ABSTRACT VERSION OF ENRIGHT'S COMPLETION

Volodymyr Mazorchuk

Uppsala University

1. Enright's completion functor

$\mathfrak{g}$ – simple complex finite-dimensional Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, root system Δ with the corresponding base π, and a fixed basis $\{X_\alpha, H_\beta : \alpha \in \Delta, \beta \in \pi\}$.

For $\alpha \in \pi$ set $T_\alpha = \{X_{-\alpha}^k : k \in \mathbb{N}\}$. This is an Ore set in $U(\mathfrak{g})$ and we denote by U_α the localization of $U(\mathfrak{g})$ with respect to T_α.

Denote by $C_\alpha : \mathfrak{g}-mod \to \mathfrak{g}-mod$ the composition of the following three functors: $U_\alpha \otimes_{U(\mathfrak{g})} \_$, taking X_α-locally finite part, and $\_|_{U(\mathfrak{g})}$. C_α is the *elementary Enright's completion functor*. It differs a little from the original functor introduced by Enright, however they coincide on weight $\mathfrak{g}$-modules with finite-dimensional weight spaces on which $X_{-\alpha}$ acts injectively. Obviously $C_\alpha \circ C_\alpha = C_\alpha$, whereas for original Enright's functor $\hat{C}_\alpha$ only $\hat{C}_\alpha \circ \hat{C}_\alpha \circ \hat{C}_\alpha = \hat{C}_\alpha \circ \hat{C}_\alpha$ is true.

$M \in \mathfrak{g} - mod$ is called *α-complete* if $C_\alpha(M) \simeq M$.

Let $S \subset \pi$. $M \in \mathfrak{g} - mod$ is called *S-complete* if M is α-complete for all $\alpha \in \pi$. π-complete modules are called *complete*.

$\mathfrak{sl}(2,\mathbb{C})$**-example.** Let $M(\lambda)$, $\lambda\in\mathbb{C}$, denote the Verma module over $\mathfrak{sl}(2,\mathbb{C})$ with highest weight λ; $L(\lambda)$ its unique simple quotient; $(M(\lambda))^*$ and $P(\lambda)$ the indecomposable projective cover of $L(\lambda)$ in the category $\mathcal{O}$. Then

$$C_\alpha(M(\lambda))=\begin{cases} M(\lambda), & \lambda\notin\{-2,-3,-4,\dots\}\\ M(-\lambda-2), & \lambda\in\{-2,-3,-4,\dots\}\end{cases}$$

$$C_\alpha(L(\lambda))=\begin{cases} L(\lambda), & \lambda\notin\mathbb{Z}\setminus\{-1\}\\ M(-\lambda-2), & \lambda\in\{-2,-3,-4,\dots\}\\ 0, & \lambda\in\mathbb{Z}_+\end{cases}$$

$$C_\alpha((M(\lambda))^*)=\begin{cases} (M(\lambda))^*\simeq M(\lambda), & \lambda\notin\mathbb{Z}\setminus\{-1\}\\ M(-\lambda-2), & \lambda\in\{-2,-3,-4,\dots\}\\ M(\lambda), & \lambda\in\mathbb{Z}_+\end{cases}$$

$$C_\alpha(P(\lambda))=P(\lambda).$$

2. Approximation with respect to an injective module.

A – finite-dimensional associative algebra and Λ the set of isomorphism classes of simple A-modules. Fix $\varnothing\neq\Gamma\subset\Lambda$.

$M\in A-mod$ is Γ-*injectively cogenerated* (resp. *copresented*) if there is an exact sequence $0\to M\to I_1$ (resp. $0\to M\to I_1\to I_2$)

with I_1 and I_2 being injective modules, all indecomposable direct summands of which are indexed by elements from Γ. Let $I(\gamma)$ be the direct sum of all indecomposable injectives indexed by Γ and A_Γ the endomorphisms ring of $I(\Gamma)$.

Theorem. (Auslander?) The full subcategory $\mathcal{C}(\Gamma)$ of Γ-injectively copresented modules is equivalent to the category of A_Γ-modules via coinduction and restriction. This gives $\mathcal{C}(\Gamma)$ an abelian structure, with respect to which the inclusion $\mathcal{C}(\Gamma) \subset A - mod$ is left exact.

The dual version for projective modules is also true.

Given $M \in A - mod$ we can first map it to $A_\Gamma - mod$ using the exact functor $Hom_A(\_, I(\Gamma))$ and then coinduce it to an injectively copresented module, say M'. On the level of $A - mod$ the first map corresponds to taking the maximal image $M \to I_1$, $I_1 \in add(I(\Gamma))$, say M_1. Then M_1 is Γ-injectively cogenerated, thus $0 \to M_1 \to I_1$ is exact. Further, $M_2 = M'$ is the intersection of all kernels of maps $I_1 \to I_2$, $I_2 \in add(I(\Gamma))$, sending M to zero. Hence $0 \to M_2 \to I_1 \to I_2$ is exact and M_2 is Γ-injectively copresented. Obviously the map $M \to M'$ is idempotent and functorial. The corresponding functor $\mathfrak{A}_\Gamma$ is the functor of *approximation with respect to* $I(\Gamma)$.

3. S-complete modules in $\mathcal{O}$.

In the BGG-category $\mathcal{O}$ for $\mathfrak{g}$ with each $\lambda \in \mathfrak{h}^*$ there associated the Verma module $M(\lambda)$ with highest weight λ; its unique simple quotient $L(\lambda)$; the corresponding projective cover $P(\lambda)$ and injective envelope $I(\lambda)$; indecomposable tilting module $T(\lambda)$.

Fix $\varnothing \neq S \subset \pi$. Denote by $\mathfrak{h}(S)$ the set of all $\lambda \in \mathfrak{h}$ satisfying $(\beta, \lambda) \notin \mathbb{N}$ for any positive root $\beta \in \Delta \cap \mathbb{Z}S$. The intersection of $\mathfrak{h}(S)$ with the integral weight lattice corresponds to the longest coset representatives of W/W_S if we index W-orbits by dominant weights (here W is the Weyl group of Δ and W_S its subgroup corresponding to S).

Main Theorem 1. $M \in \mathcal{O}$ is S-complete if and only if M is $\mathfrak{h}(S)$-injectively copresented.

Main Theorem 2. Let $S = \{\alpha\}$ then the functors C_α and $\mathfrak{A}_{\mathfrak{h}(S)}$ are naturally isomorphic.

4. Corollaries.

Let $\mathcal{O}_S$ denote the full subcategory of $\mathcal{O}$ consisting of all modules on which all $X_{-\alpha}$, $\alpha \in S$, act injectively.

Corollary 1. Let S be arbitrary, w_S be the longest element in W_S and $w_S = s_1 \dots s_k$, $s_i = s_{\alpha_i}$, be its reduced decomposition. Denote $S^{(i)} = \{\alpha_i\}$. Then $\mathfrak{A}_{\mathfrak{h}(S^{(1)})} \circ \mathfrak{A}_{\mathfrak{h}(S^{(2)})} \circ \dots \circ \mathfrak{A}_{\mathfrak{h}(S^{(k)})}$ is naturally isomorphic to $\mathfrak{A}_{\mathfrak{h}(S)}$ on $\mathcal{O}_S$.

Corollary 2. (Bouaziz, Deodhar) The functors $\{C_\alpha : \alpha \in \pi\}$ and $\{\mathfrak{A}_{\mathfrak{h}(\{\alpha\})} : \alpha \in \pi\}$ on $\mathcal{O}_S$ satisfy braid relations.

For $w_S = s_1 \dots s_k$ as above set $C_S = C_{\alpha_1} \circ \dots \circ C_{\alpha_k}$. C_S is called *Enright's S-completion functor*. Clearly $M \in \mathcal{O}$ is S-complete if and only if $C_S(M) \simeq M$.

Corollary 3. Let S be arbitrary. The functors C_S and $\mathfrak{A}_{\mathfrak{h}(S)}$ are naturally isomorphic on $\mathcal{O}_S$.

Corollary 4. Let $M \in \mathcal{O}$ be complete (i.e. $S = \pi!$). Then M has a Verma flag.

Via Soergel's double-centralizer property the last corollary gives a good lower bound for the representation type of blocks of $\mathcal{O}$ and of categories of Verma-filtered modules in these blocks (good-filtered modules for corresponding quasi-hereditary algebras).

5. Equivalence of different categories.

We recall that $\mathcal{O}$ decomposes into the following direct sum of full subcategories with respect to the action of $Z(\mathfrak{g})$: $\mathcal{O} = \oplus_{\chi \in Z(\mathfrak{g})^*} \mathcal{O}_\chi$. We denote by A_χ the basic finite-dimensional associative algebra corresponding to $\mathcal{O}_\chi$ (i.e. $\mathcal{O}_\chi \simeq A_\chi - mod$) and by e_S^χ the sum of primitive idempotents of A_χ with indexes in $\mathfrak{h}(S)$.

Let $\mathfrak{a}$ denote the Lie algebra spanned by $X_{\pm\alpha}$, $\alpha \in S$. Denote $P_S = \mathfrak{n}_+ + \mathfrak{h} + \mathfrak{a}$ the corresponding parabolic subalgebra in $\mathfrak{g}$ and by $\mathfrak{n}$ its nilpotent radical. Let $\mathcal{K}_S$ be the category of complete $\mathfrak{a}$-modules in $\mathcal{O}$ (for $\mathfrak{a}$). Denote by $\mathcal{O}(P_S, \mathcal{K}_S)$ the category of finitely generated, $\mathfrak{h}$-diagonalizable, $\mathfrak{n}$-locally finite $\mathfrak{g}$-modules, which are direct sums of modules from $\mathcal{K}_S$, when viewed as $\mathfrak{a}$-modules. $\mathcal{O}(P_S, \mathcal{K}_S)$ is a natural parabolic generalization of $\mathcal{O}$. Under our choice of $\mathcal{K}_S$ it is a subcategory of $\mathcal{O}$ (for $\mathfrak{g}$).

Let $\theta \in Z(\mathfrak{g})^*$. Denote by $\mathcal{H}_f^1(\theta)^r$ the category of finitely generated $U(\mathfrak{g})$-bimodules ($= U(\mathfrak{g} \times \mathfrak{g})$-modules), which are algebraic, i.e. direct sum of finite-dimensional modules for the canonical "diagonal" copy of $\mathfrak{g}$ ($x \mapsto (x, \sigma(x))$, σ is the Chevalley involution), and on which the right copy of $Z(\mathfrak{g})$ acts via θ and this action is diagonalizable. These are the so-called *Harish-Chandra modules for* $\mathfrak{g}$.

Theorem. The following categories are equivalent.

1. $\mathcal{O}(P_S, \mathcal{K}_S)$.
2. The category of S-complete modules in $\mathcal{O}$.
3. The category of $\mathfrak{h}(S)$-injectively copresented modules in $\mathcal{O}$.
4. The category $\mathcal{H}^1_f(\theta)^r$ with W_S being the stabilizer of θ.
5. The category $\oplus_{\chi\in Z(\mathfrak{g})^*}(e^\chi_S A_\chi e^\chi_S - mod)$.

If $\mathfrak{a} \simeq \oplus_i \mathfrak{sl}(n_i, \mathbb{C})$, one can add to this list the category $\mathcal{O}(P_S, \mathcal{G}Z)$, where $\mathcal{G}Z$ is a certain category of generic Gelfand-Zetlin modules.

6. Properties of the categories above.

Set $A^S_\chi = e^\chi_S A_\chi e^\chi_S$.

Theorem. The algebra A^S_χ is properly stratified in the sense of Dlab. If we represent A^S_χ as $\mathcal{O}(P_S, \mathcal{K}_S)$ via one of the theorems above, then parabolically induced generalized Verma modules are proper standard modules in the proper stratification and modules, parabolically induced from indecomposable projectives in $\mathcal{K}_S$, are standard modules in the proper stratification.

Theorem. (BGG-reciprocity) Let $\hat{\Lambda}$ index the set of primitive idempotents for A^S_χ and $St(\lambda)$ (resp. $Pst(\lambda)$) denote the standard (resp. proper standard) modules corresponding to $\lambda \in \hat{\Lambda}$. Then for any $\lambda, \mu \in \hat{\Lambda}$ there is the following reciprocity:

$$[P(\lambda) : St(\mu)] = [Pst(\mu) : L(\lambda)].$$

Theorem. (Double-centralizer property) There is an indecomposable projective, P, for A_χ^S, which is filtered by all standard modules. Moreover, $A_\chi^S \simeq End(P_{End_{A_\chi^S}(P)})$.

Call an A_χ^S-module *tilting* if it is filtered both by standard modules and by their duals (in this case the last is equiv. to: by duals to proper standard modules).

Theorem. Any tilting A_χ^S-module is a direct sum of indecomposable tilting A_χ^S-modules. Indecomposable tilting A_χ^S-modules are in natural bijection with standard modules (i.e. with simples).

Theorem. Generalized Verma modules in $\mathcal{O}(P_S, \mathcal{K}_S)$ (= proper standard modules for A_χ^S) are rigid, i.e. their radical and socle series coincide.

The Loewy length of generalized Verma modules and the layers of the unique Loewy filtration can be obtained from the corresponding results for Verma modules using C_S.

In contrast with $\mathcal{O}$ the *big projective modules* in $\mathcal{O}(P_S, \mathcal{K}_S)$, i.e. those indexed by antidominant weights, are not rigid in general. However, they are rigid if $S = \pi$ (in $\mathcal{K}_S$).

Conjecture. Standard modules in $\mathcal{O}(P_S, \mathcal{K}_S)$ are rigid.

The lower bound for the representation type of blocks of $\mathcal{O}$ obtained above works for $\mathcal{O}(P_S, \mathcal{K}_S)$ as well.