COMBINATORIAL CATEGORIFICATION OF $\mathfrak{sl}_k$-KNOT INVARIANTS

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1. $\mathfrak{sl}(k)$ link polynomial and trivalent colored graphs

The $\mathfrak{sl}(k)$ link polynomial $\mathbf{P}_k$ is defined via the skein relation:

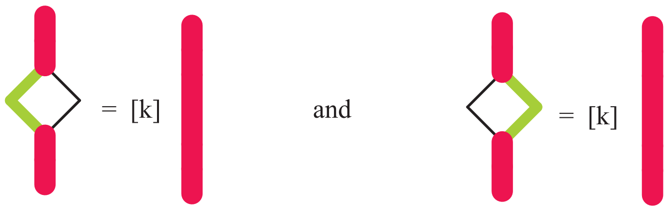
$q^k\mathbf{P}_k\Big($  $\Big) - q^{-k}\mathbf{P}_k\Big($  $\Big) = (q-q^{-1})\mathbf{P}_k\Big(\uparrow\uparrow\Big)$

and normalized by setting

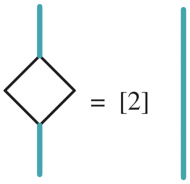
$$\mathbf{P}_k(\textbf{trivial knot}) = [k] = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

Graphical calculus for $\mathbf{P}_k$ (Murakami, Ohtsuki, Yamada):

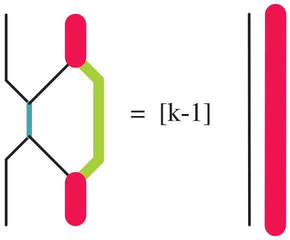
Generators:



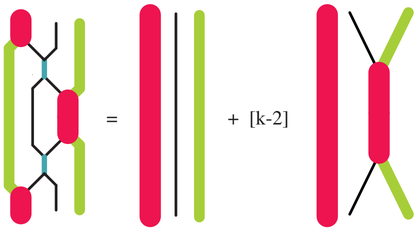
MOY Relation (I)



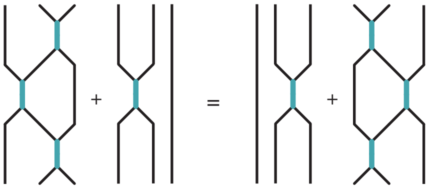
MOY Relation (II)



MOY Relation (III)



MOY Relation (IV)



MOY Relation (V)

2. Functorial invariants of trivalent colored graphs

Need: to associate a functor to every generator.

2.1 Categories

$\mathcal{O}(n)$ — **BGG category** $\mathcal{O}$ **for** $\mathfrak{sl}_n$

$\lambda = (\lambda_1, \ldots, \lambda_n)$ — **composition of** n

$\mathcal{O}(n)_\lambda$ — **singular block of** $\mathcal{O}$ **with stabilizer corr. to** λ

$\mu = (\mu_1, \ldots, \mu_n)$ — **composition of** n

$\mathcal{O}(n)^\mu$ — **parabolic subcategory of** $\mathcal{O}$ **corresp. to** μ

$\mathcal{O}(n)^\mu_\lambda$ — **parabolic part of the singular block**

$\mathcal{O}(n)^\mu_\lambda$ **is the category of modules over a Koszul algebra**

${}^{\mathbb{Z}}\mathcal{O}(n)^\mu_\lambda$ — **graded version**

Our k **means that we work with parabolic categories, which correspond to compositions with at most** k **parts.**

2.2 Functors

If ν is finer than λ we have translation *out of the wall*:

$$\theta_\lambda^\nu : {}^{\mathbb{Z}}\mathcal{O}(n)_\lambda \longrightarrow {}^{\mathbb{Z}}\mathcal{O}(n)_\nu$$

and adjoint translation *onto the wall*:

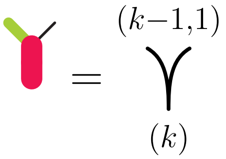
$$\theta_\nu^\lambda : {}^{\mathbb{Z}}\mathcal{O}(n)_\nu \longrightarrow {}^{\mathbb{Z}}\mathcal{O}(n)_\lambda$$

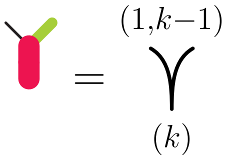
Define:

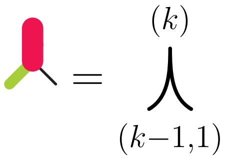
$$\overset{(i,\,j)}{\underset{(i+j)}{\curlyvee}} \;=\; \bigoplus_\mu \theta_{(i+j)}^{(i,j)} : \quad \bigoplus_\mu {}^{\mathbb{Z}}\mathcal{O}(i+j)_{(i+j)}^\mu \longrightarrow \bigoplus_\mu {}^{\mathbb{Z}}\mathcal{O}(i+j)_{(i,j)}^\mu .$$

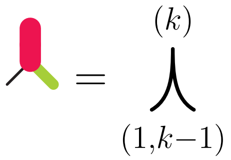
$$\overset{(i+j)}{\underset{(i,\,j)}{\curlywedge}} = \bigoplus_\mu \theta_{(i,j)}^{(i+j)} : \quad \bigoplus_\mu {}^{\mathbb{Z}}\mathcal{O}(i+j)_{(i,j)}^\mu \longrightarrow \bigoplus_\mu {}^{\mathbb{Z}}\mathcal{O}(i+j)_{(i+j)}^\mu ,$$

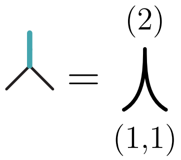
where

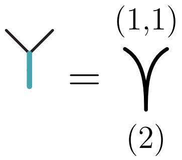












2.3 How to check relations?

q means shift of grading

Translations out of a wall and onto a wall are examples of the so-called *projective functors*. These are direct summands of the functors of the form $V \otimes_{\mathbb{C}} \_ : \mathcal{O} \to \mathcal{O}$, where V is a finite-dimensional $\mathfrak{sl}_n$-module.

Projective functors on $\mathcal{O}$ are classified:

Theorem. (BG) **Projective functors from some block of $\mathcal{O}$ to $\mathcal{O}$ are uniquely determined by the value on the dominant Verma module in the block. Indecomposable projective functors correspond to certain indecomposable projective modules**

Warning: **Projective functors are not classified on the categories we work with!**

PROBLEM: **Classify projective functors on $\mathcal{O}^\mu$.**

Idea of the proof of relations:

- **Compute the functors for both sides in $\mathcal{O}$.**
- **See that they agree modulo some "Error term".**
- **Restrict the result to our parabolic categories.**
- **Show that the error term annihilates the parabolic categories corresponding to compositions with at most k parts.**

Advantage: All calculations are purely combinatorial.

Apart from combinatorics, we need the following:

- **Classification of projective functors on $\mathcal{O}$ (BG).**
- **Projective functors admit graded lifts.**
- **Indecomposable projective functor is associated to an indecomposable projective modules and thus to a standard tableau. If this tableau has more than k rows, then the original functor annihilates any parabolic category associated to the composition with at most k rows (a basic facts from Kazhdan-Lusztig's theory).**
- **Composition of projective functors decomposes "symmetrically" with respect to the grading.**

Example:

Relation (III). **There is an isomorphism of functors**

$$\theta^{(1,k)}_{(1,1,k-1)}\,\theta^{(1,1,k-1)}_{(2,k-1)}\,\theta^{(2,k-1)}_{(1,1,k-1)}\,\theta^{(1,1,k-1)}_{(1,k)} \cong F \oplus [k-1]\,\mathrm{id}, \qquad (1)$$

where F is indecomposable and vanishes when restricted to any parabolic with at most k parts.

Proof for $k = 3$:

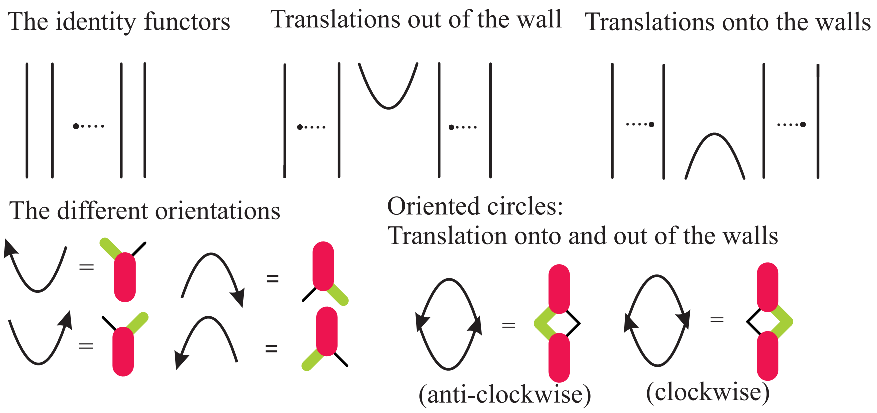
$$\begin{array}{ccccccccccc}
 & \theta^{(1,1,k-1)}_{(1,k)} & & \theta^{(2,k-1)}_{(1,1,k-1)} & & \theta^{(1,1,k-1)}_{(2,k-1)} & & & \theta^{(1,k)}_{(1,1,k-1)} & & \\
 & & & & & & & & & & e \\
e & \longrightarrow & 32 & \longrightarrow & 32 & \longrightarrow & 321 & & \longrightarrow & 321 & \\
 & & 2 & & 2 & & 21 & 32 & & 21 & e \\
 & & e & & e & & 1 & 2 & & 1 & \\
 & & & & & & & e & & e &
\end{array}$$

3. Functorial invariants of oriented tangles

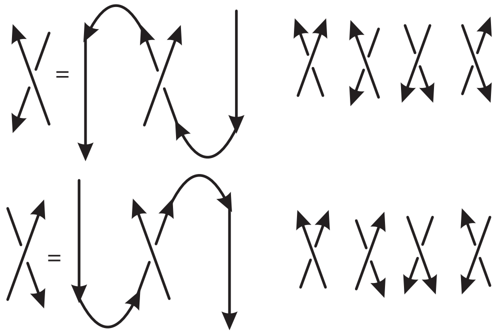
Need to associate functors with elementary tangles.

Vertical strands — identity functor

Cap and cup diagrams are handled as follows:

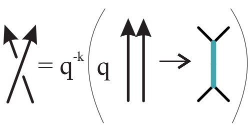


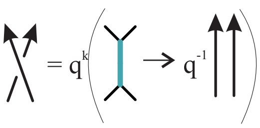
Reduction of crossings to crossings oriented up:



MAIN DIFFICULTY:

Crossings oriented up:



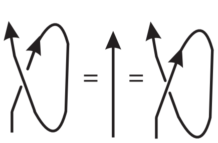


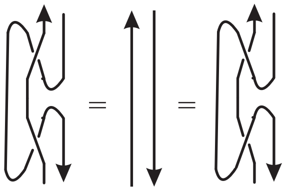
The above means that we

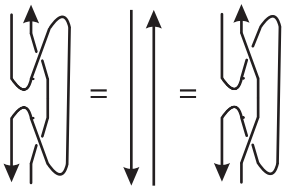
- Go to the derived category.
- Associate to the above crossings (derived) Irving's (co)shuffling functors.

Many relations to check.

Two of them (see below) require non-combinatorial arguments, that is computations with complexes (derived functors) in the derived category.







MAIN THEOREM. **The functors as above are invariants of oriented tangles and categorify P_k (that is satisfy the necessary relation in the Grothendieck group of the homotopy category of complexes of projective functors).**