COMBINATORICS OF PARTIAL BIJECTIONS

1. The symmetric group S_n

$$n \in \mathbb{N}, \qquad N_n = \{1, 2, \ldots, n\}$$

$$S_n = \{f : N_n \to N_n \ : \ f \text{ is bijective}\}$$

Elements from S_n are called *permutations* on N_n.

$$f, g \in S_n \Rightarrow f \circ g \in S_n, \quad f \circ g(x) = f(g(x)) \text{ for all } x \in N_n$$

$(S_n, \circ)$ is a group. This means the following:

1. The operation "$\circ$" is associative, i.e. $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f, g, h \in S_n$. In fact, $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x) = f(g(h(x)))$ for all $x \in N_n$.

2. S_n contains the identity permutation id, defined as follows: $id(x) = x$ for all $x \in S_n$. This transformation satisfies $id \circ f = f \circ id = f$ for all $f \in S_n$.

3. Every permutation $f \in S_n$ has an inverse in S_n, this means that there exists a permutation, $g \in S_n$, such that $g \circ f = f \circ g = id$.

Remark. It is easy to see that the identity is unique and every $f \in S_n$ has unique inverse, which is denoted by f^{-1}.

Example. The group S_3:

$$id = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$(1,2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$(1,3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$(2,3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$(1,2,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$(1,3,2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

2. The full inverse symmetric semigroup $\mathcal{IS}_n$

Let X and Y be arbitrary sets.

A partial map from X to Y is a map, f, from $A \subset X$ to $B \subset Y$. If $f : A \to B$ is a bijection it is called a **partial bijection from X to Y**.

$$\mathcal{IS}_n = \{f : N_n \to N_n \,:\, f \text{ is a partial bijection } \}.$$

Elements from $\mathcal{IS}_n$ are called *partial permutations* on N_n.

Let $f : X \to Y$ and $g : Y \to Z$ be partial maps. Define the **composition** $g \circ f : X \to Z$ as the following partial map: $g \circ f$ is defined on $x \in X$ if and only if f is defined on x and g is defined on $f(x)$; moreover, if $g \circ f$ is defined on $x \in X$, then $(g \circ f)(x) = g(f(x))$.

If $f : X \to Y$ is a partial map and $x \in X$ is such that f is not defined on x, denote this by $f(x) = \varnothing$.

$(\mathcal{IS}_n, \circ)$ is an inverse semigroup (monoid). This means the following:

1. The operation "$\circ$" is associative, i.e. $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f, g, h \in \mathcal{IS}_n$.

2. $id \in \mathcal{IS}_n$ and $id \circ f = f \circ id = f$ for all $f \in \mathcal{IS}_n$.

3. Every permutation $f \in S_n$ has a *partial inverse* in S_n, i.e. that there exists a unique permutation, $g \in S_n$, such that $g \circ f \circ g = g$ and $f \circ g \circ f = f$. This unique partial inverse is denoted by f^*.

Example. The semigroup $\mathcal{IS}_3$:

$$id = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad (1,2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad (1,3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$(2,3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad (1,2,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad (1,3,2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$[1] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 2 & 3 \end{pmatrix}, \quad (2,3)[1] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 3 & 2 \end{pmatrix}, \quad [2,3,1] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 3 & 1 \end{pmatrix}$$

$$[3,2,1] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 1 & 2 \end{pmatrix}, \quad [3,1] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 2 & 1 \end{pmatrix}, \quad [2,1] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 1 & 3 \end{pmatrix}$$

$$[2] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & \varnothing & 3 \end{pmatrix}, \quad (1,3)[2] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \varnothing & 1 \end{pmatrix}, \quad [1,3,2] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \varnothing & 2 \end{pmatrix}$$

$$[3,1,2] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \varnothing & 1 \end{pmatrix}, \quad [3,2] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & \varnothing & 2 \end{pmatrix}, \quad [1,2] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \varnothing & 3 \end{pmatrix}$$

$$[3] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & \varnothing \end{pmatrix}, \quad (1,2)[3] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & \varnothing \end{pmatrix}, \quad [1,2,3] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & \varnothing \end{pmatrix}$$

$$[2,1,3] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & \varnothing \end{pmatrix}, \quad [2,3] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & \varnothing \end{pmatrix}, \quad [1,3] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & \varnothing \end{pmatrix}$$

$$[1][2] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & \varnothing & 3 \end{pmatrix}, \quad [3,1][2] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & \varnothing & 1 \end{pmatrix}, \quad [1][3,2] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & \varnothing & 2 \end{pmatrix},$$

$$[1][3] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 2 & \varnothing \end{pmatrix}, \quad [2,1][3] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 1 & \varnothing \end{pmatrix}, \quad [1][2,3] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & 1 & \varnothing \end{pmatrix},$$

$$[2][3] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & \varnothing & \varnothing \end{pmatrix}, \quad [1,2][3] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \varnothing & \varnothing \end{pmatrix}, \quad [2][1,3] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \varnothing & \varnothing \end{pmatrix},$$

$$0 = [1][2][3] = \begin{pmatrix} 1 & 2 & 3 \\ \varnothing & \varnothing & \varnothing \end{pmatrix}.$$

3. Basic combinatorics

$$|S_n| = n \cdot (n-1) \cdot (n-2) \ldots 1 = n!.$$

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ n \text{ poss.} & n-1 \text{ poss.} & n-2 \text{ poss.} & \ldots & 2 \text{ poss.} & 1 \text{ poss.} \end{pmatrix}$$

How to calculate $|\mathcal{IS}_n|$?

$\pi \in \mathcal{IS}_n$, that is $\pi : A \to B$ is a bijection, $A, B \subset N_n$. Then

- $A = \mathrm{Dom}(\pi)$, the *domain* of π;
- $B = \mathrm{Im}(\pi)$, the *image* of π;
- $|A| = |B| = \mathrm{Rank}(\pi)$, the *rank* of π.
- $n - \mathrm{Rank}(\pi) = \mathrm{Def}(\pi)$, the *defect* of π.

- Count the number of elements of a fixed rank i separately.
- If i is fixed, the domain A can be chosen in $\binom{n}{i}$ different ways.
- If i is fixed, the image B can be chosen in $\binom{n}{i}$ different ways.
- If A and B of cardinality i are fixed, a bijection from A to B can be chosen in $i!$ different ways.

$$\text{Answer:} \qquad |S_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$$

4. Nilpotent elements

$\mathcal{IS}_n$ contains 0, that is a special element, such that $0 \circ f = f \circ 0 = 0$ for all $f \in |\mathcal{IS}_n|$.

0 is unique, $\mathrm{Dom}(0) = \mathrm{Im}(0) = \varnothing$.

$A \in \mathcal{IS}_n$ is said to be *nilpotent* provided that $A^n = 0$ for some positive integer n.

$\pi \in \mathcal{IS}_n$ is nilpotent if and only if π does not have cycles if and only if $\pi = [x_1, \dots, x_i] \dots [y_1, \dots, y_j]$.

How many nilpotent elements does $\mathcal{IS}_n$ have?

Lemma. The number of nilpotent elements of a fixed defect k, $0 < k \leq n$, equals the signless Lah number $L'_{n,k} = \frac{n!}{k!}\binom{n-1}{k-1}$.

Proof. Let $a_1, \dots, a_n$ be a permutation of $1, \dots, n$ ($n!$ ways). Choosing $k-1$ places ($\binom{n-1}{k-1}$ ways), say $m_1, \dots, m_{k-1}$ we get the following nilpotent element:

$$[a_1, \dots a_{m_1}][a_{m_1+1}, \dots a_{m_2}] \dots [a_{m_{k-1}+1}, \dots a_n].$$

This element has defect k. The permutation of chains ($k!$ ways) does not change it.

Corollary. The number of nilpotent elements in $\mathcal{IS}_n$ equals

$$\sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}.$$

5. Nilpotent subsemigroups

A semigroup, S, with zero 0 is *nilpotent* provided that $S^k = 0$ for some k.

A subsemigroup, $0 \in S \subset \mathcal{IS}_n$ is nilpotent if and only if it contains only nilpotent elements.

Question: What are the biggest (maximal) nilpotent subsemigroups in $\mathcal{IS}_n$?

Let $S \subset \mathcal{IS}_n$ be a nilpotent subsemigroup then the relation $<_S$ on N_n, defined as: $a <_S b$ if and only if there exists $\pi \in S$ such that $b = \pi(a)$, is a partial order.

Let $<$ be a partial order on N_n. Then the semigroup $S(<)$, which consists of all $\pi \in \mathcal{IS}_n$ such that $a < \pi(a)$ for all $a \in \mathrm{Dom}(\pi)$ is a nilpotent semigroup in $\mathcal{IS}_n$.

Lemma.

1. $S \subset T$ implies $<_S \subset <_T$.
2. $<_1 \subset <_2$ implies $S(<_1) \subset S(<_2)$.

Theorem [Ganyushkin-Kormysheva]. Maximal nilpotent subsemigroups in $\mathcal{IS}_n$ are exactly $S(<)$, where $<$ is a linear order on N_n. In particular, there exists exactly $n!$ nilpotent subsemigroups in $\mathcal{IS}_n$, they all are conjugated by S_n-action and hence are isomorphic.

Question: How many elements does a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ contain?

Lemma. Let $S = S(<)$ be a maximal nilpotent subsemigroup in $\mathcal{IS}_n$. Then there is a natural bijection between the elements in S and (unordered) partitions of N_n into subsets. In particular, $|S| = B_n$, the n-th Bell number.

Proof. Let $\pi \in S$, then chain decomposition of π defines a partition of N_n. Conversely, let $N_n = N_1 \cup \cdots \cup N_k$. Each N_i defined a maximal chain, ordered with respect to $<$. The product over all i defines an element in S. These two correspondences are mutually inverse bijections.

6. k-maximal subsemigroups

$$N_n = M_1 \cup \dots \cup M_k, \quad |M_i| = t_i > 0,$$

For $x, y \in N_n$ set $x < y$ if and only if $x \in M_i$, $y \in M_j$ and $i < j$.

The semigroup $S(<)$ is maximal among nilpotent subsemigroups in $\mathcal{IS}_n$ of nilpotency degree k.

Question: What is $|S(<)|$ for $<$ as above?

For $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ let the *Bell evaluation* of $f(x)$ be the following: $f(B) = a_n B_n + a_{n-1} B_{n-1} + \dots + a_1 B_1 + a_0$, where $\{B_i\}$ are Bell's numbers.

For $i \in \mathbb{N}$ set $[x]_i = x(x-1)\dots(x-i+1)$ and define

$$f_{t_1,\dots,t_k}(x) = [x]_{t_1}[x]_{t_2}\dots[x]_{t_k}.$$

Theorem [Ganyushkin-Pavlov]. $|S(<)| = f_{t_1,\dots,t_k}(B)$.

Corollary. $|S(<)|$ does not depend on the ordering of M_i's.