COMBINATORICS OF PARTIAL BIJECTIONS

1. The symmetric group $S_n$

$$n \in \mathbb{N}, \quad N_n = \{1, 2, \ldots, n\}$$

$$S_n = \{f : N_n \rightarrow N_n : f \text{ is bijective}\}$$

Elements from $S_n$ are called permutations on $N_n$.

$$f, g \in S_n \Rightarrow f \circ g \in S_n, \quad f \circ g(x) = f(g(x)) \text{ for all } x \in N_n$$

$(S_n, \circ)$ is a group. This means the following:

1. The operation “$\circ$” is associative, i.e. $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f, g, h \in S_n$. In fact, $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x) = f(g(h(x)))$ for all $x \in N_n$.

2. $S_n$ contains the identity permutation $id$, defined as follows: $id(x) = x$ for all $x \in S_n$. This transformation satisfies $id \circ f = f \circ id = f$ for all $f \in S_n$.

3. Every permutation $f \in S_n$ has an inverse in $S_n$, this means that there exists a permutation, $g \in S_n$, such that $g \circ f = f \circ g = id$.

Remark. It is easy to see that the identity is unique and every $f \in S_n$ has unique inverse, which is denoted by $f^{-1}$. 
Example. The group $S_3$:

\[
\begin{align*}
\text{id} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\
(1, 2) &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \\
(1, 3) &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \\
(2, 3) &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \\
(1, 2, 3) &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\
(1, 3, 2) &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}
\end{align*}
\]
2. The full inverse symmetric semigroup $\mathcal{IS}_n$

Let $X$ and $Y$ be arbitrary sets.

A **partial map from $X$ to $Y$** is a map, $f$, from $A \subset X$ to $B \subset Y$. If $f : A \rightarrow B$ is a bijection it is called a **partial bijection** from $X$ to $Y$.

$$\mathcal{IS}_n = \{ f : N_n \rightarrow N_n : f \text{ is a partial bijection} \}.$$ 

Elements from $\mathcal{IS}_n$ are called **partial permutations** on $N_n$.

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be partial maps. Define the **composition** $g \circ f : X \rightarrow Z$ as the following partial map: $g \circ f$ is defined on $x \in X$ if and only if $f$ is defined on $x$ and $g$ is defined on $f(x)$; moreover, if $g \circ f$ is defined on $x \in X$, then $(g \circ f)(x) = g(f(x))$.

If $f : X \rightarrow Y$ is a partial map and $x \in X$ is such that $f$ is not defined on $x$, denote this by $f(x) = \emptyset$.

$(\mathcal{IS}_n, \circ)$ is an inverse semigroup (monoid). This means the following:

1. The operation “$\circ$” is associative, i.e. $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f, g, h \in \mathcal{IS}_n$.
2. $id \in \mathcal{IS}_n$ and $id \circ f = f \circ id = f$ for all $f \in \mathcal{IS}_n$.
3. Every permutation $f \in S_n$ has a **partial inverse** in $S_n$, i.e. that there exists a unique permutation, $g \in S_n$, such that $g \circ f \circ g = g$ and $f \circ g \circ f = f$. This unique partial inverse is denoted by $f^*$. 


Example. The semigroup $I S_3$:

$$id = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad (1, 2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad (1, 3) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$(2, 3) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad (1, 2, 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad (1, 3, 2) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$[1] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & 3 \end{pmatrix}, \quad (2, 3)[1] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 3 & 2 \end{pmatrix}, \quad [2, 3, 1] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 3 & 1 \end{pmatrix}$$

$$[3, 2, 1] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & 2 \end{pmatrix}, \quad [3, 1] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & 1 \end{pmatrix}, \quad [2, 1] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & 3 \end{pmatrix}$$

$$[2] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & 3 \end{pmatrix}, \quad (1, 3)[2] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & 1 \end{pmatrix}, \quad [1, 3, 2] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & 2 \end{pmatrix}$$

$$[3, 1, 2] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & 1 \end{pmatrix}, \quad [3, 2] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & 2 \end{pmatrix}, \quad [1, 2] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & 3 \end{pmatrix}$$
\[
[3] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & \emptyset \end{pmatrix}, \quad (1, 2)[3] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & \emptyset \end{pmatrix}, \quad [1, 2, 3] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & \emptyset \end{pmatrix}
\]

\[
[2, 1, 3] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & \emptyset \end{pmatrix}, \quad [2, 3] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & \emptyset \end{pmatrix}, \quad [1, 3] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & \emptyset \end{pmatrix}
\]

\[
[1][2] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 3 \end{pmatrix}, \quad [3, 1][2] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 1 \end{pmatrix}, \quad [1][3, 2] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 2 \end{pmatrix},
\]

\[
[1][3] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & \emptyset \end{pmatrix}, \quad [2, 1][3] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & \emptyset \end{pmatrix}, \quad [1][2, 3] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & \emptyset \end{pmatrix},
\]

\[
[2][3] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{pmatrix}, \quad [1, 2][3] = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & \emptyset \end{pmatrix}, \quad [2][1, 3] = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & \emptyset \end{pmatrix}, \quad 0 = [1][2][3] = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{pmatrix}.
\]
3. Basic combinatorics

$$|S_n| = n \cdot (n - 1) \cdot (n - 2) \ldots 1 = n!.$$ 

$$
\begin{pmatrix}
1 & 2 & 3 & \ldots & n - 1 & n \\
n \text{poss.} & n - 1 \text{ poss.} & n - 2 \text{ poss.} & \ldots & 2 \text{ poss.} & 1 \text{ poss.}
\end{pmatrix}
$$

How to calculate $|\mathcal{IS}_n|$?

$\pi \in \mathcal{IS}_n$, that is $\pi : A \to B$ is a bijection, $A, B \subset N_n$. Then

- $A = \text{Dom}(\pi)$, the domain of $\pi$;
- $B = \text{Im}(\pi)$, the image of $\pi$;
- $|A| = |B| = \text{Rank}(\pi)$, the rank of $\pi$.
- $n - \text{Rank}(\pi) = \text{Def}(\pi)$, the defect of $\pi$.

- Count the number of elements of a fixed rank $i$ separately.
- If $i$ is fixed, the domain $A$ can be chosen in $\binom{n}{i}$ different ways.
- If $i$ is fixed, the image $B$ can be chosen in $\binom{n}{i}$ different ways.
- If $A$ and $B$ of cardinality $i$ are fixed, a bijection from $A$ to $B$ can be chosen in $i!$ different ways.

Answer: $$|S_n| = \sum_{i=0}^{n} \left( \binom{n}{i} \right)^2 i!$$

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4. Nilpotent elements

\( \mathcal{I} \mathcal{S}_n \) contains 0, that is a special element, such that \( 0 \circ f = f \circ 0 = 0 \) for all \( f \in |\mathcal{I} \mathcal{S}_n| \).

0 is unique, \( \text{Dom}(0) = \text{Im}(0) = \emptyset \).

\( A \in \mathcal{I} \mathcal{S}_n \) is said to be nilpotent provided that \( A^n = 0 \) for some positive integer \( n \).

\( \pi \in \mathcal{I} \mathcal{S}_n \) is nilpotent if and only if \( \pi \) does not have cycles if and only if \( \pi = [x_1, \ldots, x_i] \ldots [y_1, \ldots, y_j] \).

How many nilpotent elements does \( \mathcal{I} \mathcal{S}_n \) have?

**Lemma.** The number of nilpotent elements of a fixed defect \( k \), \( 0 < k \leq n \), equals the signless Lah number \( L'_{n,k} = \frac{n!}{k!} \binom{n-1}{k-1} \).

**Proof.** Let \( a_1, \ldots, a_n \) be a permutation of \( 1, \ldots, n \) (\( n! \) ways). Choosing \( k - 1 \) places (\( \binom{n-1}{k-1} \) ways), say \( m_1, \ldots, m_{k-1} \) we get the following nilpotent element:

\[
[a_1, \ldots a_{m_1}][a_{m_1+1}, \ldots a_{m_2}] \ldots [a_{m_{k-1}+1}, \ldots a_n].
\]

This element has defect \( k \). The permutation of chains (\( k! \) ways) does not change it.

**Corollary.** The number of nilpotent elements in \( \mathcal{I} \mathcal{S}_n \) equals

\[
\sum_{k=1}^{n} \frac{n!}{k!} \binom{n-1}{k-1}.
\]

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5. Nilpotent subsemigroups

A semigroup, $S$, with zero 0 is nilpotent provided that $S^k = 0$ for some $k$.

A subsemigroup, $0 \in S \subseteq \mathcal{I}S_n$ is nilpotent if and only if it contains only nilpotent elements.

**Question:** What are the biggest (maximal) nilpotent subsemigroups in $\mathcal{I}S_n$?

Let $S \subseteq \mathcal{I}S_n$ be a nilpotent subsemigroup then the relation $<_S$ on $N_n$, defined as: $a <_S b$ if and only if there exists $\pi \in S$ such that $b = \pi(a)$, is a partial order.

Let $<$ be a partial order on $N_n$. Then the semigroup $S(<)$, which consists of all $\pi \in \mathcal{I}S_n$ such that $a < \pi(a)$ for all $a \in \text{Dom}(\pi)$ is a nilpotent semigroup in $\mathcal{I}S_n$.

**Lemma.**

1. $S \subset T$ implies $<_S \subseteq <_T$.

2. $<_1 \subseteq <_2$ implies $S(<_1) \subseteq S(<_2)$.
Theorem [Ganyushkin-Kormysheva]. Maximal nilpotent subsemigroups in $\mathcal{LS}_n$ are exactly $S(<)$, where $<$ is a linear order on $N_n$. In particular, there exists exactly $n!$ nilpotent subsemigroups in $\mathcal{LS}_n$, they all are conjugated by $S_n$-action and hence are isomorphic.

**Question:** How many elements does a maximal nilpotent subsemigroup of $\mathcal{LS}_n$ contain?

**Lemma.** Let $S = S(<)$ be a maximal nilpotent subsemigroup in $\mathcal{LS}_n$. Then there is a natural bijection between the elements in $S$ and (unordered) partitions of $N_n$ into subsets. In particular, $|S| = B_n$, the $n$-th Bell number.

**Proof.** Let $\pi \in S$, then chain decomposition of $\pi$ defines a partition of $N_n$. Conversely, let $N_n = N_1 \cup \cdots \cup N_k$. Each $N_i$ defined a maximal chain, ordered with respect to $<$. The product over all $i$ defines an element in $S$. These two correspondences are mutually inverse bijections.
6. $k$-maximal subsemigroups

$$N_n = M_1 \cup \cdots \cup M_k, \quad |M_i| = t_i > 0,$$

For $x, y \in N_n$ set $x < y$ if and only if $x \in M_i, y \in M_j$ and $i < j$.

The semigroup $S(<)$ is maximal among nilpotent subsemigroups in $\mathcal{IS}_n$ of nilpotency degree $k$.

**Question:** What is $|S(<)|$ for $<$ as above?

For $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ let the *Bell evaluation* of $f(x)$ be the following: $f(B) = a_n B_n + a_{n-1} B_{n-1} + \cdots + a_1 B_1 + a_0$, where $\{B_i\}$ are Bell’s numbers.

For $i \in \mathbb{N}$ set $[x]_i = x(x-1)\ldots(x-i+1)$ and define

$$f_{t_1,\ldots,t_k}(x) = [x]_{t_1} [x]_{t_2} \cdots [x]_{t_k}.$$

**Theorem [Ganyushkin-Pavlov].** $|S(<)| = f_{t_1,\ldots,t_k}(B)$.

**Corollary.** $|S(<)|$ does not depend on the ordering of $M_i$'s.