GELFAND-ZETLIN MODULES
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\( \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \):
\( \{ e_{i,j} | i, j = 1, \ldots, n \} \) - matrix units, basis of \( \mathfrak{g} \).

Tableau \([ t ] = (t_{i,j})_{i,j=1}^{i=1,\ldots,n} \)

Simple f.d. \( \mathfrak{g} \)-modules \( \leftrightarrow \) \( m = (m_1, \ldots, m_n) \in \mathbb{C}^n \), \( m_i - m_{i+1} \in \mathbb{N} \)

**Theorem** (Gelfand-Zetlin, 1950) The simple f.d. \( \mathfrak{g} \)-module \( V = V(m) \), corresponding to \( m \) as above has a basis, \( B(m) \), consisting of all tableaux \([ t ] \) such that

- \( t_{n,j} = m_j \), \( j = 1, \ldots, n \);
- \( t_{i,j} - t_{i-1,j} \in \mathbb{Z}_+, i = 2, \ldots, n, j = 1, \ldots, i \);
- \( t_{i-1,j} - t_{i,j+1} \in \mathbb{N}, i = 2, \ldots, n, j = 1, \ldots, i - 1 \).

The action of \( \mathfrak{g} \) on \( B(m) \) is given by the following Gelfand-Zetlin formulae

\[
\begin{align*}
e_{i,i}[t] &= \left( \sum_{j=1}^{i} t_{i,j} - \sum_{j=1}^{i-1} t_{i-1,j} \right) [t], \\
e_{i,i+1}[t] &= -\sum_{j=1}^{i} \prod_{k=1}^{i-1}(t_{i,j} - t_{i+1,k})[t], \\
e_{i+1,i}[t] &= \sum_{j=1}^{i} \prod_{k=1}^{i-1}(t_{i,j} - t_{i,k})[t] - [\delta^{i,j}]
\end{align*}
\]

**Generic Gelfand-Zetlin modules**

Fix \([t]\) such that \( l_{i,j} - l_{i,k} \notin \mathbb{Z} \) for all \( i < n \) and \( j \neq k \).

Let \( B([t]) \) consist of all \([t]\) such that

- \( t_{n,j} = l_j \), \( j = 1, \ldots, n \);
- \( t_{i,j} - l_{i,j} \in \mathbb{Z}, i = 1, \ldots, n - 1, j = 1, \ldots, i \).

**Theorem** (Drozd-Ovsienko-Futorny, \( \sim 1989 \)) The Gelfand-Zetlin formulae define on the vectorspace \( V([t]) \), spanned by \( B([t]) \), the structure of a \( \mathfrak{g} \)-module of finite length, \( V([t]) \) is simple if and only if \( l_{i,j} - l_{i-1,k} \notin \mathbb{Z} \) for all \( i = 2, \ldots, n, j = 1, \ldots, i \) and \( k = 1, \ldots, i - 1 \).

**Gelfand-Zetlin subalgebra**

Consider standard \( \mathfrak{gl}(1, \mathbb{C}) \subset \mathfrak{gl}(2, \mathbb{C}) \subset \cdots \subset \mathfrak{gl}(n, \mathbb{C}) \).
This induces \( U(\mathfrak{gl}(1, \mathbb{C})) \subset U(\mathfrak{gl}(2, \mathbb{C})) \subset \cdots \subset U(\mathfrak{gl}(n, \mathbb{C})) \).
Set \( Z_i = Z(\mathfrak{gl}(2, \mathbb{C})), i = 1, \ldots, n \), and denote \( \Gamma = \langle Z_i | i = 1, \ldots, n \rangle \) - polynomial algebra in \( n(n+1)/2 \) variables.

**Theorem**

1. (Zhelobenko) \( B(m) \) is an eigenbasis w.r.t. \( \Gamma \), moreover, \( \Gamma \) separates elements of \( B(m) \).
2. (Drozd-Futorny-Ovsienko) \( B([t]) \) is an eigenbasis w.r.t. \( \Gamma \), moreover, \( \Gamma \) separates elements of \( B([t]) \).
THEOREM (????, Ovsienko) $\Gamma$ is a maximal commutative subalgebra of $\mathfrak{g}$.

Problem Find a transformation matrix between $B(m)$ with respect to different $\Gamma$, or, say, between $B(m)$ and the canonical basis.

Gelfand-Zetlin modules

A $\mathfrak{g}$-module, $V$, is called a **Gelfand-Zetlin module** provided it is a direct sum of non-isomorphic f.d. $\Gamma$-modules.

Examples: all f.d. modules, all weight $\mathfrak{g}$-modules with f.d. weight spaces and all generic Gelfand-Zetlin modules are Gelfand-Zetlin modules.

THEOREM (Ovsienko, 1998, preprint, yet to appear) Any character of $\Gamma$ extends to a simple $\mathfrak{g}$-module. Moreover, there exists only finitely many of such extensions up to isomorphism.

Problem When the above extension is unique? E.g., this is the case if the corresponding character of $\Gamma$ occurs in a generic Gelfand-Zetlin module (Drozd-Ovsienko-Futorny).

Realization of highest weight modules

Fix $m = (m_1, \ldots, m_n) \in \mathbb{C}^n$, such that $m_i - m_j \notin \mathbb{Z}_+$, $i < j$.

Define $[l]$ by $l_{i,j} = m_j$ for all $i, j$.

Define $C([l])$ as the set of all $[l]$ satisfying:

- $t_{n,j} = l_{n,j}$, $j = 1, \ldots, n$;
- $t_{i,j} - l_{i,j} \in \mathbb{Z}_+$ for all $i, j$;
- $t_{i,j} - t_{i-1,j} \in \mathbb{Z}_+$ for all $i, j$.

THEOREM Gelfand-Zetlin formulae define on the vector space $W([l])$, spanned by $C([l])$ the structure of a $\mathfrak{g}$-module. Moreover, $W([l])$ is a simple Verma module with highest weight $m$.

Remark that this is a precise realization for all simple Verma modules.

Realization of simple dense modules

Fix $m = (m_1, \ldots, m_n) \in \mathbb{C}^n$, such that $m_i - m_{i+1} \in \mathbb{N}$, $i = 2, \ldots, n-1$, and $x = (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$ such that $x_i - m_2 \notin \mathbb{Z}$ for all $i$.

Let $D(x, m)$ denote the set of all tableaux $[l]$ satisfying:

- $t_{n,j} = m_j$, $j = 1, \ldots, n$;
- $t_{i-1} - x_i \in \mathbb{Z}$, $i = 1, \ldots, n-1$;
- $l_{i,j} - l_{i-1,j} \in \mathbb{Z}_+$, $i = 3, \ldots, n$, $j = 2, \ldots, i$;
- $t_{i-1,j} - t_{i,j+1} \in \mathbb{N}$, $i = 3, \ldots, n$, $j = 2, \ldots, i$.

THEOREM Gelfand-Zetlin formulae define on the space $V(x, m)$, spanned by $D(x, m)$, the structure of a $\mathfrak{g}$-module. The module $V(x, m)$ is simple, dense and has f.d. weight spaces. Moreover, almost all simple dense module with f.d. weight spaces can be realized as $V(x, m)$ in the sense that all other are given by non-trivial polynomial equalities in the set of parameters.

Quantum deformation

Let $q$ be a non-zero complex non-root of unity. Consider $U_q(gl(n, \mathbb{C}))$. Substituting $(t_{i,j} - t_{i',j'})$ with $[t_{i,j} - t_{i',j'}]_q$ in Gelfand-Zetlin formulae we get quantum Gelfand-Zetlin formulae. They tend to the classical ones under $q \to 1$.

THEOREM (Jimbo) Quantum Gelfand-Zetlin formulae define on $V_q(m) = V(m)$ (space of simple f.d. $\mathfrak{g}$-module) the structure of $U_q(gl(n, \mathbb{C}))$-module, which is a deformation of $V(m)$.

THEOREM (M.-Turowska) Generic Gelfand-Zetlin modules admit quantum deformation.

THEOREM All simple dense $\mathfrak{g}$-modules with f.d. weight spaces admit quantum deformation.

THEOREM Certain $gl(k, \mathbb{C}) - gl(n, \mathbb{C})$ Harish-Chandra modules admit quantum deformation.

THEOREM Modules, parabolically induced from simple generic Gelfand-Zetlin modules (certain generalized Verma modules) admit quantum deformation.