TWISTED GENERALIZED WEYL ALGEBRAS

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Definition of algebras.

$k \in \mathbb{N}, \mathbb{N}_k = \{1, 2, \ldots, k\}$, $R$ ring with 1, $Z(R)$ – center of $R$. 
$\{\sigma_i | 1 \leq i \leq k\}$ pairwise commuting automorphisms of $R$, 
$M = (\mu_{i,j})_{i,j \in \mathbb{N}_k}$, $\mu_{i,j} \in Z(R)$ invertible and stable under all $\sigma_i$. 
$0 \neq t_i \in Z(R), i \in \mathbb{N}_k$, such that:

$$t_it_j = \mu_{i,j}\mu_{j,i}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i), \quad i, j \in \mathbb{N}_k, i \neq j.$$

$\mathcal{A}$ — unital $R$-algebra generated over $R$ by $X_i, Y_i, i \in \mathbb{N}_k$, with relations

- $X_ir = \sigma_i(r)X_i$ for any $r \in R, i \in \mathbb{N}_k$;
- $Y_ir = \sigma_i^{-1}(r)Y_i$ for any $r \in R, i \in \mathbb{N}_k$;
- $X_iY_j = \mu_{i,j}Y_jX_i$ for any $i, j \in \mathbb{N}_k, i \neq j$;
- $Y_iX_i = t_i, i \in \mathbb{N}_k$;
• \( X_i Y_i = \sigma_i(t_i), \ i \in \mathbb{N}_k. \)

\( \mathcal{A} \) is obtained from \( R, \ M, \ \{\sigma_i\} \) and \( \{t_i\} \) by twisted generalized Weyl construction (TGWC). It is \( \mathbb{Z}^k \)-graded.

Let \( R \) be commutative. The twisted generalized Weyl algebra (TGWA) \( \hat{\mathcal{A}} = \mathcal{A}(R, \sigma_1, \ldots, \sigma_k, t_1, \ldots, t_k) \) of rank \( k \) is the quotient ring \( \mathcal{A}/I \), where \( I \) is the (unique) maximal graded two-sided ideal of \( \mathcal{A} \) intersecting \( R \) trivially.

\( \mathfrak{M} = \text{max}(R) \). For \( m \in \mathfrak{M} \) and an \( \mathcal{A} \)-module (\( \hat{\mathcal{A}} \)-module) \( V \) set \( V_m = \{v \in V \mid mv = 0\} \). An \( \mathcal{A} \)-module (\( \hat{\mathcal{A}} \)-module), \( M \), will be called weight provided \( M = \sum_{m \in \mathfrak{M}} M_m \).

**Shapovalov form and weight modules.**

\( R \) — commutative. There is a unique anti-involution, \( * \), on \( A \) such that \( (X_i)^* = Y_i \) for any \( i = 1, 2, \ldots, n \) and \( r^* = r \) for any \( r \in R \).

\( A \) — TGWC. \( \mathfrak{p} : A \to A_0 \) — the graded projection on the zero component.

For \( u, v \in A \) put \( F^l(u, v) = \mathfrak{p}(u^*v) \in A_0 = R \) and \( F^r(u, v) = \mathfrak{p}(uv^*) \in A_0 = R \). We will call \( F^l \) the left Shapovalov form on \( A \) and \( F^r \) the right Shapovalov form on \( A \).

**LEMMA**

1. \( F^l : A \times A \to R \) and \( F^r : A \times A \to R \) are \( R \)-bilinear form.

2. \( F^l(xu, v) = F^l(u, x^*v) \) and \( F^r(u, vx) = F^r(ux^*, v) \) for all \( u, x, v \in A \).

3. \( F^l(u, v) = F^l(v, u) \) and \( F^r(u, v) = F^r(v, u) \) for all \( u, v \in A \).
4. $F^l(A_g, A_h) = 0$ and $F^r(A_g, A_h) = 0$ for any $g \neq h \in \mathbb{Z}^n$.

5. The ideal, generated by the intersection of the kernels of $F^l$ and $F^r$ coincides with the maximal graded ideal of $A$ intersecting $R$ trivially.

6. The intersection of the kernels of $F^l$ and $F^r$ coincides with $I$.

7. The kernel of $F^l$ coincides with $I$ (and coincides with the kernel of $F^r$).

**COROLLARY** Let $A$ be as above and $J$ be a graded two-sided ideal of $A$, stable under $*$ and intersecting $R$ trivially. Denote by $	ilde{F}$ the form induced by $F = F^l$ on the quotient $	ilde{A} = A/J$. Then $	ilde{A}$ is isomorphic to the TGWA $\hat{A}$ if and only if $\tilde{F}$ is non-degenerate on $\tilde{A}$.

Consider $\hat{A}$ as a regular left $\hat{A}$-module and fix an ideal, $m$, in $R$. Set $N(m) = \{x \in \hat{A} | \tilde{F}(x, y) \in m \text{ for any } y \in \hat{A}\}$.

**THEOREM**

1. $N(m)$ is a graded submodule of $\hat{A}$;

2. $N(m)_0 = m$;

3. If $m \in \mathfrak{M}$ then $M(m) = \hat{A}/N(m)$ is a simple graded $\hat{A}$-module.

4. Up to a shift of grading, all weight simple ($\mathbb{Z}^k$) graded $\hat{A}$-modules are exhausted by $\{M(m)\}$.
Mickelsson (step) algebras

\((g, \mathfrak{k})\) — reductive pair of complex f.d. Lie algebras, \(\Delta = \Delta^+ \cup \Delta^-\) the root system of \(\mathfrak{k}\) w.r.t. \(\mathfrak{h}\). For a \(g\)-module \(V\), \(V^+\) is the set \(\{v \in V \mid X_\alpha v = 0 \text{ for all } \alpha \in \Delta^+\}\). For the algebra \(n_+ = n_+(\mathfrak{k})\) set \(I_+ = U(g)n_+\) and \(V'(g, \mathfrak{k}) = U(g)/I_+\). Then the Mickelsson step algebra \(S(g, \mathfrak{k})\), associated with \((g, \mathfrak{k})\), is defined as \(V(g, \mathfrak{k})^+\).

A slightly more convenient algebra appears if we invert \(U(\mathfrak{h})\). Let \(D(\mathfrak{h})\) denote the fraction field of \(U(\mathfrak{h})\). Set \(U'(g) = U(g) \otimes_{U(\mathfrak{h})} D(\mathfrak{h})\), \(I'_+ = U'(g)n_+\), \(V'(g, \mathfrak{k}) = U'(g)/I'_+\) and \(Z(g, \mathfrak{k}) = V'(g, \mathfrak{k})^+\).

\(g_n = gl(n, \mathbb{C}), \mathfrak{h}_n\) the subalgebra of diagonal matrices. \(AZ_n = Z(gl(n + 1, \mathbb{C}), gl(n, \mathbb{C}) \oplus \mathbb{C}e_{n+1,n+1})\).

\(AZ_n\) has the following presentation. It is generated (over the field \(D_{n+1} = D(\mathfrak{h}_{n+1})\)) by elements \(z_i, i \in \{\pm 1, \pm 2, \ldots, \pm n\}\), with relations:

- \(z_iz_j = \alpha_{i,j}z_jz_i, i + j \neq 0;\)
- \(z_iz_{-i} = \sum_{j=1}^{n} \beta_{i,j}z_{-j}z_j + \gamma_i, i = 1, 2, \ldots, n;\)
- \([h_j, z_i] = (\epsilon_i - \epsilon_{n+1})(h_j)z_i, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n + 1;\)
- \([h_j, z_{-i}] = (\epsilon_{n+1} - \epsilon_i)(h_j)z_{-i}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n + 1;\)

where

- \(\alpha_{i,j} = \alpha_{-j,-i} = \frac{\phi_{i,j}^+}{\phi_{i,j}}, 1 \leq i < j \leq n;\)
- \(\alpha_{i,j} = 1, \text{sign}(i) \neq \text{sign}(j);\)
- \(\beta_{i,j} = \delta_i^-\gamma_{i,j}\delta_j^+;\)
- \(\gamma_i = \delta_i^-\phi_{i,n+1}^-;\)
- \(\phi_{i,j} = h_i - h_j + j - i;\)
- \(\phi_{i,j}^+ = \phi_{i,j} \pm 1;\)
- \(\gamma_{i,j} = (1 - \phi_{i,j})^{-1};\)
- \(\delta_i^\pm = \prod_{k=i+1}^{n} \phi_{i,k};\)
- \(\epsilon_i(h_j) = \delta_{i,j}, i, j = 1, 2, \ldots, n + 1.\)
Set \( t_i = z_{-i}z_i \) and denote by \( R \) the algebra, generated by \( t_1, \ldots, t_n \) over the field \( D_{n+1} \). Define \( \sigma_i, i = 1, 2, \ldots, n, \) as follows:

\[
\sigma_i(h_k) = h_k, \quad k \neq i, n + 1; \quad \sigma_i(h_i) = h_i - 1; \quad \sigma_i(h_{n+1}) = h_{n+1} + 1; \quad \sigma_i(t_j) = \frac{\phi_{i,j}^-}{\phi_{i,j}^- - 1} t_j, \quad j < i; \quad \sigma_i(t_j) = \frac{\phi_{i,j}^-}{\phi_{i,j}^-} t_j, \quad j > i; \quad \sigma_i(t_i) = \sum_{k=1}^{n} \beta_{i,k} t_k + \gamma_i.
\]

**THEOREM** \( AZ_n \) is the TGWA associated with \( R, \{\sigma_i\} \) and \( \{t_i\} \) above (all \( \mu_{i,j} = 1 \)).

The most difficult part for TGWC is to prove that \( \sigma_i \)'s commute.

To go from TGWC to TGWA use the Diamond Lemma and Shapovalov form.

**Extended OGZ-algebras**

Let \( \ell = (k - 1, k, k + 1) \). If \( \mathbb{F} \) is a field, \( \mathcal{L} = \mathcal{L}(\mathbb{F}, \ell) = \mathbb{F}^{3k} \) with elements \( [\ell] = \{l_{i,j} \mid i = 1, 2, 3; j = 1, \ldots, k + 2 - i \} \), called *tableaux*. \( \mathcal{L}_0 \) is the subset of \( \mathcal{L} \) that consists of all \( [\ell] \) such that \( l_{1,j} = 0, l_{3,j} = 0, l_{2,j} \in \mathbb{Z} \) for all \( j \).

\( \Lambda \) — rational functions in \( \lambda_{i,j} \) for all \( i, j \). \( \{[\ell] \in \mathcal{L}(\Lambda, \ell) \) defined by \( l_{i,j} = \lambda_{i,j} \). \( M = M([\ell]) \) is \( \Lambda \)-v.sp. with the basis \( v_{[\ell]}, [\ell] \in [\ell] + \mathcal{L}_0 \).
For \([t] \in [l] + \mathcal{L}_0\) and \(1 \leq j \leq k\) let

\[
a_j^\pm([t]) = \mp \frac{\prod_{m \neq j} (t_{2,m} - t_{2,j})}{\prod_{m \neq j} (t_{2,m} - t_{2,j})},
\]

define \(A\)-linear operators \(X_j^\pm : M \to M\), \(X_j^\pm v_{[l]} = a_j^\pm([t])v_{[l] \pm \delta^2 j}\), and \(H_{i,j} : M \to M\), \(H_{i,j} v_{[l]} = t_{i,j}v_{[l]}, i = 1, 2, 3\).

Let \(\mathcal{Q}\) be the localization of \(\mathbb{C}[H_{i,j}, 1 \leq i \leq n, 1 \leq j \leq r_i]\) w.r.t. the multiplicative set, generated by \(H_{2,j} - H_{2,i} + m\) for all \(j \neq l\) and \(m \in \mathbb{Z}\). The extended orthogonal Gelfand-Zetlin algebra \(\mathcal{U}\) is the \(\mathbb{C}\)-algebra, generated over \(\mathbb{C}\) by \(\mathcal{Q}\) and \(X_{i,j}^\pm, 2 \leq i \leq n - 1, 1 \leq j \leq r_i\).

\[
T_i = -\frac{\prod_{j=1}^{k+1} (H_{3,j} - H_{2,i}) \prod_{j=1}^{k-1} (H_{1,j} - H_{2,i} - 1)}{\prod_{j \neq i} (H_{2,j} - H_{2,i}) \prod_{j \neq i} (H_{2,j} - H_{2,i} - 1)},
\]

\(\sigma_i(H_{2,i}) = H_{2,i} - 1; \quad \sigma_i(H_{k,l}) = H_{k,l}, \quad k \neq 2 \text{ or } l \neq i\).

**Theorem** \(\mathcal{U}\) is the TGWA associated with \(\mathcal{Q}\), \(\{\sigma_i\}\) and \(\{T_i\}\).