Homological properties of category O, part II: Alexandru conjecture

Volodymyr Mazorchuk
(Uppsala University)

“Enveloping Algebras and Representation Theory”
August 28 – September 1, 2014, St. John’s, CANADA
Some basic homological algebra

\[ \mathcal{A} \] — an abelian category

\[ \text{Ext}_\mathcal{A}^n(N, M) : \] equivalence classes of exact sequences

\[ 0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0 \]

\[ \mathcal{B} \] — another abelian category

\( \mathcal{A} \subset \mathcal{B} \) with exact inclusion \( i \)

Fact. \( i \) induces a homomorphism \( i_n : \text{Ext}_\mathcal{A}^n(N, M) \to \text{Ext}_\mathcal{B}^n(N, M) \)

Fact. \( i_n \) is usually neither injective nor surjective
Some basic homological algebra

$\mathcal{A}$ — an abelian category

$\text{Ext}^n_{\mathcal{A}}(N, M)$: equivalence classes of exact sequences

$0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0$

$\mathcal{B}$ — another abelian category

$\mathcal{A} \subset \mathcal{B}$ with exact inclusion $i$

Fact. $i$ induces a homomorphism $i_n : \text{Ext}^n_{\mathcal{A}}(N, M) \to \text{Ext}^n_{\mathcal{B}}(N, M)$

Fact. $i_n$ is usually neither injective nor surjective
Some basic homological algebra

\[ \mathcal{A} \] — an abelian category

\[ \text{Ext}_A^n(N, M): \text{equivalence classes of exact sequences} \]

\[ 0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0 \]

\[ \mathcal{B} \] — another abelian category

\[ \mathcal{A} \subset \mathcal{B} \] with exact inclusion \( i \)

**Fact.** \( i \) induces a homomorphism \( i_n : \text{Ext}_A^n(N, M) \to \text{Ext}_B^n(N, M) \)

**Fact.** \( i_n \) is usually neither injective nor surjective
Some basic homological algebra

\[ \mathcal{A} \] — an abelian category

\[ \text{Ext}_A^n(N, M) : \text{equivalence classes of exact sequences} \]

\[ 0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0 \]

\[ \mathcal{B} \] — another abelian category

\[ \mathcal{A} \subset \mathcal{B} \text{ with exact inclusion } i \]

Fact. \( i \) induces a homomorphism \( i_n : \text{Ext}_A^n(N, M) \to \text{Ext}_B^n(N, M) \)

Fact. \( i_n \) is usually neither injective nor surjective
Some basic homological algebra

\( A \) — an abelian category

\( \text{Ext}^n_A(N, M) \): equivalence classes of exact sequences

\[
0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0
\]

\( B \) — another abelian category

\( A \subset B \) with exact inclusion \( i \)

Fact. \( i \) induces a homomorphism \( i_n : \text{Ext}^n_A(N, M) \to \text{Ext}^n_B(N, M) \)

Fact. \( i_n \) is usually neither injective nor surjective
Some basic homological algebra

$\mathcal{A}$ — an abelian category

$\text{Ext}^n_{\mathcal{A}}(N, M)$: equivalence classes of exact sequences

\[ 0 \rightarrow M \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow N \rightarrow 0 \]

$\mathcal{B}$ — another abelian category

$\mathcal{A} \subset \mathcal{B}$ with exact inclusion $i$

**Fact.** $i$ induces a homomorphism $i_n : \text{Ext}^n_{\mathcal{A}}(N, M) \rightarrow \text{Ext}^n_{\mathcal{B}}(N, M)$

**Fact.** $i_n$ is usually neither injective nor surjective
Some basic homological algebra

\[ \mathcal{A} \] — an abelian category

\[ \text{Ext}^n_{\mathcal{A}}(N, M): \text{equivalence classes of exact sequences} \]

\[ 0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0 \]

\[ \mathcal{B} \] — another abelian category

\[ \mathcal{A} \subset \mathcal{B} \text{ with exact inclusion } \iota \]

Fact. \( \iota \) induces a homomorphism \( \iota_n : \text{Ext}^n_{\mathcal{A}}(N, M) \to \text{Ext}^n_{\mathcal{B}}(N, M) \)

Fact. \( \iota_n \) is usually neither injective nor surjective
Some basic homological algebra

$\mathcal{A}$ — an abelian category

$\text{Ext}^n_{\mathcal{A}}(N, M)$: equivalence classes of exact sequences

$$0 \rightarrow M \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow N \rightarrow 0$$

$\mathcal{B}$ — another abelian category

$\mathcal{A} \subset \mathcal{B}$ with exact inclusion $i$

**Fact.** $i$ induces a homomorphism $i_n : \text{Ext}^n_{\mathcal{A}}(N, M) \rightarrow \text{Ext}^n_{\mathcal{B}}(N, M)$

**Fact.** $i_n$ is usually neither injective nor surjective
Definition. $\mathcal{A}$ is extension full in $\mathcal{B}$ provided that $i_n$ is iso for all $n$.

Note. $\text{Ext}^0$-full = full

Note. $\text{Ext}^1$-full $\sim$ Serre subcategory

Motivating? example.

- $A$ — quasi-hereditary algebra w.r.t. $e_1 < e_2 < \cdots < e_n$
- $Ae_nA$ — heredity ideal
- $B = A/Ae_nA$ (also quasi-hereditary w.r.t. $e_1 < e_2 < \cdots < e_{n-1}$)
- Theorem. (CPS) $\mathcal{B}$-mod is extension full in $\mathcal{A}$-mod
Definition. $A$ is extension full in $B$ provided that $i_n$ is iso for all $n$.

Note. $\text{Ext}^0$-full $= \text{full}$

Note. $\text{Ext}^1$-full $\sim$ Serre subcategory

Motivating? example.

- $A$ — quasi-hereditary algebra w.r.t. $e_1 < e_2 < \cdots < e_n$
- $Ae_n A$ — heredity ideal
- $B = A/Ae_n A$ (also quasi-hereditary w.r.t. $e_1 < e_2 < \cdots < e_{n-1}$)
- Theorem. (CPS) $B$-mod is extension full in $A$-mod
Definition. \( \mathcal{A} \) is **extension full** in \( \mathcal{B} \) provided that \( i_n \) is iso for all \( n \).

**Note.** \( \text{Ext}^0 \)-full = full

**Note.** \( \text{Ext}^1 \)-full \( \sim \) Serre subcategory

Motivating? example.

- \( A \) — quasi-hereditary algebra w.r.t. \( e_1 < e_2 < \cdots < e_n \)
- \( Ae_n A \) — heredity ideal
- \( B = A/Ae_n A \) (also quasi-hereditary w.r.t. \( e_1 < e_2 < \cdots < e_{n-1} \))
- **Theorem.** (CPS) \( B \)-mod is extension full in \( A \)-mod
Definition. $A$ is **extension full** in $B$ provided that $i_n$ is iso for all $n$.

Note. $\text{Ext}^0$-full = full

Note. $\text{Ext}^1$-full $\sim$ Serre subcategory

Motivating example.

- $A$ — quasi-hereditary algebra w.r.t. $e_1 < e_2 < \cdots < e_n$
- $Ae_n A$ — heredity ideal
- $B = A/Ae_n A$ (also quasi-hereditary w.r.t. $e_1 < e_2 < \cdots < e_{n-1}$)
- **Theorem. (CPS)** $B$-mod is extension full in $A$-mod
Definition. \( \mathcal{A} \) is extension full in \( \mathcal{B} \) provided that \( i_n \) is iso for all \( n \).

Note. \( \text{Ext}^0 \)-full = full

Note. \( \text{Ext}^1 \)-full \( \sim \) Serre subcategory

Motivating? example.

- \( A \) — quasi-hereditary algebra w.r.t. \( e_1 < e_2 < \cdots < e_n \)
- \( Ae_n A \) — heredity ideal
- \( B = A/Ae_n A \) (also quasi-hereditary w.r.t. \( e_1 < e_2 < \cdots < e_{n-1} \))
- Theorem. (CPS) \( B \)-mod is extension full in \( A \)-mod
Definition. $\mathcal{A}$ is extension full in $\mathcal{B}$ provided that $i_n$ is iso for all $n$.

Note. $\text{Ext}^0$-full $=$ full

Note. $\text{Ext}^1$-full $\sim$ Serre subcategory

Motivating example.

- $A$ — quasi-hereditary algebra w.r.t. $e_1 < e_2 < \cdots < e_n$
- $A e_n A$ — heredity ideal
- $B = A/A e_n A$ (also quasi-hereditary w.r.t. $e_1 < e_2 < \cdots < e_{n-1}$)
- Theorem. (CPS) $B$-mod is extension full in $A$-mod
Definition. \( A \) is \textit{extension full} in \( B \) provided that \( i_n \) is \textit{iso} for all \( n \).

Note. \( \text{Ext}^0 \)-full = full

Note. \( \text{Ext}^1 \)-full \( \sim \) Serre subcategory

Motivating? example.

- \( A \) — quasi-hereditary algebra w.r.t. \( e_1 < e_2 < \cdots < e_n \)
- \( Ae_nA \) — heredity ideal
- \( B = A/Ae_nA \) (also quasi-hereditary w.r.t. \( e_1 < e_2 < \cdots < e_{n-1} \))
- Theorem. (CPS) \( B\)-mod is extension full in \( A\)-mod
Definition. $A$ is extension full in $B$ provided that $i_n$ is iso for all $n$.

Note. $\text{Ext}^0$-full $= \text{full}$

Note. $\text{Ext}^1$-full $\sim$ Serre subcategory

Motivating? example.

- $A$ — quasi-hereditary algebra w.r.t. $e_1 < e_2 < \cdots < e_n$
- $Ae_nA$ — heredity ideal
- $B = A/Ae_nA$ (also quasi-hereditary w.r.t. $e_1 < e_2 < \cdots < e_{n-1}$)
- Theorem. (CPS) $B$-mod is extension full in $A$-mod
Definition. \( \mathcal{A} \) is extension full in \( \mathcal{B} \) provided that \( i_n \) is iso for all \( n \).

Note. \( \text{Ext}^0 \)-full = full

Note. \( \text{Ext}^1 \)-full \( \sim \) Serre subcategory

Motivating? example.

- \( \mathcal{A} \) — quasi-hereditary algebra w.r.t. \( e_1 < e_2 < \cdots < e_n \)
- \( \mathcal{A}e_n \mathcal{A} \) — heredity ideal
- \( \mathcal{B} = \mathcal{A}/\mathcal{A}e_n \mathcal{A} \) (also quasi-hereditary w.r.t. \( e_1 < e_2 < \cdots < e_{n-1} \))
- Theorem. (CPS) \( \mathcal{B} \)-mod is extension full in \( \mathcal{A} \)-mod
Extension full subcategories

Definition. $A$ is extension full in $B$ provided that $i_n$ is iso for all $n$.

Note. $\text{Ext}^0$-full $=$ full

Note. $\text{Ext}^1$-full $\sim$ Serre subcategory

Motivating? example.

- $A$ — quasi-hereditary algebra w.r.t. $e_1 < e_2 < \cdots < e_n$
- $Ae_nA$ — heredity ideal
- $B = A/Ae_nA$ (also quasi-hereditary w.r.t. $e_1 < e_2 < \cdots < e_{n-1}$)
- Theorem. (CPS) $B$-mod is extension full in $A$-mod
Some categories of $g$-modules

$g$ — semi-simple complex finite dimensional Lie algebra

$g = n_- \oplus h \oplus n_+$ — triangular decomposition

$\mathcal{O}$ — corresponding BGG category $\mathcal{O}$

$g$-Mod — the category of all $g$-modules

$\mathcal{W}$ — the category of all weight (i.e. $h$-diagonalizable) $g$-modules

$\mathcal{GW}$ — the category of all generalized weight (i.e. locally $U(h)$-finite) $g$-modules
Some categories of $\mathfrak{g}$-modules

$\mathfrak{g}$ — semi-simple complex finite dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — triangular decomposition

$\mathcal{O}$ — corresponding BGG category $\mathcal{O}$

$\mathfrak{g}$-Mod — the category of all $\mathfrak{g}$-modules

$\mathcal{W}$ — the category of all weight (i.e. $\mathfrak{h}$-diagonalizable) $\mathfrak{g}$-modules

$\mathcal{GW}$ — the category of all generalized weight (i.e. locally $U(\mathfrak{h})$-finite) $\mathfrak{g}$-modules
Some categories of $\mathfrak{g}$-modules

$\mathfrak{g}$ — semi-simple complex finite dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — triangular decomposition

$\mathcal{O}$ — corresponding BGG category $\mathcal{O}$

$\mathfrak{g}$-Mod — the category of all $\mathfrak{g}$-modules

$\mathcal{W}$ — the category of all weight (i.e. $\mathfrak{h}$-diagonalizable) $\mathfrak{g}$-modules

$\mathcal{GW}$ — the category of all generalized weight (i.e. locally $U(\mathfrak{h})$-finite) $\mathfrak{g}$-modules
Some categories of $\mathfrak{g}$-modules

$\mathfrak{g}$ — semi-simple complex finite dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — triangular decomposition

$\mathcal{O}$ — corresponding BGG category $\mathcal{O}$

$\mathfrak{g}$-Mod — the category of all $\mathfrak{g}$-modules

$\mathcal{W}$ — the category of all weight (i.e. $\mathfrak{h}$-diagonalizable) $\mathfrak{g}$-modules

$\mathcal{GW}$ — the category of all generalized weight (i.e. locally $U(\mathfrak{h})$-finite) $\mathfrak{g}$-modules
Some categories of $\mathfrak{g}$-modules

$\mathfrak{g} -$ semi-simple complex finite dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ -$ triangular decomposition

$\mathcal{O} -$ corresponding BGG category $\mathcal{O}$

$\mathfrak{g}$-Mod $-$ the category of all $\mathfrak{g}$-modules

$\mathcal{W} -$ the category of all weight (i.e. $\mathfrak{h}$-diagonalizable) $\mathfrak{g}$-modules

$\mathcal{GW} -$ the category of all generalized weight (i.e. locally $U(\mathfrak{h})$-finite) $\mathfrak{g}$-modules
Some categories of $\mathfrak{g}$-modules

$\mathfrak{g} —$ semi-simple complex finite dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ —$ triangular decomposition

$\mathcal{O} —$ corresponding BGG category $\mathcal{O}$

$\mathfrak{g}$-Mod — the category of all $\mathfrak{g}$-modules

$\mathcal{W} —$ the category of all weight (i.e. $\mathfrak{h}$-diagonalizable) $\mathfrak{g}$-modules

$\mathcal{GW} —$ the category of all generalized weight (i.e. locally $U(\mathfrak{h})$-finite) $\mathfrak{g}$-modules
Some categories of $\mathfrak{g}$-modules

$\mathfrak{g}$ — semi-simple complex finite dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — triangular decomposition

$\mathcal{O}$ — corresponding BGG category $\mathcal{O}$

$\mathfrak{g}$-Mod — the category of all $\mathfrak{g}$-modules

$\mathcal{W}$ — the category of all weight (i.e. $\mathfrak{h}$-diagonalizable) $\mathfrak{g}$-modules

$\mathcal{GW}$ — the category of all generalized weight (i.e. locally $U(\mathfrak{h})$-finite) $\mathfrak{g}$-modules
Some categories of $\mathfrak{g}$-modules

$\mathfrak{g}$ — semi-simple complex finite dimensional Lie algebra

$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ — triangular decomposition

$\mathcal{O}$ — corresponding BGG category $\mathcal{O}$

$\mathfrak{g}$-Mod — the category of all $\mathfrak{g}$-modules

$\mathcal{W}$ — the category of all weight (i.e. $\mathfrak{h}$-diagonalizable) $\mathfrak{g}$-modules

$\mathcal{GW}$ — the category of all generalized weight (i.e. locally $U(\mathfrak{h})$-finite) $\mathfrak{g}$-modules
**Thick category** $\mathcal{O}$

**Definition.** Thick category $\widetilde{\mathcal{O}}$ is the full subcategory of $\mathfrak{g}$-Mod containing all $M$ such that

- $M$ is finitely generated;
- $M$ is locally $U(\mathfrak{h})$-finite;
- $M$ is locally $U(\mathfrak{n}_+)$-finite

**Alternative to the last two:** $M$ is locally $U(\mathfrak{b})$-finite for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$

**Difference to $\mathcal{O}$:** category $\widetilde{\mathcal{O}}$ has no projectives

**Note.** $\mathcal{O}$ is not extension full in $\widetilde{\mathcal{O}}$ (not even $\text{Ext}^1$-full)

**Note.** $\widetilde{\mathcal{O}}$ is the Serre subcategory of $\mathfrak{g}$-Mod generated by $\mathcal{O}$
**Definition.** Thick category \( \widetilde{\mathcal{O}} \) is the full subcategory of \( \mathfrak{g}\text{-Mod} \) containing all \( M \) such that

- \( M \) is finitely generated;
- \( M \) is locally \( U(\mathfrak{h}) \)-finite
- \( M \) is locally \( U(n_+) \)-finite

**Alternative to the last two:** \( M \) is locally \( U(b) \)-finite for \( b = \mathfrak{h} \oplus n_+ \)

**Difference to \( \mathcal{O} \):** category \( \widetilde{\mathcal{O}} \) has no projectives

**Note.** \( \mathcal{O} \) is not extension full in \( \widetilde{\mathcal{O}} \) (not even \( \text{Ext}^1 \)-full)

**Note.** \( \widetilde{\mathcal{O}} \) is the Serre subcategory of \( \mathfrak{g}\text{-Mod} \) generated by \( \mathcal{O} \)
Definition. Thick category $\tilde{O}$ is the full subcategory of $\mathfrak{g}$-Mod containing all $M$ such that

- $M$ is finitely generated;
  - $M$ is locally $U(\mathfrak{h})$-finite
  - $M$ is locally $U(\mathfrak{n}_+)$-finite

Alternative to the last two: $M$ is locally $U(\mathfrak{b})$-finite for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$

Difference to $O$: category $\tilde{O}$ has no projectives

Note. $O$ is not extension full in $\tilde{O}$ (not even $\text{Ext}^1$-full)

Note. $\tilde{O}$ is the Serre subcategory of $\mathfrak{g}$-Mod generated by $O$
**Thick category \( \mathcal{O} \)**

**Definition.** Thick category \( \tilde{\mathcal{O}} \) is the full subcategory of \( g\text{-Mod} \) containing all \( M \) such that

- \( M \) is finitely generated;
- \( M \) is locally \( U(\mathfrak{h}) \)-finite;
- \( M \) is locally \( U(\mathfrak{n}_+) \)-finite.

Alternative to the last two: \( M \) is locally \( U(\mathfrak{b}) \)-finite for \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+ \).

**Difference to \( \mathcal{O} \):** category \( \tilde{\mathcal{O}} \) has no projectives.

**Note.** \( \mathcal{O} \) is not extension full in \( \tilde{\mathcal{O}} \) (not even \( \text{Ext}^1 \)-full).

**Note.** \( \tilde{\mathcal{O}} \) is the Serre subcategory of \( g\text{-Mod} \) generated by \( \mathcal{O} \).
**Thick category** \( \mathcal{O} \)

**Definition.** Thick category \( \mathcal{\tilde{O}} \) is the full subcategory of \( \mathfrak{g}\text{-Mod} \) containing all \( M \) such that

- \( M \) is finitely generated;
- \( M \) is locally \( U(\mathfrak{h}) \)-finite
- \( M \) is locally \( U(\mathfrak{n}_+) \)-finite

Alternative to the last two: \( M \) is locally \( U(b) \)-finite for \( b = \mathfrak{h} \oplus \mathfrak{n}_+ \)

**Difference to** \( \mathcal{O} \): category \( \mathcal{\tilde{O}} \) has no projectives

**Note.** \( \mathcal{O} \) is not extension full in \( \mathcal{\tilde{O}} \) (not even \( \text{Ext}^1 \)-full)

**Note.** \( \mathcal{\tilde{O}} \) is the Serre subcategory of \( \mathfrak{g}\text{-Mod} \) generated by \( \mathcal{O} \)
**Definition.** Thick category $\tilde{\mathcal{O}}$ is the full subcategory of $\mathfrak{g}$-$\text{Mod}$ containing all $M$ such that

- $M$ is finitely generated;
- $M$ is locally $U(\mathfrak{h})$-finite
- $M$ is locally $U(n_+)$-finite

**Alternative to the last two:** $M$ is locally $U(\mathfrak{b})$-finite for $\mathfrak{b} = \mathfrak{h} \oplus n_+$

**Difference to $\mathcal{O}$:** category $\tilde{\mathcal{O}}$ has no projectives

**Note.** $\mathcal{O}$ is not extension full in $\tilde{\mathcal{O}}$ (not even $\text{Ext}^1$-full)

**Note.** $\tilde{\mathcal{O}}$ is the Serre subcategory of $\mathfrak{g}$-$\text{Mod}$ generated by $\mathcal{O}$
**Thick category $\mathcal{O}$**

**Definition.** Thick category $\tilde{\mathcal{O}}$ is the full subcategory of $\mathfrak{g}$-Mod containing all $M$ such that

- $M$ is finitely generated;
- $M$ is locally $U(\mathfrak{h})$-finite
- $M$ is locally $U(\mathfrak{n}_+)$-finite

**Alternative to the last two:** $M$ is locally $U(\mathfrak{b})$-finite for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$

**Difference to $\mathcal{O}$:** category $\tilde{\mathcal{O}}$ has no projectives

**Note.** $\mathcal{O}$ is not extension full in $\tilde{\mathcal{O}}$ (not even $\text{Ext}^1$-full)

**Note.** $\tilde{\mathcal{O}}$ is the Serre subcategory of $\mathfrak{g}$-Mod generated by $\mathcal{O}$
**Thick category $\mathcal{O}$**

**Definition.** Thick category $\mathcal{O}$ is the full subcategory of $\mathfrak{g}$-$\text{Mod}$ containing all $M$ such that

- $M$ is finitely generated;
- $M$ is locally $U(\mathfrak{h})$-finite
- $M$ is locally $U(\mathfrak{n}_+)$-finite

**Alternative to the last two:** $M$ is locally $U(\mathfrak{b})$-finite for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$

**Difference to $\mathcal{O}$:** category $\mathcal{O}$ has no projectives

**Note.** $\mathcal{O}$ is not extension full in $\mathcal{O}$ (not even Ext$^1$-full)

**Note.** $\mathcal{O}$ is the Serre subcategory of $\mathfrak{g}$-$\text{Mod}$ generated by $\mathcal{O}$
Thick category $\mathcal{O}$

**Definition.** Thick category $\widetilde{\mathcal{O}}$ is the full subcategory of $\mathfrak{g}$-Mod containing all $M$ such that

- $M$ is finitely generated;
- $M$ is locally $U(\mathfrak{h})$-finite
- $M$ is locally $U(n_+)$-finite

**Alternative to the last two:** $M$ is locally $U(\mathfrak{b})$-finite for $\mathfrak{b} = \mathfrak{h} \oplus n_+$

**Difference to $\mathcal{O}$:** category $\widetilde{\mathcal{O}}$ has no projectives

**Note.** $\mathcal{O}$ is not extension full in $\widetilde{\mathcal{O}}$ (not even $\text{Ext}^1$-full)

**Note.** $\widetilde{\mathcal{O}}$ is the Serre subcategory of $\mathfrak{g}$-Mod generated by $\mathcal{O}$
Thick category $\mathcal{O}$

**Definition.** Thick category $\widetilde{\mathcal{O}}$ is the full subcategory of $\mathfrak{g}\text{-Mod}$ containing all $M$ such that

- $M$ is finitely generated;
- $M$ is locally $U(\mathfrak{h})$-finite
- $M$ is locally $U(\mathfrak{n}_+)$-finite

**Alternative to the last two:** $M$ is locally $U(\mathfrak{b})$-finite for $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$

**Difference to $\mathcal{O}$:** category $\widetilde{\mathcal{O}}$ has no projectives

**Note.** $\mathcal{O}$ is not extension full in $\widetilde{\mathcal{O}}$ (not even $\text{Ext}^1$-full)

**Note.** $\widetilde{\mathcal{O}}$ is the Serre subcategory of $\mathfrak{g}\text{-Mod}$ generated by $\mathcal{O}$
Main results (Coulembier-M.)

Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

Theorem 2. $\widetilde{\mathcal{O}}$ is extension full in $\mathcal{GW}$.

Theorem 3. $\mathcal{GW}$ is extension full in $\mathfrak{g}\text{-Mod}$.

Theorem 4. Theorems 1, 2 and 3 are true for basic classical Lie superalgebras.

Corollary. $\text{gl.dim} \widetilde{\mathcal{O}} = \text{gl.dim} \mathcal{GW} = \text{dim} \mathfrak{g} = \text{gl.dim} \mathfrak{g}\text{-Mod}$
Theorem 1. \( \mathcal{O} \) is extension full in \( \mathcal{W} \).

Theorem 2. \( \mathcal{O} \) is extension full in \( \mathcal{GW} \).

Theorem 3. \( \mathcal{GW} \) is extension full in \( \mathfrak{g} \text{-Mod} \).

Theorem 4. Theorems 1, 2 and 3 are true for basic classical Lie superalgebras.

Corollary. \( \text{gl.dim } \mathcal{O} = \text{gl.dim } \mathcal{GW} = \dim \mathfrak{g} \) (\( = \text{gl.dim } \mathfrak{g} \text{-Mod} \))
Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

Theorem 2. $\mathcal{O}$ is extension full in $\mathcal{GW}$.

Theorem 3. $\mathcal{GW}$ is extension full in $\mathfrak{g}$-Mod.

Theorem 4. Theorems 1, 2 and 3 are true for basic classical Lie superalgebras.

Corollary. $\text{gl.dim } \mathcal{O} = \text{gl.dim } \mathcal{GW} = \dim \mathfrak{g} (= \text{gl.dim } \mathfrak{g}$-Mod)
Main results (Coulembier-M.)

Theorem 1. \( \mathcal{O} \) is extension full in \( \mathcal{W} \).

Theorem 2. \( \tilde{\mathcal{O}} \) is extension full in \( \mathcal{GW} \).

Theorem 3. \( \mathcal{GW} \) is extension full in \( g\text{-Mod} \).

Theorem 4. Theorems 1, 2 and 3 are true for basic classical Lie superalgebras.

Corollary. \( \text{gl.dim } \tilde{\mathcal{O}} = \text{gl.dim } \mathcal{GW} = \dim g = \text{gl.dim } g\text{-Mod} \)
Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

Theorem 2. $\tilde{\mathcal{O}}$ is extension full in $\mathcal{GW}$.

Theorem 3. $\mathcal{GW}$ is extension full in $\mathfrak{g}$-Mod.

Theorem 4. Theorems 1, 2 and 3 are true for basic classical Lie superalgebras.

Corollary. $\text{gl.dim } \tilde{\mathcal{O}} = \text{gl.dim } \mathcal{GW} = \dim \mathfrak{g}$ ($= \text{gl.dim } \mathfrak{g}$-Mod)
Main results (Coulembier-M.)

Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

Theorem 2. $\widetilde{\mathcal{O}}$ is extension full in $\mathcal{GW}$.

Theorem 3. $\mathcal{GW}$ is extension full in $\mathfrak{g}\text{-Mod}$.

Theorem 4. Theorems 1, 2 and 3 are true for basic classical Lie superalgebras.

Corollary. $\text{gl.dim } \widetilde{\mathcal{O}} = \text{gl.dim } \mathcal{GW} = \dim \mathfrak{g}$ ($= \text{gl.dim } \mathfrak{g}\text{-Mod}$)
Main results (Coulembier-M.)

Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

Theorem 2. $\widetilde{\mathcal{O}}$ is extension full in $\mathcal{GW}$.

Theorem 3. $\mathcal{GW}$ is extension full in $\mathfrak{g}$-Mod.

Theorem 4. Theorems 1, 2 and 3 are true for basic classical Lie superalgebras.

Corollary. $\text{gl.dim } \widetilde{\mathcal{O}} = \text{gl.dim } \mathcal{GW} = \dim \mathfrak{g}$ ($= \text{gl.dim } \mathfrak{g}$-Mod)
Very rough idea of the proof of Theorem 1.

Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

“Easy” case: Both categories have projectives.

Use:

- Frobenius reciprocity ($=$ adjunction of $\text{Ind}$ and $\text{Res}$)
- BGG’s construction of projectives in $\mathcal{O}$
- Comparison of these projectives to projectives in $\mathcal{W}$.
- the next lemma

Lemma. Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion, both have enough projectives, and any projective $P \in \mathcal{A}$ is acyclic for the functor $\text{Hom}_\mathcal{B}(\_, K)$ for any $K \in \mathcal{A}$. Then $\mathcal{A}$ is extension full in $\mathcal{B}$.
Theorem 1. \( \mathcal{O} \) is extension full in \( \mathcal{W} \).

“Easy” case: Both categories have projectives.

Use:

- Frobenius reciprocity (\( = \) adjunction of \( \text{Ind} \) and \( \text{Res} \))
- BGG’s construction of projectives in \( \mathcal{O} \)
- Comparison of these projectives to projectives in \( \mathcal{W} \)
- the next lemma

Lemma. Assume \( \mathcal{A} \subset \mathcal{B} \) with exact inclusion, both have enough projectives, and any projective \( P \in \mathcal{A} \) is acyclic for the functor \( \text{Hom}_\mathcal{B}(\_ , K) \) for any \( K \in \mathcal{A} \). Then \( \mathcal{A} \) is extension full in \( \mathcal{B} \).
Very rough idea of the proof of Theorem 1.

**Theorem 1.** $\mathcal{O}$ is extension full in $\mathcal{W}$.

**“Easy” case:** Both categories have projectives.

**Use:**

- Frobenius reciprocity (= adjunction of $\text{Ind}$ and $\text{Res}$)
- BGG's construction of projectives in $\mathcal{O}$
- Comparison of these projectives to projectives in $\mathcal{W}$.
- the next lemma

**Lemma.** Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion, both have enough projectives, and any projective $P \in \mathcal{A}$ is acyclic for the functor $\text{Hom}_B(\_ , K)$ for any $K \in \mathcal{A}$. Then $\mathcal{A}$ is extension full in $\mathcal{B}$. 
Very rough idea of the proof of Theorem 1.

**Theorem 1.** $\mathcal{O}$ is extension full in $\mathcal{W}$.

**“Easy” case:** Both categories have projectives.

**Use:**

- Frobenius reciprocity (= adjunction of Ind and Res)
- BGG’s construction of projectives in $\mathcal{O}$
- Comparison of these projectives to projectives in $\mathcal{W}$.
- the next lemma

**Lemma.** Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion, both have enough projectives, and any projective $P \in \mathcal{A}$ is acyclic for the functor $\text{Hom}_\mathcal{B}(\_ , K)$ for any $K \in \mathcal{A}$. Then $\mathcal{A}$ is extension full in $\mathcal{B}$.
Theorem 1. \( \mathcal{O} \) is extension full in \( \mathcal{W} \).

“Easy” case: Both categories have projectives.

Use:

- Frobenius reciprocity (\( = \) adjunction of \( \text{Ind} \) and \( \text{Res} \))
- BGG’s construction of projectives in \( \mathcal{O} \)
- Comparison of these projectives to projectives in \( \mathcal{W} \).
- the next lemma

Lemma. Assume \( \mathcal{A} \subset \mathcal{B} \) with exact inclusion, both have enough projectives, and any projective \( P \in \mathcal{A} \) is acyclic for the functor \( \text{Hom}_{\mathcal{B}}(\_, K) \) for any \( K \in \mathcal{A} \). Then \( \mathcal{A} \) is extension full in \( \mathcal{B} \).
Very rough idea of the proof of Theorem 1.

Theorem 1. $\mathcal{O}$ is extension full in $\mathcal{W}$.

“Easy” case: Both categories have projectives.

Use:

- Frobenius reciprocity ($=$ adjunction of $\text{Ind}$ and $\text{Res}$)
- BGG’s construction of projectives in $\mathcal{O}$
  - Comparison of these projectives to projectives in $\mathcal{W}$.
  - the next lemma

Lemma. Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion, both have enough projectives, and any projective $P \in \mathcal{A}$ is acyclic for the functor $\text{Hom}_B(\_ , K)$ for any $K \in \mathcal{A}$. Then $\mathcal{A}$ is extension full in $\mathcal{B}$. 
Very rough idea of the proof of Theorem 1.

**Theorem 1.** $\mathcal{O}$ is extension full in $\mathcal{W}$.

**“Easy” case:** Both categories have projectives.

**Use:**

- Frobenius reciprocity ($= \text{adjunction of Ind and Res}$)
- BGG’s construction of projectives in $\mathcal{O}$
- Comparison of these projectives to projectives in $\mathcal{W}$.
- the next lemma

**Lemma.** Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion, both have enough projectives, and any projective $P \in \mathcal{A}$ is acyclic for the functor $\text{Hom}_B(\_ , K)$ for any $K \in \mathcal{A}$. Then $\mathcal{A}$ is extension full in $\mathcal{B}$. 
Theorem 1. \( \mathcal{O} \) is extension full in \( \mathcal{W} \).

“Easy” case: Both categories have projectives.

Use:

- Frobenius reciprocity (\( \cong \) adjunction of \( \text{Ind} \) and \( \text{Res} \))
- BGG’s construction of projectives in \( \mathcal{O} \)
- Comparison of these projectives to projectives in \( \mathcal{W} \).
- the next lemma

Lemma. Assume \( \mathcal{A} \subset \mathcal{B} \) with exact inclusion, both have enough projectives, and any projective \( P \in \mathcal{A} \) is acyclic for the functor \( \text{Hom}_\mathcal{B}(\_ , K) \) for any \( K \in \mathcal{A} \). Then \( \mathcal{A} \) is extension full in \( \mathcal{B} \).
Very rough idea of the proof of Theorem 1.

**Theorem 1.** $\mathcal{O}$ is extension full in $\mathcal{W}$.

**“Easy” case:** Both categories have projectives.

**Use:**

- Frobenius reciprocity (= adjunction of $\text{Ind}$ and $\text{Res}$)
- BGG’s construction of projectives in $\mathcal{O}$
- Comparison of these projectives to projectives in $\mathcal{W}$.
- the next lemma

**Lemma.** Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion, both have enough projectives, and any projective $P \in \mathcal{A}$ is acyclic for the functor $\text{Hom}_\mathcal{B}(\_ , K)$ for any $K \in \mathcal{A}$. Then $\mathcal{A}$ is extension full in $\mathcal{B}$. 
Theorem 2. $\tilde{O}$ is extension full in $\mathcal{GW}$.

Note: None of the categories have projectives.

Steps:

- Restrict the size of Jordan cells allowed for the action of $\mathfrak{h}$ to get $\tilde{O}^{(n)}$ and $\mathcal{GW}^{(n)}$
- Both $\tilde{O}^{(n)}$ and $\mathcal{GW}^{(n)}$ have projectives
- Use proof of Theorem 1 to show that $\tilde{O}^{(n)}$ is extension full in $\mathcal{GW}^{(n)}$
- Take limit $n \to \infty$
- Show that extension split into “stable” and “nilpotent” parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Very rough idea of the proof of Theorem 2.

**Theorem 2.** $\tilde{O}$ is extension full in $GW$.

**Note:** None of the categories have projectives.

**Steps:**

- Restrict the size of Jordan cells allowed for the action of $h$ to get $\tilde{O}^{(n)}$ and $GW^{(n)}$
- Both $\tilde{O}^{(n)}$ and $GW^{(n)}$ have projectives
- Use proof of Theorem 1 to show that $\tilde{O}^{(n)}$ is extension full in $GW^{(n)}$
- Take limit $n \to \infty$
- Show that extension split into "stable" and "nilpotent" parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Theorem 2. \( \tilde{O} \) is extension full in \( \mathcal{GW} \).

Note: None of the categories have projectives.

Steps:

- Restrict the size of Jordan cells allowed for the action of \( h \) to get \( \tilde{O}^{(n)} \) and \( \mathcal{GW}^{(n)} \)
- Both \( \tilde{O}^{(n)} \) and \( \mathcal{GW}^{(n)} \) have projectives
- Use proof of Theorem 1 to show that \( \tilde{O}^{(n)} \) is extension full in \( \mathcal{GW}^{(n)} \)
- Take limit \( n \to \infty \)
- Show that extension split into "stable" and "nilpotent" parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Very rough idea of the proof of Theorem 2.

**Theorem 2.** \( \tilde{O} \) is extension full in \( GW \).

**Note:** None of the categories have projectives.

**Steps:**

- Restrict the size of Jordan cells allowed for the action of \( h \) to get \( \tilde{O}^{(n)} \) and \( GW^{(n)} \).
- Both \( \tilde{O}^{(n)} \) and \( GW^{(n)} \) have projectives.
- Use proof of Theorem 1 to show that \( \tilde{O}^{(n)} \) is extension full in \( GW^{(n)} \).
- Take limit \( n \to \infty \).
- Show that extension split into “stable” and “nilpotent” parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit.
Very rough idea of the proof of Theorem 2.

**Theorem 2.**  \( \tilde{\mathcal{O}} \) is extension full in \( \mathcal{GW} \).

**Note:** None of the categories have projectives.

**Steps:**

- Restrict the size of Jordan cells allowed for the action of \( \mathfrak{h} \) to get \( \tilde{\mathcal{O}}^{(n)} \) and \( \mathcal{GW}^{(n)} \)
- Both \( \tilde{\mathcal{O}}^{(n)} \) and \( \mathcal{GW}^{(n)} \) have projectives
- Use proof of Theorem 1 to show that \( \tilde{\mathcal{O}}^{(n)} \) is extension full in \( \mathcal{GW}^{(n)} \)
- Take limit \( n \to \infty \)
- Show that extension split into “stable” and “nilpotent” parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Theorem 2. $\tilde{O}$ is extension full in $GW$.

Note: None of the categories have projectives.

Steps:

- Restrict the size of Jordan cells allowed for the action of $\mathfrak{h}$ to get $\tilde{O}(n)$ and $GW(n)$
- Both $\tilde{O}(n)$ and $GW(n)$ have projectives
  - Use proof of Theorem 1 to show that $\tilde{O}(n)$ is extension full in $GW(n)$
  - Take limit $n \to \infty$
  - show that extension split into “stable” and “nilpotent” parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Very rough idea of the proof of Theorem 2.

Theorem 2. $\tilde{O}$ is extension full in $GW$.

Note: None of the categories have projectives.

Steps:

- Restrict the size of Jordan cells allowed for the action of $\mathfrak{h}$ to get $\tilde{O}^{(n)}$ and $GW^{(n)}$
- Both $\tilde{O}^{(n)}$ and $GW^{(n)}$ have projectives
- Use proof of Theorem 1 to show that $\tilde{O}^{(n)}$ is extension full in $GW^{(n)}$
- Take limit $n \to \infty$
- show that extension split into "stable" and "nilpotent" parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Very rough idea of the proof of Theorem 2.

**Theorem 2.** \( \tilde{O} \) is extension full in \( GW \).

**Note:** None of the categories have projectives.

**Steps:**

- Restrict the size of Jordan cells allowed for the action of \( \mathfrak{h} \) to get \( \tilde{O}^{(n)} \) and \( GW^{(n)} \)
- Both \( \tilde{O}^{(n)} \) and \( GW^{(n)} \) have projectives
- Use proof of Theorem 1 to show that \( \tilde{O}^{(n)} \) is extension full in \( GW^{(n)} \)
- Take limit \( n \to \infty \)
- Show that extension split into “stable” and “nilpotent” parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Very rough idea of the proof of Theorem 2.

**Theorem 2.** $\tilde{O}$ is extension full in $GW$.

**Note:** None of the categories have projectives.

**Steps:**

- Restrict the size of Jordan cells allowed for the action of $\mathfrak{h}$ to get $\tilde{O}^{(n)}$ and $GW^{(n)}$
- Both $\tilde{O}^{(n)}$ and $GW^{(n)}$ have projectives
- Use proof of Theorem 1 to show that $\tilde{O}^{(n)}$ is extension full in $GW^{(n)}$
- Take limit $n \to \infty$
- show that extension split into “stable” and “nilpotent” parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Very rough idea of the proof of Theorem 2.

Theorem 2. \( \widetilde{O} \) is extension full in \( GW \).

Note: None of the categories have projectives.

Steps:

- Restrict the size of Jordan cells allowed for the action of \( \mathfrak{h} \) to get \( \widetilde{O}^{(n)} \) and \( GW^{(n)} \)
- Both \( \widetilde{O}^{(n)} \) and \( GW^{(n)} \) have projectives
- Use proof of Theorem 1 to show that \( \widetilde{O}^{(n)} \) is extension full in \( GW^{(n)} \)
- Take limit \( n \to \infty \)
- show that extension split into “stable” and “nilpotent” parts where the stable part gives the limit extension and the nilpotent part eventually dies when taking the limit
Theorem 3. \( \mathcal{GW} \) is extension full in \( g\)-Mod.

**Note:** \( g\)-Mod has projectives while \( \mathcal{GW} \) does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

**Lemma.** Assume \( A \subset B \) with exact inclusion.

Assume \( A \) has a full subcategory \( A_0 \) such that
- \( A \) is the Serre subcategory of \( B \) generated by \( A_0 \);
- \( A_0 \) has enough projectives.

Then \( A \) is extension full in \( B \) if and only if the natural map

\[
\text{Ext}_A^n(P, K) \rightarrow \text{Ext}_B^n(P, K)
\]

is an isomorphism for all \( n \), all projective \( P \in A_0 \) and all \( K \in A_0 \).
Very rough idea of the proof of Theorem 3.

**Theorem 3.** \( \mathcal{GW} \) is extension full in \( g\)-Mod.

**Note:** \( g\)-Mod has projectives while \( \mathcal{GW} \) does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

**Lemma.** Assume \( A \subset B \) with exact inclusion. Assume \( A \) has a full subcategory \( A_0 \) such that
- \( A \) is the Serre subcategory of \( B \) generated by \( A_0 \);
- \( A_0 \) has enough projectives.

Then \( A \) is extension full in \( B \) if and only if the natural map

\[
\text{Ext}^n_A(P, K) \to \text{Ext}^n_B(P, K)
\]

is an isomorphism for all \( n \), all projective \( P \in A_0 \) and all \( K \in A_0 \).
Theorem 3. \( GW \) is extension full in \( g\)-Mod.

Note: \( g\)-Mod has projectives while \( GW \) does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

Lemma. Assume \( A \subseteq B \) with exact inclusion.
Assume \( A \) has a full subcategory \( A_0 \) such that
- \( A \) is the Serre subcategory of \( B \) generated by \( A_0 \);
- \( A_0 \) has enough projectives.
Then \( A \) is extension full in \( B \) if and only if the natural map

\[
\text{Ext}^n_A(P, K) \to \text{Ext}^n_B(P, K)
\]

is an isomorphism for all \( n \), all projective \( P \in A_0 \) and all \( K \in A_0 \).
Very rough idea of the proof of Theorem 3.

**Theorem 3.** $\mathcal{GW}$ is extension full in $\mathfrak{g}$-Mod.

**Note:** $\mathfrak{g}$-Mod has projectives while $\mathcal{GW}$ does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

**Lemma.** Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion. Assume $\mathcal{A}$ has a full subcategory $\mathcal{A}_0$ such that

- $\mathcal{A}$ is the Serre subcategory of $\mathcal{B}$ generated by $\mathcal{A}_0$;
- $\mathcal{A}_0$ has enough projectives.

Then $\mathcal{A}$ is extension full in $\mathcal{B}$ if and only if the natural map

$$\text{Ext}^n_\mathcal{A}(P, K) \rightarrow \text{Ext}^n_\mathcal{B}(P, K)$$

is an isomorphism for all $n$, all projective $P \in \mathcal{A}_0$ and all $K \in \mathcal{A}_0$. 
Theorem 3. $GW$ is extension full in $g$-Mod.

Note: $g$-Mod has projectives while $GW$ does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

Lemma. Assume $A \subset B$ with exact inclusion. Assume $A$ has a full subcategory $A_0$ such that
- $A$ is the Serre subcategory of $B$ generated by $A_0$;
- $A_0$ has enough projectives.

Then $A$ is extension full in $B$ if and only if the natural map

$$\text{Ext}_A^n(P, K) \rightarrow \text{Ext}_B^n(P, K)$$

is an isomorphism for all $n$, all projective $P \in A_0$ and all $K \in A_0$. 
Very rough idea of the proof of Theorem 3.

**Theorem 3.** $\mathcal{GW}$ is extension full in $\mathfrak{g}$-Mod.

**Note:** $\mathfrak{g}$-Mod has projectives while $\mathcal{GW}$ does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

**Lemma.** Assume $\mathcal{A} \subset \mathcal{B}$ with exact inclusion. Assume $\mathcal{A}$ has a full subcategory $\mathcal{A}_0$ such that

- $\mathcal{A}$ is the Serre subcategory of $\mathcal{B}$ generated by $\mathcal{A}_0$;
- $\mathcal{A}_0$ has enough projectives.

Then $\mathcal{A}$ is extension full in $\mathcal{B}$ if and only if the natural map

$$\text{Ext}^n_{\mathcal{A}}(P, K) \to \text{Ext}^n_{\mathcal{B}}(P, K)$$

is an isomorphism for all $n$, all projective $P \in \mathcal{A}_0$ and all $K \in \mathcal{A}_0$. 
Very rough idea of the proof of Theorem 3.

Theorem 3. \( \mathcal{GW} \) is extension full in \( \mathfrak{g}\text{-Mod} \).

**Note:** \( \mathfrak{g}\text{-Mod} \) has projectives while \( \mathcal{GW} \) does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

**Lemma.** Assume \( \mathcal{A} \subset \mathcal{B} \) with exact inclusion. Assume \( \mathcal{A} \) has a full subcategory \( \mathcal{A}_0 \) such that

- \( \mathcal{A} \) is the Serre subcategory of \( \mathcal{B} \) generated by \( \mathcal{A}_0 \);
- \( \mathcal{A}_0 \) has enough projectives.

Then \( \mathcal{A} \) is extension full in \( \mathcal{B} \) if and only if the natural map

\[
\text{Ext}^n_A(P, K) \rightarrow \text{Ext}^n_B(P, K)
\]

is an isomorphism for all \( n \), all projective \( P \in \mathcal{A}_0 \) and all \( K \in \mathcal{A}_0 \).
Very rough idea of the proof of Theorem 3.

**Theorem 3.** $GW$ is extension full in $g$-Mod.

**Note:** $g$-Mod has projectives while $GW$ does not.

Using tricks and Frobenius reciprocity, Theorem 3 can be reduced to:

**Lemma.** Assume $A \subset B$ with exact inclusion.
Assume $A$ has a full subcategory $A_0$ such that
- $A$ is the Serre subcategory of $B$ generated by $A_0$;
- $A_0$ has enough projectives.
Then $A$ is extension full in $B$ if and only if the natural map

$$\text{Ext}_A^n(P, K) \to \text{Ext}_B^n(P, K)$$

is an isomorphism for all $n$, all projective $P \in A_0$ and all $K \in A_0$. 
Definition. \( \text{p.dim} \tilde{\mathcal{O}} M := \sup \{ k : \text{Ext}^k_{\tilde{\mathcal{O}}}(M, N) \neq 0 \text{ for some } N \in \tilde{\mathcal{O}} \} \)

Theorem 5. (Coulembier-M.) \( \text{p.dim} \tilde{\mathcal{O}} M \geq \dim \mathfrak{h} \) for \( M \in \tilde{\mathcal{O}} \)

Theorem 6. (Coulembier-M.) \( \text{p.dim} \tilde{\mathcal{O}} M = \dim \mathfrak{h} + \text{p.dim} \mathcal{O} M \) for \( M \in \mathcal{O} \)
**Definition.** $\text{p.dim}_{\tilde{O}} M := \sup \{ k : \text{Ext}^k_{\tilde{O}}(M, N) \neq 0 \text{ for some } N \in \tilde{O} \}$

**Theorem 5.** (Coulembier-M.) $\text{p.dim}_{\tilde{O}} M \geq \dim \mathfrak{h}$ for $M \in \tilde{O}$

**Theorem 6.** (Coulembier-M.) $\text{p.dim}_{\tilde{O}} M = \dim \mathfrak{h} + \text{p.dim}_O M$ for $M \in O$
Bonus: projective dimension in $\tilde{O}$

**Definition.** $\text{p.dim}_{\tilde{O}} M := \sup\{k : \text{Ext}^k_{\tilde{O}}(M, N) \neq 0 \text{ for some } N \in \tilde{O}\}$

**Theorem 5.** (Coulembier-M.) $\text{p.dim}_{\tilde{O}} M \geq \dim \mathfrak{h}$ for $M \in \tilde{O}$

**Theorem 6.** (Coulembier-M.) $\text{p.dim}_{\tilde{O}} M = \dim \mathfrak{h} + \text{p.dim}_O M$ for $M \in O$
Definition. $\text{p.dim}_\widetilde{O} M := \sup \{k : \text{Ext}^k_{\widetilde{O}}(M, N) \neq 0 \text{ for some } N \in \widetilde{O}\}$

Theorem 5. (Coulembier-M.) $\text{p.dim}_\widetilde{O} M \geq \dim \mathfrak{h}$ for $M \in \widetilde{O}$

Theorem 6. (Coulembier-M.) $\text{p.dim}_\widetilde{O} M = \dim \mathfrak{h} + \text{p.dim}_O M$ for $M \in O$
Bonus: projective dimension in $\tilde{O}$

**Definition.** $\text{p.dim}_{\tilde{O}} M := \sup \{ k : \text{Ext}_{\tilde{O}}^k(M, N) \neq 0 \text{ for some } N \in \tilde{O} \}$

**Theorem 5.** (Coulembier-M.) $\text{p.dim}_{\tilde{O}} M \geq \dim \mathfrak{h}$ for $M \in \tilde{O}$

**Theorem 6.** (Coulembier-M.) $\text{p.dim}_{\tilde{O}} M = \dim \mathfrak{h} + \text{p.dim}_O M$ for $M \in O$
Motivation: Alexandru conjecture, part I


\( \mathcal{A} \) — abelian length category

\( \text{Irr}(\mathcal{A}) \) — set of isoclasses of simple objects in \( \mathcal{A} \)

\( < \) — smallest partial order on \( \text{Irr}(\mathcal{A}) \) such that

\[
p \cdot \dim L = p \cdot \dim L' + 1 \quad \text{and} \quad \text{Ext}^1_{\mathcal{A}}(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j
\]

**Definition.** A Serre subcategory \( B \subset \mathcal{A} \) is an initial segment if \( L_j \in B \) and \( L_i < L_j \) implies \( L_i \in B \)

**Definition.** \( \mathcal{A} \) is Guichardet if any initial segment is extension full in \( \mathcal{A} \)
Motivation: Alexandru conjecture, part I


\(\mathcal{A}\) — abelian length category

\(\text{Irr}(\mathcal{A})\) — set of isoclasses of simple objects in \(\mathcal{A}\)

\(<\) — smallest partial order on \(\text{Irr}(\mathcal{A})\) such that

\[ p\cdot\text{dim} \, L = p\cdot\text{dim} \, L' + 1 \quad \text{and} \quad \text{Ext}^1_{\mathcal{A}}(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j \]

**Definition.** A Serre subcategory \(B \subset \mathcal{A}\) is an initial segment if \(L_j \in B\) and \(L_i < L_j\) implies \(L_i \in B\)

**Definition.** \(\mathcal{A}\) is Guichardet if any initial segment is extension full in \(\mathcal{A}\)
Motivation: Alexandru conjecture, part I


\( \mathcal{A} \) — abelian length category

\( \text{Irr}(\mathcal{A}) \) — set of isoclasses of simple objects in \( \mathcal{A} \)

\(<\) — smallest partial order on \( \text{Irr}(\mathcal{A}) \) such that

\[
p.\dim L = p.\dim L' + 1 \quad \text{and} \quad \text{Ext}_A^1(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j
\]

**Definition.** A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is an initial segment if \( L_j \in \mathcal{B} \) and \( L_i < L_j \) implies \( L_i \in \mathcal{B} \)

**Definition.** \( \mathcal{A} \) is Guichardet if any initial segment is extension full in \( \mathcal{A} \)

\( \mathcal{A} \) — abelian length category

\( \text{Irr}(\mathcal{A}) \) — set of isoclasses of simple objects in \( \mathcal{A} \)

< — smallest partial order on \( \text{Irr}(\mathcal{A}) \) such that

\[ \text{p.dim } L = \text{p.dim } L' + 1 \quad \text{and} \quad \text{Ext}^1_{\mathcal{A}}(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j \]

Definition. A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is an initial segment if \( L_j \in \mathcal{B} \) and \( L_i < L_j \) implies \( L_i \in \mathcal{B} \)

Definition. \( \mathcal{A} \) is Guichardet if any initial segment is extension full in \( \mathcal{A} \)
Motivation: Alexandru conjecture, part I


\( \mathcal{A} \) — abelian length category

\( \text{Irr}(\mathcal{A}) \) — set of isoclasses of simple objects in \( \mathcal{A} \)

\(<\) — smallest partial order on \( \text{Irr}(\mathcal{A}) \) such that

\[ p.\dim L = p.\dim L' + 1 \quad \text{and} \quad \text{Ext}^1_{\mathcal{A}}(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j \]

**Definition.** A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is an initial segment if \( L_j \in \mathcal{B} \) and \( L_i < L_j \) implies \( L_i \in \mathcal{B} \)

**Definition.** \( \mathcal{A} \) is Guichardet if any initial segment is extension full in \( \mathcal{A} \)
Motivation: Alexandru conjecture, part I


\[ \mathcal{A} \] — abelian length category

\[ \text{Irr}(\mathcal{A}) \] — set of isoclasses of simple objects in \( \mathcal{A} \)

\(<\) — smallest partial order on \( \text{Irr}(\mathcal{A}) \) such that

\[ p.\dim L = p.\dim L' + 1 \quad \text{and} \quad \text{Ext}^1_{\mathcal{A}}(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j \]

**Definition.** A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is an initial segment if \( L_j \in \mathcal{B} \) and \( L_i < L_j \) implies \( L_i \in \mathcal{B} \)

**Definition.** \( \mathcal{A} \) is Guichardet if any initial segment is extension full in \( \mathcal{A} \)
**Motivation: Alexandru conjecture, part I**


\( \mathcal{A} \) — abelian length category

\( \text{Irr}(\mathcal{A}) \) — set of isoclasses of simple objects in \( \mathcal{A} \)

\( < \) — smallest partial order on \( \text{Irr}(\mathcal{A}) \) such that

\[ \text{p.dim } L = \text{p.dim } L' + 1 \quad \text{and} \quad \text{Ext}^1_{\mathcal{A}}(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j \]

**Definition.** A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is an initial segment if \( L_j \in \mathcal{B} \) and \( L_i < L_j \) implies \( L_i \in \mathcal{B} \)

**Definition.** \( \mathcal{A} \) is Guichardet if any initial segment is extension full in \( \mathcal{A} \)
Motivation: Alexandru conjecture, part I


\( \mathcal{A} \) — abelian length category

\( \text{Irr}(\mathcal{A}) \) — set of isoclasses of simple objects in \( \mathcal{A} \)

\(<\) — smallest partial order on \( \text{Irr}(\mathcal{A}) \) such that

\[
p.\dim L = p.\dim L' + 1 \quad \text{and} \quad \Ext^1_{\mathcal{A}}(L, L') \neq 0 \quad \text{imply} \quad L_i < L_j
\]

Definition. A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is an initial segment if \( L_j \in \mathcal{B} \) and \( L_i < L_j \) implies \( L_i \in \mathcal{B} \)

Definition. \( \mathcal{A} \) is Guichardet if any initial segment is extension full in \( \mathcal{A} \)
Weak Alexandru conjecture: The principal block of the category of Harish-Chandra $(g, \mathfrak{t})$-modules is Guichardet.

Motivation: $\mathcal{O}_0$ is Guichardet.

Explanation:

- we know explicitly $p.\dim$ of all simples in $\mathcal{O}_0$;
- we know the quiver of $\mathcal{O}_0$;
- we can describe all initial segments in $\mathcal{O}_0$ (they are coideals in the Bruhat order on $W$);
- $\mathcal{O}_0 \cong A$-mod where $A$ is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Weak Alexandru conjecture: The principal block of the category of Harish-Chandra $(\mathfrak{g}, \mathfrak{t})$-modules is Guichardet.

Motivation: $\mathcal{O}_0$ is Guichardet.

Explanation:

- we know explicitly $\text{p.dim}$ of all simples in $\mathcal{O}_0$;
- we know the quiver of $\mathcal{O}_0$;
- we can describe all initial segments in $\mathcal{O}_0$ (they are coideals in the Bruhat order on $W$);
- $\mathcal{O}_0 \cong A\text{-mod}$ where $A$ is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Motivation: Alexandru conjecture, part II

**Weak Alexandru conjecture:** The principal block of the category of Harish-Chandra \((g, \mathfrak{t})\)-modules is Guichardet.

**Motivation:** \(\mathcal{O}_0\) is Guichardet.

**Explanation:**

- we know explicitly \(p\dim\) of all simples in \(\mathcal{O}_0\);
- we know the quiver of \(\mathcal{O}_0\);
- we can describe all initial segments in \(\mathcal{O}_0\) (they are coideals in the Bruhat order on \(W\));
- \(\mathcal{O}_0 \cong A\text{-mod}\) where \(A\) is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Weak Alexandru conjecture: The principal block of the category of Harish-Chandra \((\mathfrak{g}, \mathfrak{t})\)-modules is Guichardet.

Motivation: \(\mathcal{O}_0\) is Guichardet.

Explanation:

- we know explicitly \(\text{p.dim}\) of all simples in \(\mathcal{O}_0\);
- we know the quiver of \(\mathcal{O}_0\);
- we can describe all initial segments in \(\mathcal{O}_0\) (they are coideals in the Bruhat order on \(W\));
- \(\mathcal{O}_0 \cong A\text{-mod}\) where \(A\) is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Weak Alexandru conjecture: The principal block of the category of Harish-Chandra \((\mathfrak{g}, \mathfrak{t})\)-modules is Guichardet.

**Motivation:** \(\mathcal{O}_0\) is Guichardet.

**Explanation:**

- we know explicitly \(\text{p.dim}\) of all simples in \(\mathcal{O}_0\);
- we know the quiver of \(\mathcal{O}_0\);
- we can describe all initial segments in \(\mathcal{O}_0\) (they are coideals in the Bruhat order on \(W\));
- \(\mathcal{O}_0 \cong A\)-mod where \(A\) is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Weak Alexandru conjecture: The principal block of the category of Harish-Chandra $(\mathfrak{g}, \mathfrak{t})$-modules is Guichardet.

Motivation: $\mathcal{O}_0$ is Guichardet.

Explanation:

- we know explicitly $\text{p.dim}$ of all simples in $\mathcal{O}_0$;
- we know the quiver of $\mathcal{O}_0$;
- we can describe all initial segments in $\mathcal{O}_0$ (they are coideals in the Bruhat order on $W$);
- $\mathcal{O}_0 \cong A\text{-mod}$ where $A$ is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Weak Alexandru conjecture: The principal block of the category of Harish-Chandra \((\mathfrak{g}, \mathfrak{t})\)-modules is Guichardet.

Motivation: \(\mathcal{O}_0\) is Guichardet.

Explanation:

- we know explicitly \(p.\dim\) of all simples in \(\mathcal{O}_0\);
- we know the quiver of \(\mathcal{O}_0\);
- we can describe all initial segments in \(\mathcal{O}_0\) (they are coideals in the Bruhat order on \(W\));
- \(\mathcal{O}_0 \cong A\text{-mod}\) where \(A\) is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Weak Alexandru conjecture: The principal block of the category of Harish-Chandra \((\mathfrak{g}, \mathfrak{t})\)-modules is Guichardet.

Motivation: \(O_0\) is Guichardet.

Explanation:

- we know explicitly \(p\text{-dim}\) of all simples in \(O_0\);
- we know the quiver of \(O_0\);
- we can describe all initial segments in \(O_0\) (they are coideals in the Bruhat order on \(W\));
- \(O_0 \cong A\text{-mod}\) where \(A\) is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Motivation: Alexandru conjecture, part II

**Weak Alexandru conjecture:** The principal block of the category of Harish-Chandra \((g, \mathfrak{t})\)-modules is Guichardet.

**Motivation:** \(O_0\) is Guichardet.

**Explanation:**

- we know explicitly \(p \cdot \dim\) of all simples in \(O_0\);
- we know the quiver of \(O_0\);
- we can describe all initial segments in \(O_0\) (they are coideals in the Bruhat order on \(W\));
- \(O_0 \cong A\)-mod where \(A\) is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
**Weak Alexandru conjecture:** The principal block of the category of Harish-Chandra \((g, \mathfrak{t})\)-modules is Guichardet.

**Motivation:** \(O_0\) is Guichardet.

**Explanation:**

- we know explicitly \(p.\dim\) of all simples in \(O_0\);
- we know the quiver of \(O_0\);
- we can describe all initial segments in \(O_0\) (they are coideals in the Bruhat order on \(W\));
- \(O_0 \cong A\)-mod where \(A\) is quasi-hereditary
- to all such initial segments the theorem of CPS is applicable
Theorem 7. (Coulembier-M.) \( \tilde{\mathcal{O}}_0 \) is Guichardet.

\( \mathcal{H} \) — the category of Harish-Chandra bimodules for \( g \)

Note: \( \mathcal{H} \in g \oplus g\text{-mod} \)

BG-equivalences. \( \mathcal{O}_0 \cong {}^0 \mathcal{H}^1_0 \) and \( \tilde{\mathcal{O}}_0 \cong {}^0 \mathcal{H}_0^\infty \)

Corollary. \( {}^\infty \mathcal{H}^1_\chi, {}^1 \mathcal{H}^\infty_\chi \) and \( {}^0 \mathcal{H}_0^\infty \) are Guichardet.

Observation. \( {}^0 \mathcal{H}_0^\infty \) is not extension full in \( g \oplus g\text{-mod} \).
Theorem 7. (Coulembier-M.) $\tilde{O}_0$ is Guichardet.

$\mathcal{H}$ — the category of Harish-Chandra bimodules for $\mathfrak{g}$

Note: $\mathcal{H} \in \mathfrak{g} \oplus \mathfrak{g}$-mod

BG-equivalences. $O_0 \cong \mathcal{H}_0^1$ and $\tilde{O}_0 \cong \mathcal{H}_0^\infty$

Corollary. $\mathcal{H}_\chi^1$, $\mathcal{H}_\chi^\infty$ and $\mathcal{H}_0^\infty$ are Guichardet.

Observation. $\mathcal{H}_0^\infty$ is not extension full in $\mathfrak{g} \oplus \mathfrak{g}$-mod.
Alexandru conjecture for $\tilde{O}$ and for $\mathcal{H}$

Theorem 7. (Coulembier-M.) $\tilde{O}_0$ is Guichardet.

$\mathcal{H}$ — the category of Harish-Chandra bimodules for $\mathfrak{g}$

Note: $\mathcal{H} \in \mathfrak{g} \oplus \mathfrak{g}$-mod

BG-equivalences. $O_0 \cong _0^\infty \mathcal{H}_0^1$ and $\tilde{O}_0 \cong _0^\infty \mathcal{H}_0^\infty$

Corollary. $\chi^\infty \mathcal{H}_\chi^1$, $\chi^\infty \mathcal{H}_\chi^\infty$ and $\chi^\infty \mathcal{H}_\chi^\infty$ are Guichardet.

Observation. $\chi^\infty \mathcal{H}_\chi^\infty$ is not extension full in $\mathfrak{g} \oplus \mathfrak{g}$-mod.
Alexandru conjecture for $\tilde{O}$ and for $\mathcal{H}$

**Theorem 7. (Coulembier-M.)** $\tilde{O}_0$ is Guichardet.

$\mathcal{H}$ — the category of Harish-Chandra bimodules for $\mathfrak{g}$

**Note:** $\mathcal{H} \in \mathfrak{g} \oplus \mathfrak{g}$-mod

BG-equivalences. $O_0 \cong _0^\infty \mathcal{H}^1_0$ and $\tilde{O}_0 \cong _0^\infty \mathcal{H}_0^\infty$

**Corollary.** $^\chi \mathcal{H}^1_\chi$, $^1 \mathcal{H}_\chi^\infty$ and $^\infty_0 \mathcal{H}_0^\infty$ are Guichardet.

**Observation.** $^\infty_0 \mathcal{H}_0^\infty$ is not extension full in $\mathfrak{g} \oplus \mathfrak{g}$-mod.
Theorem 7. (Coulembier-M.) \( \widetilde{O}_0 \) is Guichardet.

\( H \) — the category of Harish-Chandra bimodules for \( g \)

Note: \( H \in g \oplus g\)-mod

BG-equivalences. \( O_0 \cong \infty H^1_0 \) and \( \widetilde{O}_0 \cong \infty H^\infty_0 \)

Corollary. \( \infty H^1, \chi H^\infty \chi \) and \( \infty H^\infty_0 \) are Guichardet.

Observation. \( \infty H^\infty_0 \) is not extension full in \( g \oplus g\)-mod.
Theorem 7. (Coulembier-M.) $\tilde{\mathcal{O}}_0$ is Guichardet.

$\mathcal{H}$ — the category of Harish-Chandra bimodules for $\mathfrak{g}$

Note: $\mathcal{H} \in \mathfrak{g} \oplus \mathfrak{g}$-mod

BG-equivalences. $\mathcal{O}_0 \cong \mathcal{H}^1_0$ and $\tilde{\mathcal{O}}_0 \cong \mathcal{H}_0^\infty$

Corollary. $\mathcal{H}_\chi^1$, $\mathcal{H}_\chi^\infty$ and $\mathcal{H}_0^\infty$ are Guichardet.

Observation. $\mathcal{H}_0^\infty$ is not extension full in $\mathfrak{g} \oplus \mathfrak{g}$-mod.
Alexandru conjecture for \( \tilde{\mathcal{O}} \) and for \( \mathcal{H} \)

**Theorem 7. (Coulembier-M.)** \( \tilde{\mathcal{O}}_0 \) is Guichardet.

\( \mathcal{H} \) — the category of Harish-Chandra bimodules for \( \mathfrak{g} \)

**Note:** \( \mathcal{H} \in \mathfrak{g} \oplus \mathfrak{g}\)-mod

**BG-equivalences.** \( \mathcal{O}_0 \cong \mathcal{H}^1_0 \) and \( \tilde{\mathcal{O}}_0 \cong \mathcal{H}^\infty_0 \)

**Corollary.** \( \mathcal{H}^1_\chi, \mathcal{H}^\infty_\chi \) and \( \mathcal{H}^\infty_0 \) are Guichardet.

**Observation.** \( \mathcal{H}^\infty_0 \) is not extension full in \( \mathfrak{g} \oplus \mathfrak{g}\)-mod.
Theorem 7. (Coulembier-M.) $\tilde{O}_0$ is Guichardet.

$\mathcal{H}$ — the category of Harish-Chandra bimodules for $\mathfrak{g}$

**Note:** $\mathcal{H} \in \mathfrak{g} \oplus \mathfrak{g}\text{-mod}$

**BG-equivalences.** $\mathcal{O}_0 \cong \mathcal{H}_0^1$ and $\tilde{O}_0 \cong \mathcal{H}_0^\infty$

**Corollary.** $\chi \mathcal{H}_1^\chi, \chi \mathcal{H}_\infty^\chi$ and $\mathcal{H}_0^\infty$ are Guichardet.

**Observation.** $\mathcal{H}_0^\infty$ is not extension full in $\mathfrak{g} \oplus \mathfrak{g}\text{-mod}$.
Observation. Singular blocks of $\mathcal{O}$ for $\mathfrak{sl}_3$ are not always Guichardet

Given by:

$$
\begin{array}{c}
1 \xleftarrow{\delta} 2 \xrightarrow{\beta} 3 \\
\end{array}
$$

with relations $\beta \gamma = 0$ and $\gamma \beta = \alpha \delta$.

Easy: $p.\dim L_1 = 1$ and $p.\dim L_2 = p.\dim L_3 = 2$

Note: $\text{Serre}(L_3)$ is an initial segment (and is semi-simple).

Note: $\text{Ext}^2_{\mathcal{O}}(L_3, L_3) \neq 0$
Observation. Singular blocks of $\mathcal{O}$ for $\mathfrak{sl}_3$ are not always Guichardet

Given by:

$1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\delta} 2 \xrightarrow{\gamma} 3$

with relations $\beta \gamma = 0$ and $\gamma \beta = \alpha \delta$.

Easy: $\text{p.dim } L_1 = 1$ and $\text{p.dim } L_2 = \text{p.dim } L_3 = 2$

Note: $\text{Serre}(L_3)$ is an initial segment (and is semi-simple).

Note: $\text{Ext}^2_{\mathcal{O}}(L_3, L_3) \neq 0$
Observation. Singular blocks of $\mathcal{O}$ for $\mathfrak{sl}_3$ are not always Guichardet

Given by:

\[
\begin{array}{c}
1 & \xleftarrow{\alpha} & 2 & \xleftarrow{\beta} & 3 \\
\delta & & \gamma \\
\end{array}
\]

with relations $\beta \gamma = 0$ and $\gamma \beta = \alpha \delta$.

Easy: $p\dim L_1 = 1$ and $p\dim L_2 = p\dim L_3 = 2$

Note: $\text{Serre}(L_3)$ is an initial segment (and is semi-simple).

Note: $\text{Ext}^2_{\mathcal{O}}(L_3, L_3) \neq 0$
Observation. Singular blocks of $\mathcal{O}$ for $\mathfrak{sl}_3$ are not always Guichardet
Given by:

\[
\begin{array}{c}
1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3 \\
\xleftarrow{\delta} \quad \xrightarrow{\gamma}
\end{array}
\]

with relations $\beta \gamma = 0$ and $\gamma \beta = \alpha \delta$.

Easy: $p \dim L_1 = 1$ and $p \dim L_2 = p \dim L_3 = 2$

Note: $\text{Serre}(L_3)$ is an initial segment (and is semi-simple).

Note: $\text{Ext}^2_{\mathcal{O}}(L_3, L_3) \neq 0$
Observation. Singular blocks of $\mathcal{O}$ for $\mathfrak{sl}_3$ are not always Guichardet

Given by:

```
1 ←[\alpha]→ 2 ←[\beta]→ 3
```

with relations $\beta \gamma = 0$ and $\gamma \beta = \alpha \delta$.

Easy: $p.\dim L_1 = 1$ and $p.\dim L_2 = p.\dim L_3 = 2$

Note: $\text{Serre}(L_3)$ is an initial segment (and is semi-simple).

Note: $\text{Ext}_2^\mathcal{O}(L_3, L_3) \neq 0$
Observation. Singular blocks of \( O \) for \( \mathfrak{sl}_3 \) are not always Guichardet

Given by:

\[
\begin{array}{ccc}
1 & \overset{\alpha}{\longrightarrow} & 2 \\
\delta & \overset{\beta}{\longrightarrow} & 3 \\
\end{array}
\]

with relations \( \beta \gamma = 0 \) and \( \gamma \beta = \alpha \delta \).

Easy: \( \text{p.dim } L_1 = 1 \) and \( \text{p.dim } L_2 = \text{p.dim } L_3 = 2 \)

Note: \( \text{Serre} \langle L_3 \rangle \) is an initial segment (and is semi-simple).

Note: \( \text{Ext}^2_\mathcal{O}(L_3, L_3) \neq 0 \)
**Observation.** Singular blocks of $\mathcal{O}$ for $\mathfrak{sl}_3$ are not always Guichardet

Given by:

\[
\begin{array}{ccc}
1 & \overset{\alpha}{\longrightarrow} & 2 \\
\delta & \Longleftarrow & \beta \\
2 & \overset{\gamma}{\Longleftarrow} & 3 \\
\end{array}
\]

with relations $\beta\gamma = 0$ and $\gamma\beta = \alpha\delta$.

**Easy:** $\text{p.dim } L_1 = 1$ and $\text{p.dim } L_2 = \text{p.dim } L_3 = 2$

**Note:** $\text{Serre}\langle L_3 \rangle$ is an initial segment (and is semi-simple).

**Note:** $\text{Ext}^2_{\mathcal{O}}(L_3, L_3) \neq 0$
Saturated Alexandru conjectures


\[ \mathcal{A} — \text{abelian length category} \]

\[ \preceq — \text{smallest partial pre-order on } \text{Irr}(\mathcal{A}) \text{ such that} \]

\[ \begin{align*}
&\quad L_i < L_j \text{ implies } L_i \preceq L_j; \\
&\quad \text{p.dim } L = \text{p.dim } L' \text{ and } \text{Ext}^1(L, L') \neq 0 \text{ or } \text{Ext}^1(L', L) \neq 0 \text{ implies } L \preceq L'.
\end{align*} \]

Definition. A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is a saturated initial segment if \( L_j \in \mathcal{B} \) and \( L_i \preceq L_j \) implies \( L_i \in \mathcal{B} \).

Definition. \( \mathcal{A} \) is saturated Guichardet if any saturated initial segment is extension full in \( \mathcal{A} \).
Saturated Alexandru conjectures


$\mathcal{A}$ — abelian length category

$\leq$ — smallest partial pre-order on $\text{Irr}(\mathcal{A})$ such that

- $L_i < L_j$ implies $L_i \leq L_j$;
- $\text{p.dim } L = \text{p.dim } L'$ and $\text{Ext}^1(L, L') \neq 0$ or $\text{Ext}^1(L', L) \neq 0$ implies $L \leq L'$.

Definition. A Serre subcategory $\mathcal{B} \subset \mathcal{A}$ is a saturated initial segment if $L_j \in \mathcal{B}$ and $L_i \leq L_j$ implies $L_i \in \mathcal{B}$

Definition. $\mathcal{A}$ is saturated Guichardet if any saturated initial segment is extension full in $\mathcal{A}$
Saturated Alexandru conjectures


\( \mathcal{A} \) — abelian length category

\( \preceq \) — smallest partial pre-order on \( \text{Irr}(\mathcal{A}) \) such that

1. \( L_i < L_j \) implies \( L_i \preceq L_j \);
2. \( \text{p.dim } L = \text{p.dim } L' \) and \( \text{Ext}^1(L, L') \neq 0 \) or \( \text{Ext}^1(L', L) \neq 0 \) implies \( L \preceq L' \).

Definition. A Serre subcategory \( B \subset \mathcal{A} \) is a saturated initial segment if \( L_j \in B \) and \( L_i \preceq L_j \) implies \( L_i \in B \).

Definition. \( \mathcal{A} \) is saturated Guichardet if any saturated initial segment is extension full in \( \mathcal{A} \).
Saturated Alexandru conjectures


\[ \mathcal{A} \text{ — abelian length category} \]

\[ \preceq \text{ — smallest partial pre-order on } \text{Irr}(\mathcal{A}) \text{ such that} \]

- \( L_i < L_j \) implies \( L_i \preceq L_j \);
- \( \text{p.dim } L = \text{p.dim } L' \) and \( \text{Ext}^1(L, L') \neq 0 \) or \( \text{Ext}^1(L', L) \neq 0 \) implies \( L \preceq L' \).

Definition. A Serre subcategory \( B \subset \mathcal{A} \) is a saturated initial segment if \( L_j \in B \) and \( L_i \preceq L_j \) implies \( L_i \in B \).

Definition. \( \mathcal{A} \) is saturated Guichardet if any saturated initial segment is extension full in \( \mathcal{A} \).
Saturated Alexandru conjectures


\[ \mathcal{A} \] — abelian length category

\[ \preceq \] — smallest partial pre-order on \( \text{Irr}(\mathcal{A}) \) such that

- \( L_i < L_j \) implies \( L_i \preceq L_j \);
- \( \text{p.dim } L = \text{p.dim } L' \) and \( \text{Ext}^1(L, L') \neq 0 \) or \( \text{Ext}^1(L', L) \neq 0 \) implies \( L \preceq L' \).

Definition. A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is a saturated initial segment if \( L_j \in \mathcal{B} \) and \( L_i \preceq L_j \) implies \( L_i \in \mathcal{B} \).

Definition. \( \mathcal{A} \) is saturated Guichardet if any saturated initial segment is extension full in \( \mathcal{A} \).
Saturated Alexandru conjectures


\( \mathcal{A} \) — abelian length category

\( \leq \) — smallest partial pre-order on \( \text{Irr}(\mathcal{A}) \) such that

- \( L_i < L_j \) implies \( L_i \leq L_j \);
- \( \text{p.dim } L = \text{p.dim } L' \) and \( \text{Ext}^1(L, L') \neq 0 \) or \( \text{Ext}^1(L', L) \neq 0 \) implies \( L \leq L' \).

Definition. A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is a saturated initial segment if \( L_j \in \mathcal{B} \) and \( L_i \leq L_j \) implies \( L_i \in \mathcal{B} \).

Definition. \( \mathcal{A} \) is saturated Guichardet if any saturated initial segment is extension full in \( \mathcal{A} \).
Saturated Alexandru conjectures


\[ A \] — abelian length category

\[ \trianglelefteq \] — smallest partial pre-order on \( \text{Irr}(A) \) such that

- \( L_i < L_j \) implies \( L_i \trianglelefteq L_j \);
- \( p.\text{dim} L = p.\text{dim} L' \) and \( \text{Ext}^1(L, L') \neq 0 \) or \( \text{Ext}^1(L', L) \neq 0 \) implies \( L \trianglelefteq L' \).

Definition. A Serre subcategory \( B \subset A \) is a saturated initial segment if \( L_j \in B \) and \( L_i \trianglelefteq L_j \) implies \( L_i \in B \)

Definition. \( A \) is saturated Guichardet if any saturated initial segment is extension full in \( A \)
Saturated Alexandru conjectures


\[ A \quad \text{— abelian length category} \]

\[ \preceq \quad \text{— smallest partial pre-order on } \text{Irr}(A) \text{ such that} \]
\[ \begin{align*}
    & L_i < L_j \text{ implies } L_i \preceq L_j; \\
    & \text{p.dim } L = \text{p.dim } L' \text{ and } \text{Ext}^1(L, L') \not= 0 \text{ or } \text{Ext}^1(L', L) \not= 0 \text{ implies } L \preceq L'.
\end{align*} \]

Definition. A Serre subcategory \( B \subset A \) is a saturated initial segment if \( L_j \in B \) and \( L_i \preceq L_j \) implies \( L_i \in B \)

Definition. \( A \) is saturated Guichardet if any saturated initial segment is extension full in \( A \)
Saturated Alexandru conjectures


\( \mathcal{A} \) — abelian length category

\( \preceq \) — smallest partial pre-order on \( \text{Irr}(\mathcal{A}) \) such that

- \( L_i < L_j \) implies \( L_i \preceq L_j \);
- \( p.\text{dim } L = p.\text{dim } L' \) and \( \text{Ext}^1(L, L') \neq 0 \) or \( \text{Ext}^1(L', L) \neq 0 \) implies \( L \preceq L' \).

Definition. A Serre subcategory \( \mathcal{B} \subset \mathcal{A} \) is a saturated initial segment if \( L_j \in \mathcal{B} \) and \( L_i \preceq L_j \) implies \( L_i \in \mathcal{B} \)

Definition. \( \mathcal{A} \) is saturated Guichardet if any saturated initial segment is extension full in \( \mathcal{A} \)
Saturated Alexandru conjectures for $O$

Observation: $O_0$ is saturated Guichardet.

Reason: Saturated initial segments and initial segments coincide.

Observation: All blocks of $O_0$ for $sl_3$ are saturated Guichardet.

Reason: Saturated initial segments are coideals in the Bruhat order.

Unknown: Is any block of $O$ saturated Guichardet?
**Observation:** $O_0$ is saturated Guichardet.

**Reason:** Saturated initial segments and initial segments coincide.

**Observation:** All blocks of $O_0$ for $sl_3$ are saturated Guichardet.

**Reason:** Saturated initial segments are coideals in the Bruhat order.

**Unknown:** Is any block of $O$ saturated Guichardet?
**Observation:** $\mathcal{O}_0$ is saturated Guichardet.

**Reason:** Saturated initial segments and initial segments coincide.

**Observation:** All blocks of $\mathcal{O}_0$ for $\mathfrak{sl}_3$ are saturated Guichardet.

**Reason:** Saturated initial segments are coideals in the Bruhat order.

**Unknown:** Is any block of $\mathcal{O}$ saturated Guichardet?
Observation: $\mathcal{O}_0$ is saturated Guichardet.

Reason: Saturated initial segments and initial segments coincide.

Observation: All blocks of $\mathcal{O}_0$ for $\mathfrak{sl}_3$ are saturated Guichardet.

Reason: Saturated initial segments are coideals in the Bruhat order.

Unknown: Is any block of $\mathcal{O}$ saturated Guichardet?
Observation: $\mathcal{O}_0$ is saturated Guichardet.

Reason: Saturated initial segments and initial segments coincide.

Observation: All blocks of $\mathcal{O}_0$ for $\mathfrak{sl}_3$ are saturated Guichardet.

Reason: Saturated initial segments are coideals in the Bruhat order.

Unknown: Is any block of $\mathcal{O}$ saturated Guichardet?
Observation: $O_0$ is saturated Guichardet.

Reason: Saturated initial segments and initial segments coincide.

Observation: All blocks of $O_0$ for $\mathfrak{sl}_3$ are saturated Guichardet.

Reason: Saturated initial segments are coideals in the Bruhat order.

Unknown: Is any block of $O$ saturated Guichardet?
Observation: $\mathcal{O}_0$ is saturated Guichardet.

Reason: Saturated initial segments and initial segments coincide.

Observation: All blocks of $\mathcal{O}_0$ for $\mathfrak{sl}_3$ are saturated Guichardet.

Reason: Saturated initial segments are coideals in the Bruhat order.

Unknown: Is any block of $\mathcal{O}$ saturated Guichardet?
Some speculations

**Why difficult:** We do not know projective dimensions of simples in $\mathcal{O}$!

**Our guess:** Some blocks of $\mathcal{O}$ are not saturated Guichardet.

**Another difficulty:** Difficult to estimate extensions in the saturated Serre subcategories.

**$\mathfrak{sl}_4$-computations:** There is a singular block for which saturated initial segments are not always given by coideals in the Bruhat order.
Some speculations

Why difficult: We do not know projective dimensions of simples in $\mathcal{O}$!

Our guess: Some blocks of $\mathcal{O}$ are not saturated Guichardet.

Another difficulty: Difficult to estimate extensions in the saturated Serre subcategories.

$\text{sl}_4$-computations: There is a singular block for which saturated initial segments are not always given by coideals in the Bruhat order.
Some speculations

**Why difficult:** We do not know projective dimensions of simples in $\mathcal{O}$!

**Our guess:** Some blocks of $\mathcal{O}$ are not saturated Guichardet.

**Another difficulty:** Difficult to estimate extensions in the saturated Serre subcategories.

**$\mathfrak{sl}_4$-computations:** There is a singular block for which saturated initial segments are not always given by coideals in the Bruhat order.
Some speculations

Why difficult: We do not know projective dimensions of simples in $O$!

Our guess: Some blocks of $O$ are not saturated Guichardet.

Another difficulty: Difficult to estimate extensions in the saturated Serre subcategories.

$sl_4$-computations: There is a singular block for which saturated initial segments are not always given by coideals in the Bruhat order.
Some speculations

Why difficult: We do not know projective dimensions of simples in $\mathcal{O}$!

Our guess: Some blocks of $\mathcal{O}$ are not saturated Guichardet.

Another difficulty: Difficult to estimate extensions in the saturated Serre subcategories.

$\mathfrak{sl}_4$-computations: There is a singular block for which saturated initial segments are not always given by coideals in the Bruhat order.
Some speculations

**Why difficult:** We do not know projective dimensions of simples in $\mathcal{O}$!

**Our guess:** Some blocks of $\mathcal{O}$ are not saturated Guichardet.

**Another difficulty:** Difficult to estimate extensions in the saturated Serre subcategories.

**$\mathfrak{sl}_4$-computations:** There is a singular block for which saturated initial segments are not always given by coideals in the Bruhat order.
THANK YOU!!!