY

$V = \mathbb{C}^n$ — $n$-dimensional complex vector space

$V^\otimes k = \underbrace{V \otimes \mathbb{C} V \otimes \mathbb{C} \cdots \otimes \mathbb{C} V}_{k \text{ factors}}$

$\text{GL}(n)$ — group of all non-degenerated complex $n \times n$ matrices

$\mathbb{C}\text{GL}(n)$ — the group algebra

$\text{GL}(n)$ acts on $V^\otimes k$ diagonally

$S_k$ — the symmetric group on $\{1, 2, \ldots, k\}$

$\mathbb{C}S_k$ — the group algebra

$S_k$ acts on $V^\otimes k$ by permuting the factors

**Schur-Weyl Duality.** The actions of $\mathbb{C}\text{GL}(n)$ and $\mathbb{C}S_k$ on $V^\otimes k$ are centralizers of each other.
2. SOME OTHER CLASSICAL SCHUR-WEYL TYPE DUALITIES

One can restrict the action of $\text{GL}(n)$ on $V$ to a subgroup $G \subset \text{GL}(n)$.

If one is lucky, one could obtain a Schur-Weyl type of duality on $V^\otimes k$ for some algebra $X = X(G)$, which is “bigger” than $\mathbb{C}S_k$:

$$g \left( V^\otimes_k \right) X$$

$X$ — is the centralizer of the $G$-action;
$\mathbb{C}G$ — is the centralizer of the $X$-action.

Some known cases:

- $G = \text{O}(n)$, the orthogonal group, then $X$ is the so-called *Brauer algebra* (R. Brauer).
- $G = S_n$, then $X$ is the so-called *partition algebra* (V. Jones, P. Martin)
3. SYMMETRIC INVERSE SEMIGROUP $\mathcal{I}S_n$

$S_n$ is the group of all bijections on $\{1, 2, \ldots, n\}$

$\mathcal{I}S_n$ is the monoid of all bijections between subsets of $\{1, 2, \ldots, n\}$

$A, B, X, Y \subset \{1, 2, \ldots, n\}$

$f : A \rightarrow B$ and $g : X \rightarrow Y$ are bijection

$g \circ f$ is a bijection from $f^{-1}(B \cap X)$ to $g(B \cap X)$ given by the composition of $f$ and $g$ whenever it makes sense to compose these maps

Example:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \circ \\
3 & 3 & \end{array}
\quad \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \\
3 & 3 & \end{array}
= \begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & \\
3 & 3 & \end{array}
\]
\( \mathcal{I}S_n \) has the *natural* faithful representation by \( n \times n \) matrices over \( \{0, 1\} \):

If \( f \in \mathcal{I}S_n \) is a bijection from \( A \) to \( B \) then the corresponding matrix \( M_f = (m_{i,j})_{i,j=1,\ldots,n} \) is defined as follows:

\[
m_{i,j} = \begin{cases} 
1, & f(j) = i; \\
0, & \text{otherwise}.
\end{cases}
\]

Hence \( \mathcal{I}S_n \) is also called the *rook monoid*

**Example:**

\[
\mathcal{I}S_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}
\]

This defines on \( V \) the *natural* structure of a module over the semigroup algebra \( \mathbb{C}\mathcal{I}S_n \).
4. DUAL SYMMETRIC INVERSE SEMIGROUP $I_n^*$

$S_n$ is the group of all bijections on \{1, 2, \ldots, n\}

$I_n^*$ is the monoid of all bijections between *quotient sets* of \{1, 2, \ldots, n\} (D. FitzGerald, J. Leech 1998)

Example of elements from $I_8^*$ and their multiplication:
The action of $\mathcal{I}_k^*$ on $V^\otimes k$ is defined as follows:

$f \in \mathcal{I}_k^*$, that is $f$ is a bijection between some decompositions (disjoint unions of non-empty subsets) 
\[
\{1, \ldots, n\} = A_1 \cup A_2 \cup \cdots \cup A_k \quad \text{and} \\
\{1, \ldots, n\} = B_1 \cup B_2 \cup \cdots \cup B_k
\]

\[
\{e_1, e_2, \ldots, e_n\} \quad \text{— the standard basis of } V
\]

\[
e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \quad \text{— a basis vector of } V^\otimes k
\]

$f(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})$ is defined as follows:

- 0 if there exists $x \neq y$ in some $A_s$ such that $e_{i_x} \neq e_{i_y}$
- $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k}$, where $j_x = i_y$ for all $x \in B_s$ and $y \in A_t$ such that $f(A_t) = B_s$.

$\mathbb{C}\mathcal{I}_k^*$ is a subalgebra of the Jones-Martin partition algebra and the above action is just the restriction of the action of partition algebra on $V^\otimes k$
5. SCHUR-WEYL DUALITY INVOLVING $\mathcal{I}S_n$ and $\mathcal{I}_k^*$

**MAIN THEOREM.**

- The action of $\mathcal{C}\mathcal{I}_k^*$ on $V^\otimes k$ gives the centralizer of the action of $\mathcal{C}\mathcal{I}S_n$.

- The action of $\mathcal{C}\mathcal{I}S_n$ on $V^\otimes k$ gives the centralizer of the action of $\mathcal{C}\mathcal{I}_k^*$.

- The representation of $\mathcal{I}S_n$ on $V^\otimes k$ is faithful.

- The representation of $\mathcal{I}_k^*$ on $V^\otimes k$ is faithful if and only if $n \geq 2$ or $k = 1$.

- The representation of $\mathcal{C}\mathcal{I}\overline{S}_n$ (the quotient modulo the zero element) on $V^\otimes k$ is faithful if and only if $k \geq n$.

- The representation of $\mathcal{C}\mathcal{I}_k^*$ on $V^\otimes k$ is faithful if and only if $k \leq n$. 


6. GENERALIZATIONS

\( \mathbb{C} \) — trivial \( \mathcal{I}S_n \)-module (all elements including zero act as the identity)

\( U = V \oplus \mathbb{C} \) — \( \mathcal{I}S_n \)-module (follows Solomon’s Schur-Weyl type dualities for \( \mathcal{I}S_n \))

There is a Schur-Weyl type duality for \( U^\otimes k \)

The object, which centralizes the action of \( \mathcal{I}S_n \) on \( U^\otimes k \) is the \textit{partial dual symmetric inverse semigroup} \( \mathcal{P}I_k^* \) (G. Kudryavtseva, V. Maltcev 2006)

Partial dual symmetric inverse semigroup consists of bijections between quotients sets of \textit{SUBSETS} of \( \{1, 2, \ldots, k\} \)

Multiplication in \( \mathcal{P}I_k^* \) can be deformed to accommodated Vernitskii’s semigroup defined in 2005.