Linear representations of semigroups from 2-categories

Volodymyr Mazorchuk
(Uppsala University)

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**Definition.** A *2-category* is a category enriched over the monoidal category $\mathbf{Cat}$ of small categories (in the latter the monoidal structure is induced by the cartesian product).

This means that a 2-category $\mathcal{C}$ is given by the following data:

- objects of $\mathcal{C}$;
- small categories $\mathcal{C}(i, j)$ of morphisms;
- functorial composition $\mathcal{C}(j, k) \times \mathcal{C}(i, j) \to \mathcal{C}(i, k)$;
- identity objects $1_j$;

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2-categories: terminology and the first example

Terminology.

- An object in $\mathcal{C}(i, j)$ is called a 1-morphism of $\mathcal{C}$.
- A morphism in $\mathcal{C}(i, j)$ is called a 2-morphism of $\mathcal{C}$.
- Composition in $\mathcal{C}(i, j)$ is called vertical and denoted $\circ_1$.
- Composition in $\mathcal{C}$ is called horizontal and denoted $\circ_0$.

Principal example. The category $\text{Cat}$ is a 2-category.

- Objects of $\text{Cat}$ are small categories.
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A monoid is the same thing as a category with one object.

Indeed: If $C$ is a category with one object $♣$, then $C(♣, ♣)$ is a monoid under composition.

If $(S, ◦, e)$ is a monoid, we can form a category $C = C(S, ◦, e)$ as follows:

- The only object of $C$ is $♣$.
- $C(♣, ♣) := S$.
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Can we extend $C$ to a 2-category?

**Naive approach to try:** Let $X \subset S$ be some submonoid.

For $s, t \in S$ set $\text{Hom}_{C(♣,♣)}(s, t) := \{x \in X : xs = t\}$.

**Note!** $S$ is just a monoid, not a group, so $\text{Hom}_{C(♣,♣)}(s, t)$ may be empty or it may contain many elements.

Composition is given by multiplication in $S$.

Is composition well-defined?
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**Is composition well-defined?**
2-categories: over monoids, part 3

Vertical: $xr = s$ and $ys = t$ implies $yxr = t$  \hspace{1cm} \text{OK}

Horizontal: $xs = t$ and $x's' = t'$ implies $xsx's' = tt'$

Need: $xx's's' = tt'$  \hspace{1cm} \text{OK if } X \subset Z(S)$

From now on: $X$ is a submonoid in the center $Z(S)$ of $S$

All compositions are well-defined!!!

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Need: \( xx's's' = tt' \) \hspace{1cm} \text{OK if} \ X \subset Z(S)

From now on: \( X \) is a submonoid in the center \( Z(S) \) of \( S \)

All compositions are well-defined!!!

Is this a 2-category?

To check: Functoriality of composition.
2-categories: over monoids, part 3

**Vertical:** \( xr = s \) and \( ys = t \) implies \( yxr = t \)  \( \text{OK} \)

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To check: Functoriality of composition.
**Vertical:** $xr = s$ and $ys = t$ implies $yxr = t$  \[\text{OK}\]

**Horizontal:** $xs = t$ and $x's' = t'$ implies $xsx's' = tt'$

**Need:** $xx'ss' = tt'$  \[\text{OK if } X \subset Z(S)\]

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2-categories: over monoids, part 3

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All compositions are well-defined!!!

Is this a 2-category?

To check: Functoriality of composition.
One way:

Conclusion 1: \((y \circ_0 y') \circ_1 (x \circ_0 x') = yy'xx'.\)
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One way:

\[
\begin{array}{ccc}
  r & \downarrow s & \circ_0 s' & \circ_1 ss' & \circ_1 \circ_0 yy' xx' \\
  x & \downarrow x' & xx' & \downarrow \circ_1 yy' xx' \\
  y & \downarrow y' & yy' & \downarrow tt' \\
  t & \downarrow t' & tt' \\
\end{array}
\]

Conclusion 1: \((y \circ_0 y') \circ_1 (x \circ_0 x') = yy' xx'\).
One way:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
    r & \\
    \downarrow & \downarrow & \\
    x & x' & xx' \\
    \downarrow & \downarrow & \\
    s & s' & ss' \\
    \downarrow & \\
    y & y' & yy' \\
    \downarrow & \\
    t & t' & tt' \\
\end{array}
\end{array}
\end{align*}
\]

Conclusion 1: \((y \circ_0 y') \circ_1 (x \circ_0 x') = yy'xx'.\)
Another way:

Conclusion 2: \((y \circ_1 x) \circ_0 (y' \circ_1 x') = yxy'x'\).
Another way:

\begin{align*}
    & r \\
    & \downarrow x \\
    s & \downarrow \circ_1 \\
    & \downarrow y \\
    t & \\

    & r' \\
    & \downarrow x' \\
    s' & \downarrow \circ_0 \\
    & \downarrow y' \\
    t' & \\

    & r \\
    & \downarrow r' \\
    & \downarrow yx \\
    & \downarrow t \\

    & r' \\
    & \downarrow r' \\
    & \downarrow y'x' \\
    & \downarrow t' \\

    & rr' \\
    & \downarrow yxy'x' \\
    & \downarrow tt'
\end{align*}

Conclusion 2: \((y \circ_1 x) \circ_0 (y' \circ_1 x') = yxy'x'.\)
2-categories: over monoids, part 5: functoriality, part 2

Another way:

\[
\begin{array}{ccccccc}
  r & \downarrow & r' & \downarrow & r & \downarrow & rr' \\
  x & \downarrow & x' & \downarrow & yx & \downarrow & yxy'x' \\
  s & \downarrow & s' & \downarrow & yx & \downarrow & t \\
  y & \downarrow & y' & \downarrow & t & \downarrow & \circ_0 y'x' \\
  t & \downarrow & t' & \downarrow & yxy'x' & \downarrow & tt' \\
\end{array}
\]

Conclusion 2: \((y \circ_1 x) \circ_0 (y' \circ_1 x') = yxy'x'.\)
Another way:

\[
\begin{array}{c}
\begin{align*}
\xymatrix{r & r' \ar[d]^-x \ar[d]^-{x'} \ar[r]^-s & s' \ar[r]^-{s'} & t \ar[d]^-{t'} \ar[r]^-t & t'}
\end{align*}
\end{array}
\]

\[
\begin{array}{c}
\begin{align*}
\xymatrix{\circ_1 & \circ_0 \ar[r]^-\circ & yx \ar[r]^-\circ_0 & y'x' \ar[r]^-\circ & yxy'x'}
\end{align*}
\end{array}
\]

Conclusion 2: \((y \circ_1 x) \circ_0 (y' \circ_1 x') = yxy'x'.\)
2-categories: over monoids, part 6: the interchange law

Need: the *interchange law* \((y \circ_1 x) \circ_0 (y' \circ_1 x') = (y \circ_0 y') \circ_1 (x \circ_0 x')\).

\[
\begin{align*}
\begin{array}{cccc}
\circ_1 & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\circ_0 & \downarrow & \downarrow & \downarrow \\
\end{array}
\end{align*}
\]

In our case: \(xyy'x' = yy'xx'\) \(\forall x, y, x', y' \in X\) \(\text{OK since } X \subset Z(S)\).

Claim. The above defines on \(C\) the structure of a 2-category if and only if \(X \subset Z(S)\).
Need: the **interchange law** \((y \circ_1 x) \circ_0 (y' \circ_1 x') = (y \circ_0 y') \circ_1 (x \circ_0 x')\).
Need: the **interchange law** \( (y \circ_1 x) \circ_0 (y' \circ_1 x') = (y \circ_0 y') \circ_1 (x \circ_0 x') \).

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Need: the **interchange law** \((y \circ_1 x) \circ_0 (y' \circ_1 x') = (y \circ_0 y') \circ_1 (x \circ_0 x')\).

\[ \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\circ_1 & & \\
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\end{array} \\
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\circ_0 & & \\
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\end{array} \\
= \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\circ_1 & & \\
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\end{array} \\
= \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\circ_0 & & \\
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\end{array} \]

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\[ 
\begin{array}{c}
\bullet \rightarrow \bullet \rightarrow \bullet \\
\circ_1
\end{array} 
\quad = 
\begin{array}{c}
\bullet \rightarrow \bullet \rightarrow \bullet \\
\end{array} 
\quad = 
\begin{array}{c}
\bullet \rightarrow \bullet \rightarrow \bullet \\
\circ_0
\end{array} 
\]

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2-categories: over monoids, part 7: ordered monoids

\[ S — \text{monoid} \]

\[ \leq — \text{compatible order on } S \text{ (i.e. } a \leq b \text{ implies } as \leq bs \text{ and } sa \leq sb) \]

Define \( C(S, \leq) — 2\)-category via

- \( C(S, \leq) \) has one object ♣
- 1-morphisms: \( C(S, \leq)(♣, ♣) = S \)
- 2-morphisms: for \( s, t \in S \) set \( \text{Hom}(s, t) = \begin{cases} (s, t), & s \leq t; \\ \emptyset, & \text{else.} \end{cases} \)
- horizontal composition is given by multiplication in \( S \);
- vertical composition is uniquely defined.
2-categories: over monoids, part 7: ordered monoids

$S$ — monoid

$\leq$ — compatible order on $S$ (i.e. $a \leq b$ implies $as \leq bs$ and $sa \leq sb$)

Define $C_{(S,\leq)}$ — 2-category via

- $C_{(S,\leq)}$ has one object ♣
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2-categories: over monoids, part 7: ordered monoids

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\mathcal{A} \text{ and } \mathcal{C} — two 2-categories

**Definition.** A 2-functor \( F : \mathcal{A} \to \mathcal{C} \) is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

**Example.** For \( i \in \mathcal{C} \) the functor \( \mathcal{C}(i, -) : \mathcal{C} \to \text{Cat} \) sends

- an object \( j \in \mathcal{C} \) to the category \( \mathcal{C}(i, j) \),
- a 1-morphism \( F \in \mathcal{C}(j, k) \) to the functor \( F \circ - : \mathcal{C}(i, j) \to \mathcal{C}(i, k) \),
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2-representations: 2-functors, part 1

\[ \mathcal{A} \text{ and } \mathcal{C} \text{ — two 2-categories} \]

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2-representations: 2-functors, part 1

\(\mathcal{A}\) and \(\mathcal{C}\) — two 2-categories

**Definition.** A 2-functor \(F : \mathcal{A} \to \mathcal{C}\) is a functor which sends 1-morphisms to 1-morphisms and 2-morphisms to 2-morphisms in a way that is coordinated with all the categorical structures (domains, codomains, identities and compositions).

**Example.** For \(i \in \mathcal{C}\) the functor \(\mathcal{C}(i, _) : \mathcal{C} \to \textbf{Cat}\) sends

- an object \(j \in \mathcal{C}\) to the category \(\mathcal{C}(i, j)\),
- a 1-morphism \(F \in \mathcal{C}(j, k)\) to the functor \(F \circ _{-} : \mathcal{C}(i, j) \to \mathcal{C}(i, k)\),
- a 2-morphism \(\alpha : F \to G\) to the natural transformation \(\alpha \circ _0 _{-} : F \circ _{-} \to G \circ _{-}\).
2-representations: 2-functors, part 2

\[
\begin{array}{ccc}
  j & \xrightarrow{F_0} & k \\
\downarrow_{\mathcal{C}(i,j)} & \searrow & \downarrow_{\mathcal{C}(i,k)} \\
  i & \downarrow_{\alpha_0} & i \\
\end{array}
\]

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2-representations: 2-functors, part 2

\[ j \xrightarrow{F_0} k \]

\[ C(i,j) \xleftarrow{i} \xrightarrow{k} C(i,k) \]

\[ j \xleftarrow{G_0} k \]

\[ C(i,j) \xleftarrow{i} \xrightarrow{k} C(i,k) \]

Volodymyr Mazorchuk
2-representations: 2-functors, part 2
Definition: A 2-representation of a 2-category $\mathcal{C}$ is a 2-functor from $\mathcal{C}$ to some “classical” 2-category.

Example: $\mathcal{C}(i, -)$ is the principal 2-representation of $\mathcal{C}$ in $\text{Cat}$.

“Classical” 2-representations:

- in $\text{Cat}$;
- in the 2-category $\text{Add}$ of additive categories and additive functors;
- in the 2-subcategory $\text{add}$ of $\text{Add}$ consisting of all fully additive categories with finitely many isoclasses of indecomposable objects;
- a the 2-category $\text{ab}$ of abelian categories and exact functors.
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Note: If \(\mathcal{A}\) is idempotent split with finitely many indecomposables, then \([\mathcal{A}]\) is free abelian of finite rank with indecomposables/iso as basis.

Definition. A 2-category \(\mathcal{C}\) is called locally finitary over a field \(\mathbb{k}\) if each \(\mathcal{C}(i,j)\) is \(\mathbb{k}\)-linear, additive, idempotent split with finitely many indecomposables.

\(\mathcal{C} \rightsquigarrow\) locally finitary

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Decategorification: Grothendieck category

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Decategorification: linear algebra

Main point: Forget the 2-level.

Note: For $k$-linear categories indecomposability is defined on the 2-level (an object in indecomposable iff its endomorphism algebra is local).

Assume: $C$ — locally finitary; $F$ — 2-representation of $C$ s.t.

- object $i \mapsto$ additive (abelian, triangulated) category $C_i$
- 1-morphism $\mapsto$ additive (exact, triangulated) functor
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Then: The category $[C]$ acts on $[C]$

In particular: If $C$ has 1 object ♣ then the monoid $[C](♣, ♣)$ acts on the abelian group $[C]$

Extending scalars: The algebra $k[C](♣, ♣)$ acts on the vector space $k[C]$, that is we get a linear representation of the monoid $[C](♣, ♣)$. 
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Extending scalars: The algebra $k[\mathcal{C}](\heartsuit, \heartsuit)$ acts on the vector space $k[C]$, that is we get a linear representation of the monoid $[\mathcal{C}](\heartsuit, \heartsuit)$. 
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Extending scalars: The algebra \( k[C](♣, ♣)\) acts on the vector space \( k[C] \), that is we get a linear representation of the monoid \([C](♣, ♣)\).
Decategorification: advantages

Assume: $\mathcal{C}$ is 2-represented on $\mathcal{C}$

Decategorify: $[\mathcal{C}]$ acts on $[\mathcal{C}]$

Main point: $\mathcal{C}$ has non-trivial structure

Example 1: The group $[\mathcal{C}]$ might have many natural bases (e.g. given by simple, injective, projective or tilting modules).

Example 2: The category $\mathcal{C}$ could have stratifications, e.g. by Gelfand-Kirillov dimension of objects. This gives rise to filtrations on $[\mathcal{C}]$.

Example 3: The category $\mathcal{C}$ could be graded, which would give a “layered upgrade” of $[\mathcal{C}]$ (e.g. Jones polynomial $\rightarrow$ Khovanov homology).
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Decategorification: advantages

**Assume:** $C$ is 2-represented on $C$

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Decategorify: \( [\mathcal{C}] \) acts on \( [\mathcal{C}] \)

Main point: \( \mathcal{C} \) has non-trivial structure

Example 1: The group \([\mathcal{C}]\) might have many natural bases (e.g. given by simple, injective, projective or tilting modules).

Example 2: The category \( \mathcal{C} \) could have stratifications, e.g. by Gelfand-Kirillov dimension of objects. This gives rise to filtrations on \([\mathcal{C}]\).

Example 3: The category \( \mathcal{C} \) could be graded, which would give a “layered upgrade” of \([\mathcal{C}]\) (e.g. Jones polynomial \( \rightarrow \) Khovanov homology).
Decategorification: advantages

Assume: $C$ is 2-represented on $C$

Decategorify: $[C]$ acts on $[C]$

Main point: $C$ has non-trivial structure

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Decategorification: advantages

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Hecke-Kiselman semigroups: definition

Assume: $\Gamma$ — simple digraph (no loops or multiple edges in the same direction)

Definition: The Hecke-Kiselman monoid $HK_\Gamma$ has generators $e_i$ where $i$ is a vertex of $\Gamma$ and relations

- $e_i e_j e_i = e_j e_i e_j$ if $i \xrightarrow{\hspace{5cm}} j$;
- $e_i e_j e_i = e_j e_i e_j = e_i e_j$ if $i \xrightarrow{} j$;
- $e_i e_j = e_j e_i$ if $i \xrightarrow{} j$.

Examples:

- $\Gamma$ — no edges $\Rightarrow HK_\Gamma$ is the Boolean of $\Gamma_0$;
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Hecke-Kiselman semigroups: Catalan monoid

**Catalan monoid:** $C_n$ — order preserving (i.e. $a \leq b \Rightarrow f(a) \leq f(b)$) and order decreasing (i.e. $f(a) \leq a$) transformations of $\{0, 1, \ldots, n\}$.

$|C_n| = \frac{1}{n+1} \binom{2n}{n}$ — the $n$-th Catalan number

$$
\Gamma = \Gamma_n := 1 \rightarrow 2 \rightarrow \cdots \rightarrow n
$$

**Theorem (A. Solomon):** $HK\Gamma_n \cong C_n$

**Standard effective representations** $\Phi$ of $C_n$:
$v = (v_1, v_2, \ldots, v_n)$ basis of $\mathbb{k}^n$, action

$$
e_i(v_j) = \begin{cases} 
  v_j, & j \neq i; \\
  v_{j-1}, & j = i > 1; \\
  0, & j = i = 1.
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$$\Gamma = \Gamma_n := \begin{array}{cccccccccc}
1 & 2 & \cdots & \cdots & \cdots & n \\
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Path categories

$\Gamma$ — acyclic quiver (no loops but multiple edges allowed)

$k\Gamma$ — path category of $\Gamma$

- objects: vertices of $\Gamma$
- morphisms: linear combinations of paths in $\Gamma$
- composition: concatenation of paths

Representation of $k\Gamma$ — functor to $k$-vector spaces, i.e.

- objects $\mapsto$ vector space
- paths in $\Gamma$ $\mapsto$ linear map
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$k\Gamma$-mod — category of locally finite dimensional representations
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- concatenation of paths $\mapsto$ composition of linear maps

$k\Gamma$-mod — category of locally finite dimensional representations
(morphisms $=$ natural transformations of functors)
Projection functors

\[ \Gamma \rightarrow \text{acyclic quiver}, \ i \in \Gamma \]

\[ F_i : \mathbb{k} \Gamma \text{-mod} \rightarrow \mathbb{k} \Gamma \text{-mod} \rightarrow \text{projection functor} \]

“factor out the maximal possible \( \mathbb{k} \Gamma \)-invariant subspace at vertex \( i \)”

**Theorem (Greensing).** Projections functors satisfy:

1. \( F_i F_j \cong F_j F_i \) if \( i \) and \( j \) are not connected in \( \Gamma \);
2. \( F_i F_j F_i \cong F_j F_i F_j \cong F_i F_j \) if there is an arrow from \( i \) to \( j \) in \( \Gamma \).

**Difficulty.** Projections functors are not exact.

**Fact.** Projections functors send injectives to injectives.

**Way out.** Let \( G_i \) be the unique left exact functor whose action on the additive category of injective modules is isomorphic to that of \( F_i \).

**Fact:** \( G_i \) is exact
Projection functors

\( \Gamma \) — acyclic quiver, \( i \in \Gamma \)

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Projection functors

\(\Gamma\) — acyclic quiver, \(i \in \Gamma\)

\(F_i : \kappa\Gamma\text{-mod} \to \kappa\Gamma\text{-mod}\) — projection functor

“factor out the maximal possible \(\kappa\Gamma\)-invariant subspace at vertex \(i\)”

**Theorem (Grensing).** Projections functors satisfy:

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Categorification of the Catalan monoid

Γ — acyclic quiver, Θ — underlying simple digraph

Definition: 2-category $\mathcal{C}_\Theta,\Gamma$.

- Object: ♣ := $k\Gamma$-mod;
- 1-morphisms: Endofunctors on $k\Gamma$-mod isomorphic to a direct sum of direct summands of compositions of the $G_i$’th
- 2-morphisms: natural transformations of functors

The 2-category $\mathcal{C}_\Theta,\Gamma$ is given by its defining 2-representation, that is a functorial action on $k\Gamma$-mod.

Theorem (Grensing-M): $[\mathcal{C}_{\Gamma_n,\Gamma_n}](♣, ♣) \cong \mathbb{Z}[C_n]$.

Corollary: In the basis of simple modules, the action of $[\mathcal{C}_{\Gamma_n,\Gamma_n}](♣, ♣)$ on $[k\Gamma$-mod] gives $\Phi$.

Consequence: In the basis of projective (injective) modules we get two new (but equivalent) effective linear representations of $C_n$. 
Categorification of the Catalan monoid

Γ — acyclic quiver, Θ — underlying simple digraph

**Definition:** 2-category \( C_{\Theta, \Gamma} \).

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Categorification of the Catalan monoid

\( \Gamma \) — acyclic quiver, \( \Theta \) — underlying simple digraph

**Definition:** 2-category \( \mathcal{C}_\Theta, \Gamma \).

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Categorification of the Catalan monoid

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**Definition:** 2-category $\mathcal{C}_Θ, Γ$.

- **Object:** $♣ := kΓ\text{-mod}$;
- **1-morphisms:** Endofunctors on $kΓ\text{-mod}$ isomorphic to a direct sum of direct summands of compositions of the $G_i$’th;
- **2-morphisms:** natural transformations of functors

The 2-category $\mathcal{C}_Θ, Γ$ is given by its *defining 2-representation*, that is a functorial action on $kΓ\text{-mod}$.

**Theorem (Grensing-M):** $[\mathcal{C}_Γ, Γ](♣, ♣) \cong \mathbb{Z}[C_n]$.

**Corollary:** In the basis of simple modules, the action of $[\mathcal{C}_Γ, Γ](♣, ♣)$ on $[kΓ\text{-mod}]$ gives $Φ$.

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Volodymyr Mazorchuk  
Linear representations of semigroups from 2-categories
Categorification of the Catalan monoid

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Volodymyr Mazorchuk
Linear representations of semigroups from 2-categories 21/22
Other Hecke-Kiselman monoids

$\Gamma, \Theta$ as above

**Fact:** Mapping $e_i$ to $G_i$ gives a weak functorial action of $\text{HK}_\Theta$ on $k\Gamma\text{-mod}$.

**Example:** From [Kudryavtseva-M] it follows that if $\Theta$ is the full graph on $\{1, 2, \ldots, n\}$ oriented from smaller to bigger vertices (i.e. $\text{HK}_\Theta$ is the Kiselman semigroup), then there exists $\Gamma$ such that this action is faithful.

**Difficulty:** Composition of the $G_i$’s may decompose!

**Problem:** What are indecomposable 1-morphisms in $\mathcal{C}_{\Theta, \Gamma}$?

**Known full answer:** For $\Gamma_n$ any composition of the $G_i$’s is indecomposable.

**Known partial answer:** For a Dynkin quiver of type $A$ and any orientation, indecomposable 1-morphisms in $\mathcal{C}_{\Theta, \Gamma}$ form a monoid $T$ (under composition) generated by idempotents (each $\rightarrow \bullet \rightarrow$ contributes with one generator and each $\rightarrow \bullet \leftarrow$ and $\leftarrow \bullet \rightarrow$ with two generators). There is a presentation for $T$ and a realization of $\text{HK}_\Theta$ inside $\mathbb{Z}[T]$. 
Other Hecke-Kiselman monoids

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\[ \Gamma, \Theta \text{ as above} \]

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Other Hecke-Kiselman monoids

Γ, Θ as above

Fact: Mapping $e_i$ to $G_i$ gives a weak functorial action of $HK_\Theta$ on $k\Gamma$-mod.

Example: From [Kudryavtseva-M] it follows that if Θ is the full graph on \{1, 2, \ldots, n\} oriented from smaller to bigger vertices (i.e. $HK_\Theta$ is the Kiselman semigroup), then there exists Γ such that this action is faithful.

Difficulty: Composition of the $G_i$’s may decompose!

Problem: What are indecomposable 1-morphisms in $C_\Theta,\Gamma$?

Known full answer: For Γ, any composition of the $G_i$’s is indecomposable.

Known partial answer: For a Dynkin quiver of type $A$ and any orientation, indecomposable 1-morphisms in $C_\Theta,\Gamma$ form a monoid $T$ (under composition) generated by idempotents (each $\rightarrow \bullet \rightarrow$ contributes with one generator and each $\rightarrow \bullet \leftarrow$ and $\leftarrow \bullet \rightarrow$ with two generators). There is a presentation for $T$ and a realization of $HK_\Theta$ inside $\mathbb{Z}[T]$. 
Other Hecke-Kiselman monoids

\( \Gamma, \Theta \) as above

**Fact:** Mapping \( e_i \) to \( G_i \) gives a weak functorial action of \( \text{HK}_\Theta \) on \( k\Gamma\text{-mod} \).

**Example:** From [Kudryavtseva-M] it follows that if \( \Theta \) is the full graph on \( \{1, 2, \ldots, n\} \) oriented from smaller to bigger vertices (i.e. \( \text{HK}_\Theta \) is the Kiselman semigroup), then there exists \( \Gamma \) such that this action is faithful.

**Difficulty:** Composition of the \( G_i \)'s may decompose!

**Problem:** What are indecomposable 1-morphisms in \( C_\Theta, \Gamma \)?

**Known full answer:** For \( \Gamma_n \) any composition of the \( G_i \)'s is indecomposable.

**Known partial answer:** For a Dynkin quiver of type \( A \) and any orientation, indecomposable 1-morphisms in \( C_\Theta, \Gamma \) form a monoid \( T \) (under composition) generated by idempotents (each \( \rightarrow \bullet \rightarrow \) contributes with one generator and each \( \rightarrow \bullet \leftarrow \) and \( \leftarrow \bullet \rightarrow \) with two generators). There is a presentation for \( T \) and a realization of \( \text{HK}_\Theta \) inside \( \mathbb{Z}[T] \).
Other Hecke-Kiselman monoids

Γ, Θ as above

**Fact:** Mapping $e_i$ to $G_i$ gives a weak functorial action of $\text{HK}_\Theta$ on $k\Gamma\text{-mod}$.

**Example:** From [Kudryavtseva-M] it follows that if Θ is the full graph on \{1, 2, \ldots, n\} oriented from smaller to bigger vertices (i.e. $\text{HK}_\Theta$ is the Kiselman semigroup), then there exists Γ such that this action is faithful.

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\( \Gamma, \Theta \) as above

**Fact:** Mapping \( e_i \) to \( G_i \) gives a weak functorial action of \( \text{HK}_\Theta \) on \( k\Gamma\text{-mod} \).

**Example:** From [Kudryavtseva-M] it follows that if \( \Theta \) is the full graph on \( \{1, 2, \ldots, n\} \) oriented from smaller to bigger vertices (i.e. \( \text{HK}_\Theta \) is the Kiselman semigroup), then there exists \( \Gamma \) such that this action is faithful.

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