**SL_2-CATEGORIZATION**

1. \( n \in \{1, 2, 3, \ldots, 3 \} \) fixed.
   \[ C_1 = C_n := \mathbb{C}[X_1, \ldots, X_n] / (\text{homog. } S_n \text{-inv., deg} \geq 0) \]
   \[ \text{COINVARIANTS} \]

2. \( S_n = \langle s_{i, i+1} s_{i+1, i} s_{i, i+1} \rangle \leq S_n \)
   \[ \text{COXETER GENERATORS} \]

For \( i = 0, 1, \ldots, n \) define:

\[ C_i^n := (C_n)^i := C \]

\[ S_i^n = \langle s_{i, i+1} s_{i+1, i} s_{i, i+1} \rangle \leq S_n \]

\[ C_i^n := S_i^n \text{-invariants in } C_n \]

For \( I = \{i_1, \ldots, i_k \} \subset \{1, \ldots, n-1\} \)

\[ S_I^n = S_I^{i_1, \ldots, i_k} := \langle s_j \mid j \not\in I \rangle \]

\[ C_I^n := C_{i_1, \ldots, i_k} : = S_I^n \text{-invariants in } C_n \]

**Fact:**

1. \( C_I^n \) symmetric
2. \( C_I^n \) is free both as a \( C_n \) and as a \( C_i^{i+1} \) module.
3. \( \dim C_i^n = \binom{n}{i} \)
2. **Category** $C_n$

**Objects:** $-n, -n+2, \ldots, n-2, n$

$C_n^{\text{mod}} \rightarrow C_n^{\text{mod}} \rightarrow C_n^{\text{mod}} \rightarrow C_n^{\text{mod}} \rightarrow C_n^{\text{mod}}$

**Generating 1-morphisms:**

$\text{Res} \circ \text{Ind} \circ \text{Ind} \circ \text{Res}$

**Claim:** $C_n$ is FIAT

**Remark:** The fact that $C_n^{\text{til}}$ and $C_n^{\text{til}}$ are symmetric makes induction biadjoint to the restriction.
ALTERNATIVE DESCRIPTION

\[ \mathfrak{gl}_n = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \]

\( \mathfrak{O} = \text{BGG - CATEGORY } \mathfrak{O} \text{ FOR } \mathfrak{gl}_n \)

(\( \mathfrak{h} \)-DIAGONALIZABLE, \( \mathfrak{n}_+ \)-LOC. FINITELY GENERATED).

For \( i = 0, 1, \ldots, n \) define block \( \mathfrak{O}^i \) as the block with simples

\[ \text{L}(\lambda), \ \lambda = (\varepsilon_1, \ldots, \varepsilon_n), \ \varepsilon_j \in \{0, 1\} \forall j \]

\[ \text{if } j : \varepsilon_j = 13 \text{ then } i \]

\( \mathfrak{O}^i \) has unique indecomposable projective-injective \( P_i \).

\[ \text{Fact (Soergel)} \quad \text{End}(P_i) = C_i \]

\[ \Rightarrow C_i \text{-mod} \approx \text{MODULES WITH PROJ. -INS. PRESENTATION IN } \mathfrak{O}^i \]

\[ \text{Res}^i_0 \left( \left( \bigotimes C^n \right) \right) \]

\[ \text{Res}^{i+1} \left( \left( \bigotimes C^n \right) \right) \]

To get \( C_n \) we restrict this picture to proj-inj presented modules.
**Rem:** $C_n$ is defined via $U$ by Bernstein - Frenkel - Khovanov. Reformulation in $C_n^i$-language follows Frenkel - Khovanov - Stroppel “Equivalent” category appears in Chuang - Rouquier (the latter differs by a special choice of 2-morphisms). More general versions! Lauda, Khovanov - Lauda - Rouquier.

**4)** Indecomposable 1-mor in $C_n$

$$C^i_n \text{-mod} \xrightarrow{E_i^{(k)}} C^j_n \text{-mod}$$

Define $F_j^{(k)}$ as adjoint to $E_i^{(k)}$

**Lemma.** (1) Functors $E_i^{(k)}$ and $F_j^{(k)}$ are indecomposable

(2) $E_i^{(k)} \circ E_i^{(k)} \circ \ldots \circ E_i^{(k)} = k! \cdot E_i^{(k)}$
THM [BFK]

The following is a complete and irredundent list of indecomp. 1-mor in $C_n$: (from $A$ to $M$)

Ex $n=2$ 2-sided cells!
**Theorem [BFK]**

We have functorial $SL_2$-relations!

\[
E_{\lambda-2}^{(n)} \circ F_{\lambda}^{(n)} = q \cdot \text{Id} \oplus F_{\lambda+2}^{(n)} E_{\lambda}^{(n)}, \lambda > 0
\]

\[
E_{\lambda-2}^{(n)} \circ F_{\lambda}^{(n)} \oplus (-q) \cdot \text{Id} = F_{\lambda+2}^{(n)} E_{\lambda}^{(n)}, \lambda \leq 0
\]

**Remark:** Chuang-Rouquier's and Lauda's categorification of the idempotent completion $U(sl_2)$ maps into $C_n$ when considering the action on the minimal model for the categorification of the $\mathfrak{sl}_2$-dimensional simple $SL_2$-module (the latter is the defining representation of $C_n$).

Categorification of $U(sl_n)$ and other 2-Kac-Moody algebras are not flat (infinitely many objects).
Prop [BEK, CR]

(a) \(1_\lambda\) and \(1_{-\lambda}\) are in the same 2-sided cell

(b) There is a bijection

\[
\begin{align*}
\text{2-sided cells} & \quad \leftrightarrow \quad \lambda \in \{-n, -n+2, \ldots, n\} \\
\text{2-sided cell} & \quad \text{containing } 1_{\lambda}
\end{align*}
\]

Lemma

(a) Two different left cells inside the same 2-sided cell are not comparable w.r.t. \(\leq\)

(b) \(|L \cap K| = 1 \quad \forall L, K \subset J\)

(c) \(F \mapsto m_{F,F}\) is constant on right cells.

Cor.

(a) \(\forall L, L' \subset J \implies \overline{\Gamma}_L\) equivalent to \(\overline{\Gamma}_{L'}\)

(b) The defining \(\overline{\text{RER}}\) is equivalent to \(\overline{\text{C}}_L\) for \(L \geq 1_n\)
OTHER CELL MODULES

\[ S_1, S_2, \ldots, S_{n-1}, S_{n-1}' \]

\[ S_1', S_2', S_3', \ldots, S_{n-2}' \]

\[ S_1, S_2, S_3' \]

\[ S_1', S_2', S_3' \]

\[ C_{n-2} \text{ is a quotient of } C_n \text{ via } x_1 x_n = 0 \]

\[ C_{n-2} \text{ -mod } C_n \text{ -mod} \]

\[ \bigoplus_{i=0}^{n-1} C_{n-2} \text{ -mod } \bigoplus_{i=0}^{n-1} C_{n} \text{ -mod} \]

\[ \text{DEFINING REP} \]

\[ \text{DOES NOT WORK.} \]

\[ \text{BUT: } C_n \rightarrow C_{n-2} \text{ AND } C_{n-1} \rightarrow C_{n-2} \]

\[ \text{DEFINES} \]

\[ \text{Ker } C_n \rightarrow C_n \rightarrow C_{n-2} \]

**LEMMA** Ker contains \( \text{id}_{1n} \) and hence \( \text{id}_F \) for \( F \subset \text{id}_{1n} \).

**Q:** Ker is generated by \( \text{id}_{1n} \).

**NO!**
Cor. Other cell modules for \( C_n \) are equivalent to the defining representation of \( C_k \), \( k = n-2, n-4, \ldots \).

Rem: We have:
\[ C_0 \leftarrow C_2 \leftarrow C_4 \leftarrow \ldots \]

\( U(sl_2) \) maps (subjects?) compatibly into every component.

Note Chuang-Rouquier and Lauda specify some 2-mor for their 2-categories. We take all!

Cor [CR] Uniqueness of the minimal model for the category of the simple \((n+1)\)-dim module.

Rem Graded version replaces \( S_n \) by \( H_n(q) \) and \( sl_2 \) by \( U_q(sl_2) \).