NEW PROPERTIES AND APPLICATIONS OF GELFAND-ZETLIN MODULES

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1. Gelfand-Zetlin formulae

\[ \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}), \{e_{i,j}\} — \text{matrix units} \]

\textbf{tableau:} \([l] = (l_{i,j}), i = 1, \ldots, n, j = 1, \ldots, i.\]

\([\delta^{i,j}] — \text{Kronecker tableau.}\]

\textbf{Theorem. (Gelfand-Zetlin)} Let \( V \) be a simple finite-dimensional \( \mathfrak{g} \)-module and \((m_1, \ldots, m_n) \in \mathbb{C}^n, m_i - m_{i+1} \in \mathbb{Z}_+, \) be its highest weight. Then \( V \) has a basis, indexed by tableaux \([l]\), satisfying:

- \( l_{n,j} = m_j; \)
- \( l_{i,j} - l_{i-1,j} \in \mathbb{Z}_+; \)
- \( l_{i-1,j} - l_{i,j+1} \in \mathbb{N}. \)

Moreover, the action of the generators of \( \mathfrak{g} \) on the basis elements is given by the following \textbf{Gelfand Zetlin formulae:}
\[ e_{i,i}[l] = \left( \sum_{j=1}^{i} l_{i,j} - \sum_{j=1}^{i-1} l_{i-1,j} \right)[l] \]

\[ i = 1, \ldots, n. \]

\[ e_{i,i+1}[l] = \left( -\sum_{j=1}^{i} \prod_{k=1}^{i+1} \frac{\prod_{k=1}^{i} (l_{i,j} - l_{i+1,k})}{\prod_{k \neq i} (l_{i,j} - l_{i,k})} \right)[l + \delta^{i,j}] \]

\[ e_{i+1,i}[l] = \left( \sum_{j=1}^{i-1} \prod_{k=1}^{i} \frac{\prod_{k=1}^{i} (l_{i,j} - l_{i-1,k})}{\prod_{k \neq i} (l_{i,j} - l_{i,k})} \right)[l - \delta^{i,j}] \]

\[ i = 1, \ldots, n - 1. \]
2. Generic Gelfand-Zetlin modules

Let $[t]$ be a tableau, satisfying
\[ l_{i,j} - l_{i,k} \notin \mathbb{Z}, \quad i = 1, \ldots, n-1; j \neq k. \]

\[ P([t]) = \{ [\ell] : l_{n,j} = t_{n,j}, j = 1, \ldots, n, \text{ and } l_{i,j} - t_{i,j} \in \mathbb{Z} \text{ for all } i, j \} \]

**Theorem.** (Drozd-Futorny-Ovsienko) Gelfand-Zetlin formulae define on the vector space $V([t])$ with the basis $P([t])$ the structure of a $\mathfrak{g}$-module of finite length. The module $V([t])$ is simple if and only if $l_{i,j} - t_{i+1,k} \notin \mathbb{Z}$ for all $i = 1, \ldots, n-1, j = 1, \ldots, i$, $k = 1, \ldots, i + 1$.

$V([t])$ is called generic Gelfand-Zetlin module

The family of generic Gelfand-Zetlin module is big. The modules are parameterized by $\frac{n(n+1)}{2}$ “independent” parameters. Moreover, the following analogue of the Harish-Chandra Theorem is true.

**Theorem.** (M.) For every $u \in U(\mathfrak{g})$ there exists a generic Gelfand-Zetlin module, $V([t])$, such that $u \not\in \text{Ann}_{U(\mathfrak{g})}(V([t]))$. 

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3. Gelfand-Zetlin subalgebra

The proof of the DFO Theorem requires the following notion:

Consider \( \mathfrak{gl}(1, \mathbb{C}) \subset \mathfrak{gl}(2, \mathbb{C}) \subset \cdots \subset \mathfrak{gl}(n, \mathbb{C}) \) with respect to the left upper corner.

\[ \Gamma = \langle Z(\mathfrak{gl}(i, \mathbb{C})), i = 1, \ldots, n \rangle \subset U(\mathfrak{g}). \]

**Lemma.** (Zhelobenko?) \( \Gamma \) has a simple spectrum on every simple finite-dimensional \( \mathfrak{g} \)-module.

**Corollary.** Let \( V([t]) \) be a generic Gelfand-Zetlin module. Then \( P([t]) \) is an eigenbasis of \( V([t]) \) with respect to the action of \( \Gamma \). Moreover, this action separates the elements of \( P([t]) \).

This corollary is the main ingredient for the proof of the “finite-length part” of the DFO Theorem.

**Theorem.** (“general nonsense”—Cherednik?–Ovsienko) \( \Gamma \) is a maximal commutative subalgebra in \( U(\mathfrak{g}) \).
4. Gelfand-Zetlin modules

For a $\mathfrak{g}$-module, $V$, and $\chi \in \Gamma^*$ set

$$V_{\chi} = \{v \in V : uv = \chi(u)v \text{ for all } u \in \Gamma\}$$

$$V^\chi = \{v \in V : \text{ for all } u \in \Gamma \text{ there exists } k \in \mathbb{N} \text{ such that } (u - \chi(u))^kv = 0\}.$$ 

A $\mathfrak{g}$-module, $V$, is called a **Gelfand-Zetlin module** provided that

$$V = \bigoplus_{\chi \in \Gamma^*} V^\chi, \quad \dim(V^\chi) < \infty \text{ for all } \chi.$$ 

**Examples.**

1. Finite-dimensional modules are Gelfand-Zetlin modules.
2. Generic Gelfand-Zetlin modules are Gelfand-Zetlin modules.
3. Weight modules (with respect to a Cartan subalgebra) with finite-dimensional weight spaces are Gelfand-Zetlin modules. In particular, all highest weight modules are Gelfand-Zetlin modules.
Theorem. (Ovsienko) For every $\chi \in \Gamma^*$ there exists a simple Gelfand-Zetlin module, $V$, such that $V^\chi \neq 0$.

Theorem. (Ovsienko) For every $\chi \in \Gamma^*$ there number of pairwise non-isomorphic simple simple Gelfand-Zetlin module $V$ such that $V^\chi \neq 0$ is finite.

Theorem. (Drozd-Futorny-Ovsienko) Let $[t]$ be a tableaux defining a generic Gelfand-Zetlin module and $\chi \in \Gamma^*$ be such that $u[t] = \chi(u)[t], u \in \Gamma$. Then the number of pairwise non-isomorphic simple simple Gelfand-Zetlin module $V$ such that $V^\chi \neq 0$ equals 1 and this unique simple Gelfand-Zetlin module is a subquotient of $V([t])$.

Theorem. (Ovsienko) Let $[t]$ be a tableaux such that $l_{i,j} - l_{i,k} \not\in \mathbb{Z} \setminus 0$, $i = 1, \ldots, n$, $1 \leq j, k \leq i$ and $\chi$ be the corresponding character of $\Gamma$ (which can be computed by Zhelobenko’s formulae). Then the number of pairwise non-isomorphic simple simple Gelfand-Zetlin module $V$ such that $V^\chi \neq 0$ equals 1.

Problem. For which $\chi \in \Gamma^*$ the number of pairwise non-isomorphic simple simple Gelfand-Zetlin module $V$ such that $V^\chi \neq 0$ equals 1?
5. Connection with a localization of the Weyl algebra

For $N = n(n+1)/2$ consider the Weyl algebra $A_N$ with generators 
$\{x_{ij}, \partial_{ij} | 1 \leq j \leq i \leq n\}$ and relations $[\partial_{ij}, x_{ij}] = 1, 1 \leq j \leq i \leq n$ 
(all other commutators equal zero).

Let $S$ be the multiplicative (Ore) subset of $A_N$ generated by 
$\{\partial_{ik}, \partial_{ik}x_{ik} - \partial_{ij}x_{ij} + m | 1 \leq i \leq n-1, 1 \leq j < k \leq i, m \in \mathbb{Z}\}$.

Set $D = (A_N)_S$.

**Theorem. (Khomenko)** For $i = 1, \ldots, n-1$ set $E^+_i = e_{i,i+1}$ and $E^-_i = e_{i+1,i}$. There exists an injective algebra morphism $\varphi : U(\mathfrak{g}) \to D$ such that 
$$
\varphi(E^\pm_i) = \sum_{j=1}^i \mp \partial_{ij}^{\pm 1} \frac{\prod_{k} (\partial_{i\pm 1,k}x_{i\pm 1,k} - \partial_{ij}x_{ij})}{\prod_{k \neq j} (\partial_{ik}x_{ik} - \partial_{ij}x_{ij})}.
$$

**Corollary.** The annihilator of a simple generic Gelfand-Zetlin module $V([t])$ is generated by the central character of $V([t])$.

**Corollary. (Duflo Theorem)** The annihilator of a Verma module $V([t])$ is generated by its central character.
6. Admissible categories of Gelfand-Zetlin modules

Let $V([t])$ be a simple generic Gelfand-Zetlin module.

denote by $\mathcal{F}([t])$ the full subcategory in $\mathfrak{g}$-mod, consisting of all sub-quotients of the modules $F \otimes V([t])$, where $F$ is finite-dimensional.

**Lemma. (Futorny-König-M.)** For every finite-dimensional module $F$ the module $F \otimes V([t])$ always has length $\dim(V([t]))$ and all simple subquotients of this module are simple generic Gelfand-Zetlin modules.

**Theorem. (Futorny-König-M.)** The category $\mathcal{F}([t])$ decomposes into a direct sum of module categories for local finite-dimensional associative algebras.

**Theorem. (König-M.)** The finite-dimensional associative algebras above are isomorphic to the endomorphism algebras of the “big projective” modules in the category $\mathcal{O}$ for $\mathfrak{g}$.

**Theorem. (König-M.)** The category $\mathcal{F}([t])$ is equivalent a sub-category of the category $\mathcal{O}$ consisting of module, complete in the sense of Enright.
7. Parabolic categories $\mathcal{O}(p, \Lambda)$ and generalized Verma modules

$\mathfrak{g}$ – semi-simple finite-dimensional complex Lie algebra, $p \subset g$ – parabolic subalgebra with the Levi decomposition $p = g \oplus h^g \oplus n$.

Denote $\mathcal{O}(p, \Lambda)$ the full subcategory in $\mathfrak{g}$-mod, which consists of all finitely generated, $h^g$-diagonalizable and $U(n)$-finite modules, which decompose into a (possibly infinite) direct sum of modules from $\Lambda = \mathcal{F}([t])$ as $g$-modules.

Typical objects in $\mathcal{O}(p, \Lambda)$: the **generalized Verma module**

$$M(p, \lambda, [l]) = U((g)) \otimes_{U(p)} V([l]),$$

where $\lambda \in (h^g)^*$, $V([l])$ is a simple generic Gelfand-Zetlin module from $\mathcal{F}([t])$ and the $p$-module structure on $V([l])$ is defined as follows: $g$ acts in the natural way, $h^g$ acts via $\lambda$, $n$ acts trivially.

**Theorem. (Khomenko)** The annihilator of $M(p, \lambda, [l])$ is generated by its central character.

**Theorem. (Khomenko-M.-Ovsienko)** Let $V'$ be a simple Verma $g$-module with the same central character as $V([l])$. Then the module $M(p, \lambda, [l])$ is simple if and only if the Verma module $M(p, \lambda, V') = U((g)) \otimes_{U(p)} V'$ is simple.
Theorem. (Futorny-König-M.)
The category $\mathcal{O}(\mathfrak{p}, \Lambda)$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a finite-dimensional properly stratified (in the sense of Dlab) algebra. Moreover, each of these direct summands is also equivalent to a direct summand in the category of Harish-Chandra bimodules for $\hat{\mathfrak{g}}$ with a right scalar action of the center.

Corollary. There is a map, $f : \{(\lambda, [\ell])\} \to \mathfrak{h}^*$, such that the following is true:

$$(M(\mathfrak{p}, \lambda, [\ell]) : M(\mathfrak{p}, \lambda', [\ell'])) = (M(f(\lambda, [\ell])) : M(f(\lambda', [\ell']))).$$

Remark that the latter multiplicities are given by KL-polynomials).

Theorem. (Khomenko-M.) The modules $M(\mathfrak{p}, \lambda, [\ell])$ are rigid, that is their radical and socle filtrations coincide. The multiplicities of simple subquotients in these filtrations are coefficients in the corresponding KL-polynomials.
8. Remarks and open problems.

Remarks.

1. Almost all results can be naturally generalized to orthogonal algebras \((B_n\) and \(D_n\) cases), where the classical result is due to Gelfand and Zetlin as well.

2. Gelfand-Zetlin formula for finite-dimensional modules in \(C_n\) case are recently obtained by Molev. However it is not quite clear even how to construct simple generic Gelfand-Zetlin modules in this case.

3. There is a natural analogue of the construction and the theory for (generic) Gelfand-Zetlin modules over \(U_q(\mathfrak{gl}(n, \mathbb{C}))\).

4. Gelfand-Zetlin type formulae were used to classify unitarizable modules over many \(*\)-algebras, associated with \(U(\mathfrak{gl}(n, \mathbb{C}))\) and \(U_q(\mathfrak{gl}(n, \mathbb{C}))\) (see Ottoson, Klimyk, Jorgov and others).

5. For non-standard quantum deformations of, say, orthogonal algebras, Gelfand-Zetlin method does not give a complete classification of for example unitarizable modules, whereas for Drinfeld-Jimbo deformation is usually works perfectly.
Problems.

1. Classify and give a precise construction of all simple Gelfand-Zetlin modules.

2. Find sufficient and necessary conditions for a character of $\Gamma$ to have only one extension to a simple $g$-module.

3. Let $F$ be a simple finite-dimensional $g$-modules. Consider the usual Gelfand-Zetlin basis in $F$ and the Gelfand-Zetlin basis with respect to the “backward” Gelfand-Zetlin subalgebra, associated with inclusions of $gl(i, \mathbb{C})$ into $gl(i+1, \mathbb{C})$ with respect to the right lower corner. Find a transformation matrix between these two bases. Equivalently, find a transformation matrix between the Gelfand-Zetlin basis and, for example, the canonical basis.

4. Describe the subquotients in the module $F \otimes V$, where $F$ is a finite-dimensional and $V$ is a simple Gelfand-Zetlin module.

5. Gelfand-Zetlin method for exceptional algebras????