TWISTED GENERALIZED WEYL ALGEBRAS

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Definition of algebras.

$k \in \mathbb{N}$, $\mathbb{N}_k = \{1, 2, \dots, k\}$, R ring with 1, $Z(R)$ – center of R.

$\{\sigma_i \mid 1 \le i \le k\}$ pairwise commuting automorphisms of R,

$M = (\mu_{i,j})_{i,j \in \mathbb{N}_k}$, $\mu_{i,j} \in Z(R)$ invertible and stable under all σ_i.

$0 \ne t_i \in Z(R)$, $i \in \mathbb{N}_k$, such that:

$$t_i t_j = \mu_{i,j}\mu_{j,i}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i), \quad i, j \in \mathbb{N}_k, i \ne j.$$

$\mathcal{A}$ — unital R-algebra generated over R by X_i, Y_i, $i \in \mathbb{N}_k$, with relations

- $X_i r = \sigma_i(r) X_i$ for any $r \in R$, $i \in \mathbb{N}_k$;
- $Y_i r = \sigma_i^{-1}(r) Y_i$ for any $r \in R$, $i \in \mathbb{N}_k$;
- $X_i Y_j = \mu_{i,j} Y_j X_i$ for any $i, j \in \mathbb{N}_k$, $i \ne j$;
- $Y_i X_i = t_i$, $i \in \mathbb{N}_k$;

- $X_iY_i = \sigma_i(t_i)$, $i \in \mathbb{N}_k$.

$\mathcal{A}$ is obtained from R, M, $\{\sigma_i\}$ and $\{t_i\}$ by *twisted generalized Weyl construction* (TGWC). It is $\mathbb{Z}^k$-graded.

Let R be commutative. The *twisted generalized Weyl algebra* (TGWA) $\hat{\mathcal{A}} = \mathcal{A}(R, \sigma_1, \dots, \sigma_k, t_1, \dots, t_k)$ of rank k is the quotient ring $\mathcal{A}/I$, where I is the (unique) maximal graded two-sided ideal of $\mathcal{A}$ intersecting R trivially.

$\mathfrak{M} = max(R)$. For $\mathfrak{m} \in \mathfrak{M}$ and an $\mathcal{A}$-module ($\hat{\mathcal{A}}$-module) V set $V_{\mathfrak{m}} = \{v \in V \mid \mathfrak{m}v = 0\}$. An $\mathcal{A}$-module ($\hat{\mathcal{A}}$-module), M, will be called *weight* provided $M = \sum_{\mathfrak{m} \in \mathfrak{M}} M_{\mathfrak{m}}$.

Shapovalov form and weight modules.

R – commutative. There is a unique anti-involution, $*$, on A such that $(X_i)^* = Y_i$ for any $i = 1, 2, \dots, n$ and $r^* = r$ for any $r \in R$.

A — TGWC. $\mathfrak{p} : A \to A_0$ — the graded projection on the zero component.

For $u, v \in A$ put $F^l(u, v) = \mathfrak{p}(u^*v) \in A_0 = R$ and $F^r(u, v) = \mathfrak{p}(uv^*) \in A_0 = R$. We will call F^l the *left Shapovalov form* on A and F^r the *right Shapovalov form* on A.

LEMMA

1. $F^l : A \times A \to R$ and $F^r : A \times A \to R$ are R-bilinear form.
2. $F^l(xu, v) = F^l(u, x^*v)$ and $F^r(u, vx) = F^r(ux^*, v)$ for all $u, x, v \in A$.
3. $F^l(u, v) = F^l(v, u)$ and $F^r(u, v) = F^r(v, u)$ for all $u, v \in A$.

4. $F^l(A_g, A_h) = 0$ and $F^r(A_g, A_h) = 0$ for any $g \neq h \in \mathbb{Z}^n$.
5. The ideal, generated by the intersection of the kernels of F^l and F^r coincides with the maximal graded ideal of A intersecting R trivially.
6. The intersection of the kernels of F^l and F^r coincides with I.
7. The kernel of F^l coincides with I (and coincides with the kernel of F^r).

COROLLARY Let A be as above and J be a graded two-sided ideal of A, stable under $*$ and intersecting R trivially. Denote by $\tilde{F}$ the form induced by $F = F^l$ on the quotient $\tilde{A} = A/J$. Then $\tilde{A}$ is isomorphic to the TGWA $\hat{A}$ if and only if $\tilde{F}$ is non-degenerate on $\tilde{A}$.

Consider $\hat{A}$ as a regular left $\hat{A}$-module and fix an ideal, $\mathfrak{m}$, in R. Set $N(\mathfrak{m}) = \{x \in \hat{A} \,|\, \tilde{F}(x, y) \in \mathfrak{m}$ for any $y \in \hat{A}\}$.

THEOREM

1. $N(\mathfrak{m})$ is a graded submodule of $\hat{A}$;
2. $N(\mathfrak{m})_0 = \mathfrak{m}$;
3. If $\mathfrak{m} \in \mathfrak{M}$ then $M(\mathfrak{m}) = \hat{A}/N(\mathfrak{m})$ is a simple graded $\hat{A}$-module.
4. Up to a shift of grading, all weight simple ($\mathbb{Z}^k$-) graded $\hat{A}$-modules are exhausted by $\{M(\mathfrak{m})\}$.

Mickelsson (step) algebras

$(\mathfrak{g}, \mathfrak{k})$ — reductive pair of complex f.d. Lie algebras, $\Delta_{\mathfrak{k}} = \Delta_{\mathfrak{k}}^{+} \cup \Delta_{\mathfrak{k}}^{-}$ the root system of $\mathfrak{k}$ w.r.t. $\mathfrak{h}$. For a $\mathfrak{g}$-module V, V^{+} is the set $\{v \in V \mid X_\alpha v = 0 \text{ for all } \alpha \in \Delta_{\mathfrak{k}}^{+}\}$. For the algebra $\mathfrak{n}_{+} = \mathfrak{n}_{+}(\mathfrak{k})$ set $I_{+} = U(\mathfrak{g})\mathfrak{n}_{+}$ and $V(\mathfrak{g}, \mathfrak{k}) = U(\mathfrak{g})/I_{+}$. Then the *Mickelsson step algebra* $S(\mathfrak{g}, \mathfrak{k})$, associated with $(\mathfrak{g}, \mathfrak{k})$, is defined as $V(\mathfrak{g}, \mathfrak{k})^{+}$.

A slightly more convenient algebra appears if we invert $U(\mathfrak{h})$. Let $D(\mathfrak{h})$ denote the fraction field of $U(\mathfrak{h})$. Set $U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} D(\mathfrak{h})$, $I'_{+} = U'(\mathfrak{g})\mathfrak{n}_{+}$, $V'(\mathfrak{g}, \mathfrak{k}) = U'(\mathfrak{g})/I'_{+}$ and $Z(\mathfrak{g}, \mathfrak{k}) = V'(\mathfrak{g}, \mathfrak{k})^{+}$.

$\mathfrak{g}_n = \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{h}_n$ the subalgebra of diagonal matrices. $AZ_n = Z(\mathfrak{gl}(n+1, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}) \oplus \mathbb{C}e_{n+1,n+1})$.

AZ_n has the following presentation. It is generated (over the field $D_{n+1} = D(\mathfrak{h}_{n+1})$) by elements z_i, $i \in \{\pm 1, \pm 2, \ldots, \pm n\}$, with relations:

- $z_i z_j = \alpha_{i,j} z_j z_i$, $i + j \neq 0$;
- $z_i z_{-i} = \sum_{j=1}^{n} \beta_{i,j} z_{-j} z_j + \gamma_i$, $i = 1, 2, \ldots, n$;
- $[h_j, z_i] = (\varepsilon_i - \varepsilon_{n+1})(h_j) z_i$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n+1$;
- $[h_j, z_{-i}] = (\varepsilon_{n+1} - \varepsilon_i)(h_j) z_{-i}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n+1$;

where

$$\alpha_{i,j} = \alpha_{-j,-i} = \frac{\phi_{i,j}^{+}}{\phi_{i,j}}, 1 \leq i < j \leq n; \quad \alpha_{i,j} = 1, \mathrm{sign}(i) \neq \mathrm{sign}(j);$$

$$\beta_{i,j} = \delta_i^{-} \gamma_{i,j} \delta_j^{+}; \quad \gamma_i = \delta_i^{-} \phi_{i,n+1}^{-}; \quad \phi_{i,j} = h_i - h_j + j - i, \quad \phi_{i,j}^{\pm} = \phi_{i,j} \pm 1;$$

$$\gamma_{i,j} = (1 - \phi_{i,j})^{-1}; \quad \delta_i^{\pm} = \prod_{k=i+1}^{n} \frac{\phi_{i,k}^{\pm}}{\phi_{i,k}}; \quad \varepsilon_i(h_j) = \delta_{i,j}, i, j = 1, 2, \ldots, n+1.$$

Set $t_i = z_{-i}z_i$ and denote by R the algebra, generated by $t_1, \ldots, t_n$ over the field D_{n+1}. Define σ_i, $i = 1, 2, \ldots, n$, as follows:

$$\sigma_i(h_k) = h_k, k \neq i, n+1; \quad \sigma_i(h_i) = h_i - 1; \sigma_i(h_{n+1}) = h_{n+1} + 1;$$

$$\sigma_i(t_j) = \frac{\phi_{i,j}^-}{\phi_{i,j}^- - 1} t_j, \quad j < i; \quad \sigma_i(t_j) = \frac{\phi_{i,j}}{\phi_{i,j}^-} t_j, \quad j > i;$$

$$\sigma_i(t_i) = \sum_{k=1}^{n} \beta_{i,k} t_k + \gamma_i.$$

THEOREM AZ_n is the TGWA associated with R, $\{\sigma_i\}$ and $\{t_i\}$ above (all $\mu_{i,j} = 1$).

The most difficult part for TGWC is to prove that σ_i's commute.

To go from TGWC to TGWA use the Diamond Lemma and Shapovalov form.

Extended OGZ-algebras

Let $r = (k-1, k, k+1)$. If $\mathbb{F}$ is a field, $\mathcal{L} = \mathcal{L}(\mathbb{F}, r) = \mathbb{F}^{3k}$ with elements $[l] = \{l_{i,j} \mid i = 1, 2, 3; j = 1, \ldots, k+2-i\}$, called *tableaux*. $\mathcal{L}_0$ is the subset of $\mathcal{L}$ that consists of all $[l]$ such that $l_{1,j} = 0$, $l_{3,j} = 0$, $l_{2,j} \in \mathbb{Z}$ for all j.
Λ — rational functions in $\lambda_{i,j}$ for all i, j. $[\mathfrak{l}] \in \mathcal{L}(\Lambda, r)$ defined by $\mathfrak{l}_{i,j} = \lambda_{i,j}$. $M = M([\mathfrak{l}])$ is Λ-v.sp. with the basis $v_{[t]}$, $[t] \in [\mathfrak{l}] + \mathcal{L}_0$.

For $[t] \in [\mathfrak{l}] + \mathcal{L}_0$ and $1 \leq j \leq k$ let

$$a_j^{\pm}([t]) = \mp \frac{\prod_m (t_{2\pm 1,m} - t_{2,j})}{\prod_{m \neq j} (t_{2,m} - t_{2,j})},$$

define Λ-linear operators $X_j^{\pm} : M \to M$, $X_j^{\pm} v_{[t]} = a_j^{\pm}([t]) v_{[t] \pm [\delta^{2,j}]}$, and $H_{i,j} : M \to M$, $H_{i,j} v_{[t]} = t_{i,j} v_{[t]}$, $i = 1, 2, 3$.

Let $\mathcal{Q}$ be the localization of $\mathbb{C}[H_{i,j}, 1 \leq i \leq n, 1 \leq j \leq r_i]$ w.r.t. the multiplicative set, generated by $H_{2,j} - H_{2,l} + m$ for all $j \neq l$ and $m \in \mathbb{Z}$. The *extended orthogonal Gelfand-Zetlin algebra* $\mathcal{U}$ is the $\mathbb{C}$-algebra, generated over $\mathbb{C}$ by $\mathcal{Q}$ and $X_{i,j}^{\pm}$, $2 \leq i \leq n-1$, $1 \leq j \leq r_i$.

$$T_i = -\frac{\prod_{j=1}^{k+1} (H_{3,j} - H_{2,i}) \prod_{j=1}^{k-1} (H_{1,j} - H_{2,i} - 1)}{\prod_{j \neq i} (H_{2,j} - H_{2,i}) \prod_{j \neq i} (H_{2,j} - H_{2,i} - 1)},$$

$$\sigma_i(H_{2,i}) = H_{2,i} - 1; \quad \sigma_i(H_{k,l}) = H_{k,l}, \quad k \neq 2 \text{ or } l \neq i.$$

THEOREM $\mathcal{U}$ is the TGWA associated with $\mathcal{Q}$, $\{\sigma_i\}$ and $\{T_i\}$.