ON FINITISTIC DIMENSION OF STRATIFIED ALGEBRAS

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1. Notation

$k$ — algebraically closed field.

$A$ — finite-dimensional associative $k$-algebra.

$A$-mod — category of all finite-dimensional $A$-modules.

$\{e_1, \ldots, e_n\}$ — a complete set of primitive idempotents.

$L(i), P(i), I(i), i = 1, \ldots, n,$ — the corresponding simple, projective and injective modules.

\[ L = \bigoplus_{i=1}^{n} L(i), \quad P = \bigoplus_{i=1}^{n} P(i), \quad I = \bigoplus_{i=1}^{n} I(i). \]
2. (Generalized) tilting module

**Definition.** $T \in A$-mod is called a (generalized) *tilting* module provided that

1. $\text{Ext}^i_A(T, T) = 0$, $i > 0$;
2. $\text{p.d.}(T) < \infty$;
3. there exists a coresolution $0 \to P \to T_0 \to \cdots \to T_k \to 0$, where $T_i \in \text{Add}(T)$ for all $i$.

**Remark.** Minimal $k$ above equals $\text{p.d.}(T)$.

3. Duality

**Definition.** The algebra $A$ is said to have a (simple preserving) *duality*, if there exists a contravariant exact equivalence on $A$-mod, which preserves the iso-classes of simple modules.

**Example.** Any isomorphism $\varphi : A \cong A^{opp}$, such that $\varphi(e_i) = e_i$ for all $i$, gives rise to a duality.
3. Finitistic dimension

*Global dimension of $A$:*

$$\text{gl.d.}(A) = \max_{M \in A\text{-mod}} \text{p.d.}(M).$$

$\mathcal{P}^{<\infty}(A)$ — the full subcategory of $A\text{-mod}$, consisting of all modules of finite projective dimension.

*Projectively defined finitistic dimension of $A$:*

$$\text{fin.d.}(A) = \max_{M \in \mathcal{P}^{<\infty}(A)} \text{p.d.}(M).$$

**Finitistic dimension conjecture.** $\text{fin.d.}(A) < \infty$ for every $A$. 
4. Finitistic dimension algebras with duality and self-dual tilting modules

Lemma. Assume that \( \text{p.d.}(I) < \infty \). Then \( \text{fin.d.}(A) = \text{p.d.}(I) \).

Proof. Let \( M \in \mathcal{P}^{<\infty}(A) \) be such that \( \text{p.d.}(M) = \text{fin.d.}(A) = m \). Choose \( M \hookrightarrow \hat{I} \twoheadrightarrow K \) and apply \( \text{Hom}_A(\_, S) \). One gets the exact sequence

\[
\cdots \to \text{Ext}_A^m(\hat{I}, S) \to \text{Ext}_A^m(M, S) \neq 0 \to \text{Ext}_A^{m+1}(K, S) = 0.
\]

Hence \( \text{Ext}_A^m(\hat{I}, S) \neq 0 \) and therefore \( \text{p.d.}(I) = \text{p.d.}(\hat{I}) = m = \text{fin.d.}(A) \). Q.E.D.

Theorem A. [M.-Ovsienko] Assume that

(i) \( A \) has a duality, \( \circ \).

(ii) There is a (generalized) tilting module, \( T \), such that all indecomposable summands of \( T \) are self-dual with respect to \( \circ \).

(iii) \( \text{fin.d.}(A) < \infty \).

Then \( \text{fin.d.}(A) = 2 \cdot \text{p.d.}(T) \)
Proof. Let
\[ 0 \to P \to T_0 \to \cdots \to T_k \to 0 \]  \hspace{1cm} (1)
be a minimal tilting coresolution of \( P \). Remark that \( k = \text{p.d.}(T) \).
Apply \( \circ \) form (i) and use (ii) to obtain a tilting resolution for \( I \):
\[ 0 \to T_k \to \cdots \to T_0 \to I \to 0. \]  \hspace{1cm} (2)
In particular, \( \text{p.d.}(I) < \infty \). Hence Lemma implies that \( \text{fin.d.}(A) \)
equals the maximal \( m \) such that \( \text{Ext}^m_A(I, P) \neq 0 \). We calculate such \( m \) using (1) and (2).

In \( D^b(A) \) we can substitute \( P \) and \( I \) by tilting complexes \( \mathcal{T}_1^\bullet \)
and \( \mathcal{T}_2^\bullet \) obtained from (1) and (2) respectively. Then the extensions
can be calculated as the usual homomorphisms between the shifted complexes up to homotopy.

If \( t > 2k \), we have the following picture for the homomorphisms from \( \mathcal{T}_2^\bullet[-t] \) to \( \mathcal{T}_1^\bullet \):
\[ \cdots \to 0 \to 0 \to 0 \to \cdots \to T_k \to \cdots \to g_{k-1} T_0 \to \cdots \]
\[ \cdots \to T_0 \to T_k \to 0 \to \cdots \to 0 \to 0 \to \cdots \to 0 \to \cdots \]
Hence \( \text{Ext}^t_A(I, P) = 0 \) for all \( t > 2k \).

If \( t > 2k \), we have the following non-trivial homomorphism:
\[ \cdots \to 0 \to 0 \to T_k \to T_{k-1} \to \cdots \to T_0 \to \cdots \]
\[ \cdots \to T_0 \to T_{k-1} \to T_k \to 0 \to \cdots \to 0 \to \cdots \]
Minimality of the resolution implies that it is not homotopic to zero, giving a non-trivial extension of degree \( 2k \) between \( I \) and \( P \).
\ \textbf{Q.E.D.}
5. Various stratified algebras

For $i = 1, \ldots, n$ define:

- **standard modules** $\Delta(i)$ as the maximal quotient of $P(i)$ such that 
  $[\Delta(i) : L(j)] = 0, j > i$;

- **proper standard modules** $\overline{\Delta}(i)$ as the maximal quotient of $\Delta(i)$
  such that $[\overline{\Delta}(i) : L(i)] = 1$;

- **costandard modules** $\nabla(i)$ as the maximal submodule of $I(i)$ such that 
  $[\nabla(i) : L(j)] = 0, j > i$;

- **proper costandard modules** $\overline{\nabla}(i)$ as the maximal submodule of $\nabla(i)$
  such that $[\overline{\nabla}(i) : L(i)] = 1$.

**Definition.** A is called **strongly standardly stratified** provided that for every $i$
the kernel of $P(i) \rightarrow \Delta(i)$ has a filtration with subquotients $\Delta(j), j > i$.

**Definition.** A is called **properly stratified** provided that it is strongly
standardly stratified and each $\Delta(i)$ has a filtration with subquotients $\overline{\Delta}(i)$.

**Definition.** A is called **quasi-hereditary** provided that it is properly
stratified and $\Delta(i) = \overline{\Delta}(i)$ for all $i$. 
6. Application of Theorem A to quasi-hereditary algebras

$A$ — quasi-hereditary with duality.

$\mathcal{F}(\Delta)$ — category of modules having a standard filtration.

$\mathcal{F}(\nabla)$ — category of modules having a costandard filtration.

**Fact.** $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \text{Add}(T)$, where $T$ is a (generalized) tilting module with self-dual indecomposable summands.

**Corollary.** $\text{gl.dim.}(A) = 2 \cdot \text{p.d.}(T)$.

**Corollary.** $\text{gl.dim.}(A) = 2 \cdot \dim_{\Delta}(A) = 2 \cdot \text{gl.dim.}(B)$, where $\dim_{\Delta}(A)$ is the $\Delta$-filtration dimension of $A$, and $B$ is an exact Borel subalgebra of some $A' \simeq_{\text{Morita}} A$. 
7. Application of Theorem A to properly stratified algebras

$A$ — properly stratified with duality.

$\mathcal{F}(\Delta), \mathcal{F}(\nabla)$ as above.

$\mathcal{F}(\overline{\Delta})$ — category of modules having a proper standard filtration.

$\mathcal{F}(\overline{\nabla})$ — category of modules having a proper costandard filtration.

**Fact.** $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \text{Add}(T)$, where $T$ is a (generalized) tilting module.

**Fact.** $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla) = \text{Add}(C)$, where $C$ is a (generalized) cotilting module.

**Fact.** If $T = C$ then all indecomposable summands of $T$ are self-dual.

**Corollary.** Assume $T = C$. Then $\text{fin.d.}(A) = 2 \cdot \text{p.d.}(T)$.

**Conjecture.** [M.-Parker] $\text{fin.d.}(A) = 2 \cdot \text{p.d.}(T)$ for any properly stratified algebra $A$ with duality.
7. New generalized tilting module for strongly stratified algebras

$A$ — strongly stratified.

$T$ — characteristic tilting module for $A$.

$R = \text{End}_A(T)$ — the Ringel dual of $A$.

$F = \text{Hom}_A(T, \_): A \text{- mod} \rightarrow R \text{- mod}$ — the Ringel duality functor.

**Fact.** $F : \mathcal{F}(\nabla^{(A)}) \rightarrow \mathcal{F}(\Delta^{(R)})$ is an exact equivalence.

**Theorem.** [Frisk-M.] Assume that $R$ is properly stratified, then $H = F^{-1}(T^{(R)})$ is a (generalized) tilting module for $A$.

**Corollary.** Assume that $R$ is properly stratified. Then

$$\text{fin.d.}(A) = \text{p.d.}(H).$$

**Corollary.** Assume that $R$ is properly stratified. Then $\mathcal{P}^{<\infty}(A)$ is contravariantly finite.
7. Two-step duality for strongly stratified algebras

$A$ — strongly stratified.

Assume that $R$ is properly stratified.

$H$ — new (two-step) tilting module for $A$.

**Theorem.** [Frisk-M.]

1. $B = \text{End}_A(H)^{opp}$ is strongly stratified.
2. The Ringel dual of $B$ is properly stratified.
3. The two-step dual for $B$ is Morita equivalent to $A^{opp}$.

$G = \text{Hom}_k(\text{Hom}_A(-, H), k) : A-\text{mod} \to B-\text{mod}$ — the two-step duality functor.

**Corollary.** $G : \mathcal{P}^{<\infty}(A) \to \mathcal{I}^{<\infty}(B)$ is an exact equivalence.
8. Finitistic dimension for strongly stratified algebras

**Theorem.** [Frisk-M.] Assume that both $A$ and $R$ are properly stratified with duality. Then

$$\text{fin.d.}(A) = 2 \cdot \text{p.d.}(T).$$

**Theorem.** [Frisk-M.] Assume that $A$ is properly stratified with duality and $R$ is properly stratified. Then

$$\text{fin.d.}(A) = 2 \cdot \text{p.d.}(T^{(R)}).$$