TWISTING, COMPLETING AND APPROXIMATING CATEGORY $\mathcal{O}$

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1. Category $\mathcal{O}$

$\mathfrak{g}$ — simple finite-dimensional Lie algebra over $\mathbb{C}$

$U(\mathfrak{g})$ — the universal enveloping algebra of $\mathfrak{g}$

$Z(\mathfrak{g})$ — the center of $U(\mathfrak{g})$

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — fixed triangular decomposition of $\mathfrak{g}$

**Definition:** (BGG) Category $\mathcal{O}$ is the full subcategory in $\mathfrak{g}$-mod that consists of all modules, which are

- finitely generated;
- $\mathfrak{h}$-diagonalizable;
- locally $U(\mathfrak{n}_+)$-finite.
With respect to the action of $Z(\mathfrak{g})$ the category $\mathcal{O}$ decomposes:

$$\mathcal{O} = \bigoplus_{\chi \in Z(\mathfrak{g})^*} \mathcal{O}_\chi.$$

Every $\mathcal{O}_\chi$ is equivalent to the module category of a finite-dimensional, associative, quasi-hereditary algebra.

Simple modules in $\mathcal{O}_\chi$ are indexed (sometimes non-bijectively) by the elements of the Weyl group $W$.

**Example:** Indecomposable modules in the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$: 

- $L(\lambda)$: 
  
  ![Diagram for L(\lambda)]

- $M(\lambda)$: 
  
  ![Diagram for M(\lambda)]

- $M(s_\alpha \cdot \lambda)$: 
  
  ![Diagram for M(s_\alpha \cdot \lambda)]

- $M(\lambda)^*$: 
  
  ![Diagram for M(\lambda)^*]

- $P(s_\alpha \cdot \lambda)$: 
  
  ![Diagram for P(s_\alpha \cdot \lambda)]
2. Twisting functors on $\mathcal{O}$

Let $\alpha$ be a simple root and $X_{-\alpha}$ be a non-zero root vector.

Let $U_\alpha$ denote the (Ore) localization of $(\mathfrak{g})$ with respect to 
\[ \{X_{-\alpha}^l : l \geq 0\}. \]

$B_\alpha = U_\alpha / U(\mathfrak{g})$ is the twisting $U(\mathfrak{g})$-bimodule (Arkhipov).

Let $\Phi_\alpha$ be the inner automorphism of $\mathfrak{g}$, corresponding to $\alpha$.

**Definition:** (Arkhipov) The twisting functor $T_\alpha : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ is defined as the functor $\Phi_\alpha\left( B_\alpha \otimes_{U(\mathfrak{g})} - \right)$.

$T_\alpha$ preserves all integral blocks of $\mathcal{O}$.

$T_\alpha$ is right exact.

**Theorem.** (Arkhipov?, Andersen?, Andersen-Lauritzen?, Khomenko-M.) Functors $T_\alpha$, $\alpha$ simple, (weakly) satisfy braid relations on the integral blocks of $\mathcal{O}$. 
**Example:** Action of $T_\alpha$ on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$: 

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$T_\alpha$
3. Enright-Deodhar’s completion functor on $\mathcal{O}$

Let $\alpha$ and $U_\alpha$ be as above.

**Definition:** The Enright-Deodhar’s completion functor $E_\alpha : \mathfrak{g}\text{-mod} \rightarrow \mathfrak{g}\text{-mod}$ is defined as the composition of the following functors:

1. $U_\alpha \otimes U(\mathfrak{g})$;
2. restriction to $U(\mathfrak{g})$;
3. taking $\mathfrak{g}_\alpha$-locally finite part.

$E_\alpha$ is left exact and idempotent.

**Example:** Action of $E_\alpha$ on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$:
4. Enright-Joseph’s completion functor on $\mathcal{O}$

For $M, N \in U(\mathfrak{g})$-mod denote by $\mathcal{L}(M, N)$ the space of all locally ad-$\mathfrak{g}$-finite linear maps from $M$ to $N$.

$M(\lambda)$ is the Verma module with highest weight $\lambda \in \mathfrak{h}^*$.

**Definition:** The Enright-Joseph’s completion functor $J_\alpha : \mathfrak{g}$-mod $\to \mathfrak{g}$-mod is defined as the functor

$$J_\alpha = \mathcal{L}(M(s_\alpha \cdot \lambda), -) \otimes_{U(\mathfrak{g})} M(\lambda),$$

where $M(\lambda)$ is the dominant Verma module in $\mathcal{O}_\lambda$.

$J_\alpha$ is left exact and $J_\alpha^3 \cong J_\alpha^2$.

**Example:** Action of $J_\alpha$ on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$:
5. **Approximation functor**

\[ A \] — finite-dimensional associative algebra.

\[ \Upsilon \] — a set of primitive pairwise orthogonal idempotents.

\[ P(\Upsilon), I(\Upsilon) \] — the corresponding projective and injective modules respectively.

**Definition:** (Auslander?) The approximation functor \( c_\Upsilon : A\text{-mod} \to A\text{-mod} \) is defined as

\[
c_\Upsilon = \text{Hom}_{\text{End}_A(P(\Upsilon))}(\text{Hom}_A(P(\Upsilon), A), \text{Hom}_A(P(\Upsilon), \_)).\]

\( c_\Upsilon \) is left exact and idempotent.

\( c_\Upsilon \) can be viewed as the composition of the following two procedures. Start with \( M \in A\text{-mod}. \)

1. Take the maximal possible image \( M_1 \) of \( M \) in some \( I(\Upsilon)^n \).

2. Make the maximal possible coextension of \( M_1 \) inside \( I(\Upsilon)^n \) with non-\( \Upsilon \) simples.

For a simple root \( \alpha \) and a block \( O_\chi \) we let \( \Upsilon \) denote the set of \( \alpha \)-antidominant simples.
**Example:** Action of $c_Y$ on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$:

The coapproximation functor $\tilde{c}_Y$ is defined dually.

**Theorem.** (Auslander?) The functor $\tilde{c}_Y$ is left adjoint to $c_Y$. 
6. Functor of partial approximation

$A, \Upsilon, P(\Upsilon), I(\Upsilon)$ — as above.

$I$ — injective generator of $A$-mod.

**Definition:** (Khomenko-M.) The functor of partial approximation $\mathfrak{d}_\Upsilon : A$-mod $\rightarrow A$-mod is defined as the composition of the following three procedures. Start with $M \in A$-mod.

1. Take a minimal injective envelope $M \subset I_M$ of $M$.
2. Make the maximal possible coextension of $M$ inside $I_M$ with non-$\Upsilon$ simples obtaining $M_1$.
3. Take the maximal possible image of $M_1$ in some $I(\Upsilon)^n$.

$\mathfrak{d}_\Upsilon$ is left exact and $\mathfrak{d}_\Upsilon^3 = \mathfrak{d}_\Upsilon^2$.

The coapproximation functor $\tilde{\mathfrak{d}}_\Upsilon$ is defined dually.

**Theorem.** (Khomenko-M.) The functor $\tilde{\mathfrak{d}}_\Upsilon$ is left adjoint to $\mathfrak{d}_\Upsilon$.

For a simple root $\alpha$ and a block $\mathcal{O}_\chi$ we let $\Upsilon$ denote the set of $\alpha$-antidominant simples.
**Example:** Action of $c_T$ on the regular block of $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$:
7. Relations between these functors (Khomenko-M.)

Let $\mathcal{O}_\chi$ be integral and regular.

**Theorem:** The functors $E_\alpha$ and $c_v$ are isomorphic.

**Theorem:** The functors $J_\alpha$ and $d_v$ are isomorphic.

**Theorem:** There is a non-trivial natural transformation from $T_\alpha$ to the identity functor.

**Theorem:** The functors $T_\alpha$ and $\tilde{d}_v$ are isomorphic.

**Corollary:** The functor $T_\alpha$ is left adjoint to the functor $J_\alpha$.

**Corollary:** The functor $T_\alpha$ is left adjoint to the functor $\star \circ T_\alpha \circ \star$.

**Corollary:** The functor $J_\alpha$ is left adjoint to the functor $\star \circ J_\alpha \circ \star$.

**Corollary:** (Joseph) The functors $J_\alpha$, $\alpha$ simple, satisfy braid relations.

**Corollary:** (Deodhar, Bouaziz) The functors $E_\alpha$, $\alpha$ simple, satisfy braid relations on the full subcategory of $\mathcal{O}_\chi$, which consists of all modules, torsion free with respect to all $g_{-\beta}$, $\beta$ positive.
8. $T_\alpha$ and the Kazhdan-Lusztig conjecture

Let $\mathcal{O}_\chi$ be integral and regular.

Let $L(\lambda) \in \mathcal{O}_\chi$ be a simple module such that $T_\alpha(L(\lambda)) \neq 0$ (that is $\lambda$ is $\alpha$-antidominant).

**Theorem:** (Andersen-Stroppel) The Kazhdan-Lusztig conjecture is equivalent to the following statement: The kernel of the natural morphism $T_\alpha(L(\lambda)) \rightarrow L(\lambda)$ is semi-simple for all $\alpha$ and $\lambda$ as above.