NEW PROPERTIES AND APPLICATIONS OF GELFAND-ZETLIN MODULES

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1. Gelfand-Zetlin formulae

$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $\{e_{i,j}\}$ — matrix units

tableau: $[l] = (l_{i,j})$, $i = 1, \dots, n$, $j = 1, \dots, i$.

$[\delta^{i,j}]$ — Kronecker tableau.

Theorem. (Gelfand-Zetlin) Let V be a simple finite-dimensional $\mathfrak{g}$-module and $(m_1, \dots, m_n) \in \mathbb{C}^n$, $m_i - m_{i+1} \in \mathbb{Z}_+$, be its highest weight. Then V has a basis, indexed by tableaux $[l]$, satisfying:

- $l_{n,j} = m_j$;
- $l_{i,j} - l_{i-1,j} \in \mathbb{Z}_+$;
- $l_{i-1,j} - l_{i,j+1} \in \mathbb{N}$.

Moreover, the action of the generators of $\mathfrak{g}$ on the basis elements is given by the following **Gelfand Zetlin formulae:**

$$e_{i,i}[l] = \left(\sum_{j=1}^{i} l_{i,j} - \sum_{j=1}^{i-1} l_{i-1,j} \right) [l]$$

$$i = 1, \ldots, n.$$

$$e_{i,i+1}[l] = \left(- \sum_{j=1}^{i} \frac{\prod_{k=1}^{i+1} (l_{i,j} - l_{i+1,k})}{\prod_{k \neq i} (l_{i,j} - l_{i,k})} \right) [l + \delta^{i,j}];$$

$$e_{i+1,i}[l] = \left(\sum_{j=1}^{i} \frac{\prod_{k=1}^{i-1} (l_{i,j} - l_{i-1,k})}{\prod_{k \neq i} (l_{i,j} - l_{i,k})} \right) [l - \delta^{i,j}]$$

$$i = 1, \ldots, n-1.$$

2. Generic Gelfand-Zetlin modules

Let $[t]$ be a tableau, satisfying

$$l_{i,j} - l_{i,k} \notin \mathbb{Z}, \quad i = 1, \ldots, n-1; j \neq k.$$

$$P([t]) = \{[l] : l_{n,j} = t_{n,j}, j = 1, \ldots, n, \text{ and } l_{i,j} - t_{i,j} \in \mathbb{Z} \text{ for all } i, j\}$$

Theorem. (Drozd-Futorny-Ovsienko) Gelfand-Zetlin formulae define on the vector space $V([t])$ with the basis $P([t])$ the structure of a $\mathfrak{g}$-module of finite length. The module $V([t])$ is simple if and only if $l_{i,j} - t_{i+1,k} \notin \mathbb{Z}$ for all $i = 1, \ldots, n-1$, $j = 1, \ldots, i$, $k = 1, \ldots, i+1$.

$V([t])$ is called **generic Gelfand-Zetlin module**

The family of generic Gelfand-Zetlin module is big. The modules are parameterized by $\frac{n(n+1)}{2}$ "independent" parameters. Moreover, the following analogue of the Harish-Chandra Theorem is true.

Theorem. (M.) For every $u \in U(\mathfrak{g})$ there exists a generic Gelfand-Zetlin module, $V([t])$, such that $u \notin \mathrm{Ann}_{U(\mathfrak{g})}(V([t]))$.

3. Gelfand-Zetlin subalgebra

The proof of the DFO Theorem requires the following notion:

Consider $\mathfrak{gl}(1,\mathbb{C}) \subset \mathfrak{gl}(2,\mathbb{C}) \subset \cdots \subset \mathfrak{gl}(n,\mathbb{C})$ with respect to the left upper corner.

$\Gamma = \langle Z(\mathfrak{gl}(i,\mathbb{C})), i = 1,\ldots,n\rangle \subset U(\mathfrak{g})$.

Lemma. (Zhelobenko?) Γ has a simple spectrum on every simple finite-dimensional $\mathfrak{g}$-module.

Corollary. Let $V([t])$ be a generic Gelfand-Zetlin module. Then $P([t])$ is an eigenbasis of $V([t])$ with respect to the action of Γ. Moreover, this action separates the elements of $P([t])$.

This corollary is the main ingredient for the proof of the "finite-length part" of the DFO Theorem.

Theorem. ("general nonsense"–Cherednik?–Ovsienko) Γ is a maximal commutative subalgebra in $U(\mathfrak{g})$.

4. Gelfand-Zetlin modules

For a $\mathfrak{g}$-module, V, and $\chi \in \Gamma^*$ set

$$V_\chi = \{v \in V : uv = \chi(u)v \text{ for all } u \in \Gamma\}$$

$$V^\chi = \{v \in V : \text{ for all } u \in \Gamma \text{ there exists } \\ k \in \mathbb{N} \text{ such that } (u - \chi(u))^k v = 0\}.$$

A $\mathfrak{g}$-module, V, is called a **Gelfand-Zetlin module** provided that

$$V = \bigoplus_{\chi \in \Gamma^*} V^\chi, \qquad \dim(V^\chi) < \infty \text{ for all } \chi.$$

Examples.

1. Finite-dimensional modules are Gelfand-Zetlin modules.
2. Generic Gelfand-Zetlin modules are Gelfand-Zetlin modules.
3. Weight modules (with respect to a Cartan subalgebra) with finite-dimensional weight spaces are Gelfand-Zetlin modules. In particular, all highest weight modules are Gelfand-Zetlin modules.

Theorem. (Ovsienko) For every $\chi \in \Gamma^*$ there exists a simple Gelfand-Zetlin module, V, such that $V^\chi \neq 0$.

Theorem. (Ovsienko) For every $\chi \in \Gamma^*$ there number of pairwise non-isomorphic simple simple Gelfand-Zetlin module V such that $V^\chi \neq 0$ is finite.

Theorem. (Drozd-Futorny-Ovsienko) Let $[t]$ be a tableaux defining a generic Gelfand-Zetlin module and $\chi \in \Gamma^*$ be such that $u[t] = \chi(u)[t]$, $u \in \Gamma$. Then the number of pairwise non-isomorphic simple simple Gelfand-Zetlin module V such that $V^\chi \neq 0$ equals 1 and this unique simple Gelfand-Zetlin module is a subquotient of $V([t])$.

Theorem. (Ovsienko) Let $[t]$ be a tableaux such that $l_{i,j} - l_{i,k} \notin \mathbb{Z} \setminus 0$, $i = 1, \dots, n$, $1 \leq j, k \leq i$ and χ be the corresponding character of Γ (which can be computed by Zhelobenko's formulae). Then the number of pairwise non-isomorphic simple simple Gelfand-Zetlin module V such that $V^\chi \neq 0$ equals 1.

Problem. For which $\chi \in \Gamma^*$ the number of pairwise non-isomorphic simple simple Gelfand-Zetlin module V such that $V^\chi \neq 0$ equals 1?

5. Connection with a localization of the Weyl algebra

For $N = n(n+1)/2$ consider the Weyl algebra A_N with generators $\{x_{ij}, \partial_{ij} | 1 \leq j \leq i \leq n\}$ and relations $[\partial_{ij}, x_{ij}] = 1$, $1 \leq j \leq i \leq n$ (all other commutators equal zero).

Let S be the multiplicative (Ore) subset of A_N generated by

$$\{\partial_{ik}, \partial_{ik}x_{ik} - \partial_{ij}x_{ij} + m | 1 \leq i \leq n-1, 1 \leq j < k \leq i, m \in \mathbb{Z}\}.$$

Set $D = (A_N)_{\mathcal{S}}$.

Theorem. (Khomenko) For $i = 1, \ldots, n-1$ set $E_i^+ = e_{i,i+1}$ and $E_i^- = e_{i+1,i}$. There exists an injective algebra morphism $\varphi : U(\mathfrak{g}) \to D$ such that

$$\varphi(E_i^{\pm}) = \sum_{j=1}^{i} \mp \partial_{ij}^{\mp 1} \frac{\prod_k (\partial_{i\pm 1,k}x_{i\pm 1,k} - \partial_{ij}x_{ij})}{\prod_{k \neq j}(\partial_{ik}x_{ik} - \partial_{ij}x_{ij})}.$$

Corollary. The annihilator of a simple generic Gelfand-Zetlin module $V([t])$ is generated by the central character of $V([t])$.

Corollary. (Duflo Theorem) The annihilator of a Vemra module $V([t])$ is generated by its central character.

6. Admissible categories of Gelfand-Zetlin modules

Let $V([t])$ be a simple generic Gelfand-Zetlin module.

denote by $\mathcal{F}([t])$ the full subcategory in $\mathfrak{g}$-mod, consisting of all subquotients of the modules $F \otimes V([t])$, where F is finite-dimensional.

Lemma. (Futorny-König-M.) For every finite-dimensional module F the module $F \otimes V([t])$ always has length $\dim(V([t]))$ and all simple subquotients of this module are simple generic Gelfand-Zetlin modules.

Theorem. (Futorny-König-M.) The category $\mathcal{F}([t])$ decomposes into a direct sum of module categories for local finite-dimensional associative algebras.

Theorem. (König-M.) The finite-dimensional associative algebras above are isomorphic to the endomorphism algebras of the "big projective" modules in the category $\mathcal{O}$ for $\mathfrak{g}$.

Theorem. (König-M.) The category $\mathcal{F}([t])$ is equivalent a subcategory of the category $\mathcal{O}$ consisting of module, complete in the sense of Enright.

7. Parabolic categories $\mathcal{O}(\mathfrak{p}, \Lambda)$ and generalized Verma modules

$\tilde{\mathfrak{g}}$ – semi-simple finite-dimensional complex Lie algebra, $\mathfrak{p} \subset \mathfrak{g}$ – parabolic subalgebra with the Levi decomposition $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{h}^{\mathfrak{g}} \oplus \mathfrak{n}$.

Denote $\mathcal{O}(\mathfrak{p}, \Lambda)$ the full subcategory in $\tilde{\mathfrak{g}}$-mod, which consists of all finitely generated, $\mathfrak{h}^{\mathfrak{g}}$-diagonalizable and $U(\mathfrak{n})$-finite modules, which decompose into a (possibly infinite) direct sum of modules from $\Lambda = \mathcal{F}([t])$ as $\mathfrak{g}$-modules.

Typical objects in $\mathcal{O}(\mathfrak{p}, \Lambda)$: the **generalized Verma module**

$$M(\mathfrak{p}, \lambda, [l]) = U((g)) \bigotimes_{U(\mathfrak{p})} V([l]),$$

where $\lambda \in (\mathfrak{h}^{\mathfrak{g}})^*$, $V([l])$ is a simple generic Gelfand-Zetlin module from $\mathcal{F}([t])$ and the $\mathfrak{p}$-module structure on $V([l])$ is defined as follows: $\mathfrak{g}$ acts in the natural way, $\mathfrak{h}^{\mathfrak{g}}$ acts via λ, $\mathfrak{n}$ acts trivially.

Theorem. (Khomenko) The annihilator of $M(\mathfrak{p}, \lambda, [l])$ is generated by its central character.

Theorem. (Khomenko-M.-Ovsienko) Let V' be a simple Verma $\mathfrak{g}$-module with the same central character as $V([l])$. Then the module $M(\mathfrak{p}, \lambda, [l])$ is simple if and only if the Verma module $M(\mathfrak{p}, \lambda, V') = U((g)) \otimes_{U(\mathfrak{p})} V'$ is simple.

Theorem. (Futorny-König-M.)
The category $\mathcal{O}(\mathfrak{p}, \Lambda)$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a finite-dimensional properly stratified (in the sense of Dlab) algebra. Moreover, each of these direct summands is also equivalent to a direct summand in the category of Harish-Chandra bimodules for $\tilde{\mathfrak{g}}$ with a right scalar action of the center.

Corollary. There is a map, $f : \{(\lambda, [l])\} \to \mathfrak{h}^*$, such that the following is true:

$$(M(\mathfrak{p}, \lambda, [l]) : M(\mathfrak{p}, \lambda', [l'])) = (M(f(\lambda, [l])) : M(f(\lambda', [l']))).$$

(remark that the latter multiplicities are given by KL-polynomials).

Theorem. (Khomenko-M.) The modules $M(\mathfrak{p}, \lambda, [l])$ are rigid, that is their radical and socle filtrations coincide. The multiplicities of simple subquotients in these filtrations are coefficients in the corresponding KL-polynomials.

8. Remarks and open problems.

Remarks.

1. Almost all results can be naturally generalized to orthogonal algebras (B_n and D_n cases), where the classical result is due to Gelfand and Zetlin as well.

2. Gelfand-Zetlin formula for finite-dimensional modules in C_n case are recently obtained by Molev. However it is not quite clear even how to construct simple generic Gelfand-Zetlin modules in this case.

3. There is a natural analogue of the construction and the theory for (generic) Gelfand-Zetlin modules over $U_q(\mathfrak{gl}(n, \mathbb{C}))$.

4. Gelfand-Zetlin type formulae were used to classify unitarizable modules over many $*$-algebras, associated with $U(\mathfrak{gl}(n, \mathbb{C}))$ and $U_q(\mathfrak{gl}(n, \mathbb{C}))$ (see Ottoson, Klimyk, Jorgov and others).

5. For non-standard quantum deformations of, say, orthogonal algebras, Gelfand-Zetlin method does not give a complete classification of for example unitarizable modules, whereas for Drinfeld-Jimbo deformation is usually works perfectly.

Problems.

1. Classify and give a precise construction of all simple Gelfand-Zetlin modules.
2. Find sufficient and necessary conditions for a character of Γ to have only one extension to a simple $\mathfrak{g}$-module.
3. Let F be a simple finite-dimensional $\mathfrak{g}$-modules. Consider the usual Gelfand-Zetlin basis in F and the Gelfand-Zetlin basis with respect to the "backward" Gelfand-Zetlin subalgebra, associated with inclusions of $\mathfrak{gl}(i, \mathbb{C})$ into $\mathfrak{gl}(i+1, \mathbb{C})$ with respect to the right lower corner. Find a transformation matrix between these two bases. Equivalently, find a transformation matrix between the Gelfand-Zetlin basis and, for example, the canonical basis.
4. Describe the subquotients in the module $F \otimes V$, where F is a finite-dimensional and V is a simple Gelfand-Zetlin module.
5. Gelfand-Zetlin method for exceptional algebras????