GELFAND-ZETLIN MODULES

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$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$;

$\{e_{i,j} | i, j = 1, \dots, n\}$ – matrix units, basis of $\mathfrak{g}$.

tableau $[l] = (l_{i,j})_{i=1,\dots,n}^{j=1,\dots,i}$

simple f.d. $\mathfrak{g}$-modules $\leftrightarrow \mathfrak{m} = (m_1, \dots, m_n) \in \mathbb{C}^n$, $m_i - m_{i+1} \in \mathbb{N}$

THEOREM (Gelfand-Zetlin, 1950) The simple f.d. $\mathfrak{g}$-module $V = V(\mathfrak{m})$, corresponding to $\mathfrak{m}$ as above has a basis, $\mathcal{B}(\mathfrak{m})$, consisting of all tableaux $[t]$ such that

- $t_{n,j} = m_j$, $j = 1, \dots, n$;
- $t_{i,j} - t_{i-1,j} \in \mathbb{Z}_+$, $i = 2, \dots, n$, $j = 1, \dots, i$;
- $t_{i-1,j} - t_{i,j+1} \in \mathbb{N}$, $i = 2, \dots, n$, $j = 1, \dots, i-1$.

The action of $\mathfrak{g}$ on $\mathcal{B}(\mathfrak{m})$ is given by the following *Gelfand-Zetlin formulae*

$$e_{i,i}[t] = \left(\sum_{j=1}^{i} t_{i,j} - \sum_{j=1}^{i-1} t_{i-1,j} \right) [t],$$

$$e_{i,i+1}([t]) = -\sum_{j=1}^{i} \frac{\prod_{k=1}^{i+1}(t_{i,j}-t_{i+1,k})}{\prod_{k\neq j}(t_{i,j}-t_{i,k})}([t]+[\delta^{i,j}]),$$

$$e_{i+1,i}([t]) = \sum_{j=1}^{i} \frac{\prod_{k=1}^{i-1}(t_{i,j}-t_{i-1,k})}{\prod_{k\neq j}(t_{i,j}-t_{i,k})}([t]-[\delta^{i,j}]).$$

Generic Gelfand-Zetlin modules

fix $[l]$ such that $l_{i,j} - l_{i,k} \notin \mathbb{Z}$ for all $i < n$ and $j \neq k$.

Let $B([l])$ consist of all $[t]$ such that

- $t_{n,j} = l_j$, $j = 1, \ldots, n$;
- $t_{i,j} - l_{i,j} \in \mathbb{Z}$, $i = 1, \ldots, n-1$, $j = 1, \ldots, i$.

THEOREM (Drozd-Ovsienko-Futorny, $\sim$ 1989) The Gelfand-Zetlin formulae define on the vectorspace $V([l])$, spanned by $B([l])$, the structure of a $\mathfrak{g}$-module of finite length. $V([l])$ is simple if and only if $l_{i,j} - l_{i-1,k} \notin \mathbb{Z}$ for all $i = 2, \ldots, n$, $j = 1, \ldots, i$ and $k = 1, \ldots, i-1$.

Gelfand-Zetlin subalgebra

Consider standard $\mathfrak{gl}(1,\mathbb{C}) \subset \mathfrak{gl}(2,\mathbb{C}) \subset \cdots \subset \mathfrak{gl}(n,\mathbb{C})$.

This induces $U(\mathfrak{gl}(1,\mathbb{C})) \subset U(\mathfrak{gl}(2,\mathbb{C})) \subset \cdots \subset U(\mathfrak{gl}(n,\mathbb{C}))$.

Set $Z_i = Z(\mathfrak{gl}(2,\mathbb{C}))$, $i = 1,\dots,n$, and denote $\Gamma = \langle Z_i | i = 1,\dots,n\rangle$ – polynomial algebra in $n(n+1)/2$ variables.

THEOREM

1. (Zhelobenko ?) $\mathcal{B}(\mathfrak{m})$ is an eigenbasis w.r.t. Γ, moreover, Γ separates elements of $\mathcal{B}(\mathfrak{m})$.
2. (Drozd-Futorny-Ovsienko) $B([l])$ is an eigenbasis w.r.t. Γ, moreover, Γ separates elements of $B([l])$.

THEOREM (????, Ovsienko) Γ is a maximal commutative subalgebra of $\mathfrak{g}$.

Problem Find a transformation matrix between $\mathcal{B}(\mathfrak{m})$ with respect to different Γ, or, say, between $\mathcal{B}(\mathfrak{m})$ and the canonical basis.

Gelfand-Zetlin modules

A $\mathfrak{g}$-module, V, is called *Gelfand-Zetlin module* provided it is a direct sum of non-isomorphic f.d. Γ-modules.

Examples: all f.d. modules, all weight $\mathfrak{g}$-modules with f.d. weight spaces and all generic Gelfand-Zetlin modules are Gelfand-Zetlin modules.

THEOREM (Ovsienko, 1998, preprint, yet to appear) Any character of Γ extends to a simple $\mathfrak{g}$-module. Moreover, there exists only finitely many of such extensions up to isomorphism.

Problem When the above extension is unique? E.g. this is the case if the corresponding character of Γ occurs in a generic Gelfand-Zetlin module (Drozd-Ovsienko-Futorny).

Realization of highest weight modules

Fix $\mathfrak{m} = (m_1, \ldots, m_n) \in \mathbb{C}^n$, such that $m_i - m_j \notin \mathbb{Z}_+$, $i < j$.

Define $[l]$ by $l_{i,j} = m_j$ for all i, j.

Define $C([l])$ as the set of all $[t]$ satisfying:

- $t_{n,j} = l_{n,j}$, $j = 1, \ldots, n$;
- $l_{i,j} - t_{i,j} \in \mathbb{Z}_+$ for all i, j;
- $t_{i,j} - t_{i-1,j} \in \mathbb{Z}_+$ for all i, j.

THEOREM Gelfand-Zetlin formulae define on the vector space $W([l])$, spanned by $C([l])$ the structure of a $\mathfrak{g}$-module. Moreover, $W([l])$ is a simple Verma module with highest weight $\mathfrak{m}$.

Remark that this is a precise realization for all simple Verma modules.

Realization of simple dense modules

A $\mathfrak{g}$-module is called *weight* if it is diagonalizable w. r. t. a Cartan subalgebra. We fix $\mathfrak{h} = \langle e_{i,i} | i = 1, \dots, n \rangle$.

A weight $\mathfrak{g}$-module, V, is called *dense* if its support, i.e. the set of $\mathfrak{h}$-weights, coincides with a coset of $\mathfrak{h}^*$ modulo $\mathbb{Z}\Delta$, where Δ is the root system of $\mathfrak{g}$.

Any simple weight $\mathfrak{g}$-module is either dense or parabolically induced (Fernando-Futorny).

Simple dense modules with f.d. weight spaces exist only for A_n and C_n algebras (Fernando).

All simple dense modules with f.d. weight spaces are recently classified (Mathieu, 2000).

Fix $\mathfrak{m} = (m_1, \dots, m_n) \in \mathbb{C}^n$, such that $m_i - m_{i+1} \in \mathbb{N}$, $i = 2, \dots, n-1$, and $x = (x_1, \dots, x_{n-1}) \in \mathbb{C}^{n-1}$ such that $x_i - m_2 \notin \mathbb{Z}$ for all i.

Let $D(x, \mathfrak{m})$ denote the set of all tableaux $[t]$ satisfying:

- $t_{n,j} = m_j$, $j = 1, \dots, n$;
- $t_{i,1} - x_i \in \mathbb{Z}$, $i = 1, \dots, n-1$;
- $l_{i,j} - l_{i-1,j} \in \mathbb{Z}_+$, $i = 3, \dots, n$, $j = 2, \dots, i$;
- $l_{i-1,j} - l_{i,j+1} \in \mathbb{N}$, $i = 3, \dots, n$, $j = 2, \dots, i$.

THEOREM Gelfand-Zetlin formulae define on the space $V(x, \mathfrak{m})$, spanned by $D(x, \mathfrak{m})$, the structure of a $\mathfrak{g}$-module. The module $V(x, \mathfrak{m})$ is simple, dense and has f.d. weight spaces. Moreover, almost all simple dense module with f.d. weight spaces can be realized as $V(x, \mathfrak{m})$ in the sense that all other are given by non-trivial polynomial equalities in the set of parameters.

Quantum deformation

Let q be a non-zero complex non-root of unity. Fix some $h \in \mathbb{C}$ such that $q = \exp(h)$ and for $x \in \mathbb{C}$ put

$$[x]_q = \frac{\exp(hx) - \exp(-hx)}{q - q^{-1}}.$$

$U_q(\mathfrak{gl}(n, \mathbb{C}))$ is generated by $K_i = \tilde{e}_{i,i}$, $i = 1, \ldots, n$, $E_i = \tilde{e}_{i,i+1}$, $F_i = \tilde{e}_{i+1,i}$, $i = 1, \ldots, n-1$; subject to the following relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1},$$
$$K_i E_j = q^{\delta_{i,j} - \delta_{i,j+1}} E_j K_i, \quad K_i F_j = q^{-\delta_{i,j} + \delta_{i,j+1}} F_j K_i,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}},$$
$$[E_i, E_j] = [F_i, F_j] = 0, \quad |i - j| \geq 2,$$
$$E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 = 0,$$
$$F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 = 0.$$

Substituting $(t_{i,j} - t_{i',j'})$ with $[t_{i,j} - t_{i',j'}]_q$ in Gelfand-Zetlin formulae we get *quantum Gelfand-Zetlin formulae*. They tend to the classical ones under $q \to 1$.

THEOREM (Jimbo) Quantum Gelfand-Zetlin formulae define on $V_q(\mathfrak{m}) = V(\mathfrak{m})$ (space of simple f.d. $\mathfrak{g}$-module) the structure of $U_q(\mathfrak{g}l(n, \mathbb{C}))$-module, which is a deformation of $V(\mathfrak{m})$.

THEOREM (M.-Turowska) Generic Gelfand-Zetlin modules admit quantum deformation.

THEOREM All simple dense $\mathfrak{g}$-modules with f.d. weight spaces admit quantum deformation.

THEOREM Certain $\mathfrak{g}l(k, \mathbb{C})-\mathfrak{g}l(n, \mathbb{C})$ Harish-Chandra modules admit quantum deformation.

THEOREM Modules, parabolically induced from simple generic Gelfand-Zetlin modules (certain generalized Verma modules) admit quantum deformation.