CATEGORY $\mathcal{O}$ AS A SOURCE FOR CATEGORIFICATION

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$\mathfrak{g}$ — semi-simple finite-dimensional Lie algebra over $\mathbb{C}$

$\mathcal{O}$ — Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for $\mathfrak{g}$

$\mathcal{O}_0$ — the regular block of $\mathcal{O}$

$\mathcal{O}_0 \cong A$-mod, where

$A$ — finite-dimensional associative algebra over $\mathbb{C}$

$W$ — Weyl group of $\mathfrak{g}$

$A$-simples $\leftrightarrow$ elements in $W$
$K_0(\mathcal{O}_0)$ — Grothendieck group of $\mathcal{O}_0$

Projective functors are exact and thus induce endomorphisms of $K_0(\mathcal{O}_0)$

This is a categorification of the regular right $\mathbb{Z}W$-module

Twisting functors satisfy braid relations

They are NOT equivalences of $\mathcal{O}_0$

Derived twisting functors ARE equivalences of $\mathcal{D}^b(\mathcal{O}_0)$

This gives a categorification of the regular left $\mathbb{Z}W$-module

Twisting and projective functors commute

This gives a categorification of $\mathbb{Z}W \mathbb{Z}W \mathbb{Z}W$
$A$ admits a $\mathbb{Z}$-grading

$A$-gmod — category of graded $A$-modules

all our functors admit graded lifts

$\mathcal{H}$ — Hecke algebra of $W$

effect on categorification: change $\mathbb{Z}W$ by $\mathcal{H}$

$A$ is Koszul
Taking certain subcategories one produces categorifications of other modules

\( \mathfrak{p} \) — some parabolic subalgebra of \( \mathfrak{g} \)

\( W^p \) — the corresponding parabolic subgroup of \( W \)

\( \mathcal{O}_0^p \) — the corresponding parabolic subcategory of \( \mathcal{O}_0 \)

Projective functors preserve \( \mathcal{O}_0^p \)

This gives a categorification of the \( \mathbb{Z}W \)-module, induced from the sign \( W^p \)-module
\( \mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \)

\( \lambda \) — partition corresponding to \( p \)

\( \lambda' \) — the conjugate partition

\( Q \) — basic projective-injective module in \( \mathcal{O}_0^p \)

End\((Q)\)-mod can be viewed as a subcategory of \( \mathcal{O}_0^p \)

projective functors preserve End\((Q)\)-mod

This gives a categorification of the Specht module corresponding to \( \lambda' \)
\( W(p) \) — longest coset representatives in \( W^p \setminus W \)

\( e_w \) — primitive idempotent of \( A \) corresponding to \( w \in W \)

\[ e_p = \sum_{w \in W(p)} e_w \]

\[ B = e_p A e_p \]

\( B \)-mod can be realized as a subcategory of \( A \)-mod

projective functors preserve \( B \)-mod

This gives a categorification of the permutation module corresponding to \( W \) and \( W^p \)
$\mathfrak{R} \rightleftharpoons$ right cell in $W$

for $w \in \mathfrak{R}$ set $P^R(w) = P(w)/X$, where

$X \subset P(w)$ is generated by all $L(v), v \not\leq_{\text{right}} w$

$P^R = \bigoplus_{w \in \mathfrak{R}} P^R(w)$

$C = \text{End}(P^R)$

$C$-mod can be realized as a subcategory of $A$-mod

projective functors preserve $C$-mod

This gives a categorification of the cell module corresponding to $\mathfrak{R}$