STRUCTURE OF GENERALIZED VERMA MODULES

Volodymyr Mazorchuk

Uppsala University

1. Verma modules and their structure

$\mathfrak{g}$ – complex finite-dimensional Lie algebra with a triangular decomposition, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$.

For $\lambda \in \mathfrak{h}^*$ define $U(\mathfrak{h} \oplus \mathfrak{n}_+)$-module $\mathbb{C}_\lambda = \mathbb{C}$ by

$$(h+n)(v) = \lambda(h)v, \quad v \in \mathbb{C}_\lambda, h \in \mathfrak{h}, n \in \mathfrak{n}.$$

Verma module: $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$.

$M(\lambda)$ has unique simple quotient $L(\lambda)$.

BGG Theorem. Let $\mathfrak{g}$ be finite-dimensional. Then the following statements are equivalent:

1. $M(\mu) \subset M(\lambda)$.
2. $[M(\lambda) : L(\mu)] \neq 0$.
3. There exists a sequence of reflections, $s_1, \dots, s_k$, in the Weyl group W, such that

   $$\mu \leq s_1(\mu) \leq s_2 s_1(\mu) \leq \cdots \leq (s_k \dots s_2 s_1)(\mu) = \lambda,$$

   where $\leq$ is the standard partial order on $\mathfrak{h}^*$, associated with the chosen basis of the root system.

Corollary. Let $\mathfrak{g}$ be finite-dimensional and $\lambda \in \mathfrak{h}^*$. Then $M(\lambda)$ is simple if and only if $(\lambda, \alpha) \notin \mathbb{Z}_+$ for all positive roots α.

KL Conjecture-Theorem. Let $\mathfrak{g}$ be finite-dimensional, $\lambda \in \mathfrak{h}^*$ integral anti-dominant, and $w_1, w_2 \in W$. Denote by w_0 the longest element in W. Then $[M(w_1 \cdot \lambda) : L(w_2 \cdot \lambda)] = P_{w_0 w_1, w_0 w_2}(1)$, where $P_{w_0 w_1, w_0 w_2}(q)$ is the corresponding KL-polynomial.

Theorem. (Irving) Let $\mathfrak{g}$ be finite-dimensional and $\lambda \in \mathfrak{h}^*$ integral regular. Then the Verma module $M(\lambda)$ is rigid, that is, its socle and radical filtrations coincide. Moreover, the coefficients of KL-polynomials determine the multiplicities of simple highest weights modules in the subquotients of the unique Loewy filtration (this is the socle filtration in this case) of $M(\lambda)$: for λ anti-dominant integral, and $w_1, w_2 \in W$ one has:

$$P_{w_0 w_1, w_0 w_2}(q) = \sum_j [soc_{l(w_2)+1+2j} M(w_1 \cdot \lambda) : L(w_2 \cdot \lambda)] q^j.$$

Theorem (Soergel). KL-conjecture is true for non-integral $\lambda \in \mathfrak{h}^*$ as well.

Theorem (Beilinson-Ginzburg-Soergel). All Verma modules are rigid and the multiplicities of simple highest weights modules in the subquotients of Loewy filtrations for Verma modules are given by KL-polynomials.

2. Generalized Verma modules

$\mathfrak{p}$ – parabolic subalgebra of $\mathfrak{g}$, containing $\mathfrak{h} \oplus \mathfrak{n}_+$.

$\mathfrak{p} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{h}^\perp$ – Levi decomposition, $\mathfrak{n}$ – radical, $\mathfrak{a}' = \mathfrak{a} \oplus \mathfrak{h}^\perp$– reductive part, $\mathfrak{a}$ – semi-simple, $\mathfrak{h}^\perp$ – center of $\mathfrak{a}'$.

For a simple $\mathfrak{a}'$-module V define $\mathfrak{n}V = 0$, making it $U(\mathfrak{p})$-module.

Generalized Verma module: $M(\mathfrak{p}, V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$.

$M(\mathfrak{p}, V)$ has unique simple quotient $L(\mathfrak{p}, V)$.

Questions:

- What can be said about the structure of $M(\mathfrak{p}, V)$?
- When $M(\mathfrak{p}, V) = L(\mathfrak{p}, V)$?
- When $M(\mathfrak{p}, V)$ has finite length?
- When $M(\mathfrak{p}, V)$ is rigid?
- What are the simple subquotients of $M(\mathfrak{p}, V)$?
- What are the multiplicities of these simple subquotients?

Example. $\mathfrak{g}, V$ – finite-dimensional.

1. BGG-Theorem — Rocha-Caridi.
2. KL-Conjecture — Casian, Collingwood.
3. partial information about Loewy filtrations of generalized Verma modules (Irving).

Several known results for special classes of V.

1. $\mathfrak{g}$ – f.d.; V – f.d.: Rocha-Caridi, Casian, Collingwood, Irving (BGG Theorem, multiplicities).
2. $\mathfrak{g}$ – f.d., V – Whittaker: McDowell, Soergel, Milicic, Backelin (multiplicities).
3. $\mathfrak{g}$ – f.d., $\mathfrak{a} \simeq \mathfrak{sl}(2, \mathbb{C})$, V – simple weight without highest and lowest weight: Futorny, Khomenko, Mazorchuk (BGG Theorem).
4. $\mathfrak{g}$ – f.d., V – simple weight with finite-dimensional weight spaces: Mathieu (multiplicities).
5. $\mathfrak{g}$ – f.d., V – simple generic Gelfand-Zetlin module: Futorny, König, Mazorchuk, Ovsienko, (BGG Theorem, multiplicities).

Remark. The approach of Kac and Kazhdan, which uses the Shapovalov form and the Jantzen filtration on $M(\lambda)$, allows one to transfer the BGG Theorem to many infinite-dimensional Lie algebra, e.g. Kac-Moody Lie algebras, contragradient Lie algebras and the Virasoro Lie algebra.

3. First general result

Verma module, associated with $M(p, V)$. Assume that $\mathfrak{a}$ is finite-dimensional. Then V has a central character as an $\mathfrak{a}'$-module. Let W be a simple Verma module over $\mathfrak{a}'$, having the same central character, as V. Set

$$\mathfrak{f}(M(\mathfrak{p}, V)) = M(\mathfrak{p}, W).$$

Then $\mathfrak{f}(M(\mathfrak{p}, V))$ is a Verma module over $\mathfrak{g}$.

Theorem. (Khomenko-Mazorchuk) Let $\mathfrak{g}$ be contragradient and $\mathfrak{p}$ be such that $\mathfrak{a} \simeq \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{g}$ is an integrable $\mathfrak{a}$-module. Let V be a simple infinite-dimensional $\mathfrak{a}'$-module. Then the module $M(\mathfrak{p}, V)$ is simple if and only if the module $\mathfrak{f}(M(\mathfrak{p}, V))$ is simple.

4. Second general result

Theorem. (Khomenko-Mazorchuk) Let $\mathfrak{g}$ be finite-dimensional and V be a simple module, whose annihilator is generated by the central character (i.e. the annihilator is a minimal primitive ideal). Then the module $M(\mathfrak{p}, V)$ is simple if and only if the module $\mathfrak{f}(M(\mathfrak{p}, V))$ is simple.

Theorem. (Khomenko-Mazorchuk) Let $\mathfrak{g}$ be finite-dimensional and V_1, V_2 be simple module, whose annihilators are minimal primitive ideals. Then $[M(\mathfrak{p}, V_1) : L(\mathfrak{p}, V_2)] = [\mathfrak{f}(M(\mathfrak{p}, V_1)), L(\mu)]$, where μ is easily computed from V_2.

Remark. This result COVERS all known cases for

1. $\mathfrak{a} \simeq \mathfrak{sl}(2, \mathbb{C})$.
2. V – simple Whittaker.
3. V – generic Gelfand-Zetlin module.

Remark. This result DOES NOT cover the case when V is simple weight with f.d. weigth spaces for $\mathfrak{a} \not\simeq \mathfrak{sl}(2, \mathbb{C})$.

Idea of the proof. Extend arguments of Soergel and Milicic and transfer the question to the corresponding question for Harish-Chandra bimodules.

Difficulties. It is not known if $M(\mathfrak{p}, V)$ as above has finite length. Most probably it does not have finite length in general. The arguments of Soergel and Milicic embed $M(\mathfrak{p}, V)$ into a category, $\mathcal{O}_{\mathfrak{p},V}$, which shows only the "rough" structure of $M(\mathfrak{p}, V)$. One can see only simple subquotients $L(\mathfrak{p}, W)$ in $M(\mathfrak{p}, V)$, which correspond to simple W, whose annihilator is again a minimal primitive ideal.

The main part of the proof is actually to show that the category $\mathcal{O}_{\mathfrak{p},V}$ is equivalent to a category of Harish-Chandra bimodules.

5. Rigidity

Theorem. (Khomenko-Mazorchuk) Let $\mathfrak{g}$ be finite-dimensional and V be a simple module, whose annihilator is a minimal primitive ideal. The "rough" structure of $M(\mathfrak{p}, V)$ is rigid, that is the module $M(\mathfrak{p}, V)$ is rigid, as an object of $\mathcal{O}_{\mathfrak{p},V}$. Moreover, the multiplicities of $L(\mathfrak{p}, W)$ in the subquotients of the unique Loewy filtration of $M(\mathfrak{p}, V)$ are given by coefficients of certain KL-polynomials.

6. Relative GK-dimension

Theorem. (Ovsienko-Mazorchuk) Let $\mathfrak{g}$ be finite-dimensional and V be a simple generic GZ-module. Then the growth of the length of $M(\mathfrak{p}, V)$, considered as an $\mathfrak{a}$-module, is polynomial.

Corollary. Let $\mathfrak{g}$ be finite-dimensional and V_1, V_2 be simple generic GZ-modules. Then

$$\dim(\mathrm{Hom}_{\mathfrak{g}}(M(\mathfrak{p}, V_1), M(\mathfrak{p}, V_2))) \leq 1.$$