Simple supermodules for classical Lie superalgebras

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(Uppsala University)

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"Full" answer: Only for \( \mathfrak{sl}_2 \), R. Block 1979, — reduces to description of equivalence classes of irreducible elements in a non-commutative Euclidean ring

Some partial answers:

- Finite dimensional modules: E. Cartan 1913
- Whittaker modules: B. Kostant 1978
- Weight modules with fin.-dim. weight spaces: O. Mathieu 2000

Some other classes of simple modules:

- Simple modules for exotic Whittaker pairs: J. Nilsson 2013
Classification of simple modules for semi-simple Lie algebras

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Classical Lie superalgebras

\[ g = g_0 \oplus g_1 \]

\( g_0 \) — finite dimensional reductive

\( g_1 \) — finite dimensional and semi-simple over \( g_0 \)

Some examples:

- General linear Lie superalgebra \( \mathfrak{gl}(m|n) \)
- Queer Lie superalgebra \( \mathfrak{q}(n) \)
- Generalized Takiff Lie superalgebra \( \mathfrak{g}_a, \mathcal{V} \) where \( g_0 = a \),
  \( g_1 = \mathcal{V} \in a\text{-mod} \) and \([\mathcal{V}, \mathcal{V}] = 0\).

Main problem: Classification of simple \( g \)-supermodules

Reduction: Modulo classification of simple \( g_0 \)-modules
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Some examples:

▶ General linear Lie superalgebra gl(m|n)
▶ Queer Lie superalgebra q(n)
▶ Generalized Takiff Lie superalgebra g_\alpha, V where g_0 = \alpha,
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Classification of simple supermodules for classical Lie superalgebras

“Full” answer:

- $\mathfrak{gl}(1, 1)$ and $\mathfrak{q}(1)$ — exercise
- $\mathfrak{osp}(1, 2)$: V. Bavula, F. van Oystaeyen 2000
- $\mathfrak{p}(2)$: V. Serganova 2002
- $\mathfrak{sl}(1, 2)$: V. Serganova 2003
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Special cases:

- Typical generic modules for basic: I. Penkov 1994
- Strongly typical modules for basic: M. Gorelik 2002
- Weight modules with fin.-dim. weight spaces for type I: D. Grantcharov 2003
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Further reduction

$L$ — simple $g$-supermodule

$\text{Ann}_{U(g)}(L)$ — the annihilator of $L$ in $U(g)$

$\text{Ann}_{U(g)}(L)$ is a primitive ideal of $U(g)$

**Theorem.** (I. Musson 1992) There is a simple highest weight $g$-supermodule $L(\lambda)$ such that $\text{Ann}_{U(g)}(L) = \text{Ann}_{U(g)}(L(\lambda))$.

$L(\lambda)$ is of finite length over $U(g_0)$

Take any $\mu$ such that $L^{g_0}(\mu)$ is a simple $g_0$-submodule of $L(\lambda)$

**Note:** $\mu$ is not uniquely defined
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\( L(\lambda) \) is of finite length over \( U(g_{\bar{0}}) \)

Take any \( \mu \) such that \( L^{g_{\bar{0}}}(\mu) \) is a simple \( g_{\bar{0}} \)-submodule of \( L(\lambda) \)

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Further reduction

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$\text{Ann}_{U(\mathfrak{g})}(L)$ — the annihilator of $L$ in $U(\mathfrak{g})$

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\( L(\lambda) \) is of finite length over \( U(g^-_0) \)

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Harish-Chandra bimodules

$X$ — $g_0$-module

$Y$ — $g$-supermodule

$L(X, Y)$ — the set of locally $\text{ad}(g_0)$-finite linear maps from $X$ to $Y$

$L(X, Y)$ is a $U(g) - U(g_0)$-bimodule (a Harish-Chandra bimodule)

$L(X, Y) \otimes_{U(g_0)} - : U(g_0)\text{-mod} \to U(g)\text{-smod}$
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Harish-Chandra bimodules

\[ X \rightarrow g_0\text{-module} \]

\[ Y \rightarrow g\text{-supermodule} \]

\[ \mathcal{L}(X, Y) \rightarrow \text{the set of locally } \text{ad}(g_0)\text{-finite linear maps from } X \text{ to } Y \]

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\[ X \rightarrow g_0\text{-module} \]

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\[ \mathcal{L}(X, Y) \otimes_{U(g_0)} : U(g_0)\text{-mod} \rightarrow U(g)\text{-smod} \]
The main conjecture

$L$ — simple $g$-supermodule

$\mathcal{I} := \text{Ann}_{U(g)}(L)$

$L(\lambda)$ — a simple highest weight module with $\mathcal{I} = \text{Ann}_{U(g)}(L(\lambda))$

$L^{g_0}(\mu)$ — a simple $U(g_0)$-submodule of $L(\lambda)$

$J := \text{Ann}_{U(g_0)}(L^{g_0}(\mu))$

$\mathcal{L} := \mathcal{L}(L^{g_0}(\mu), L(\lambda))$

Main conjecture. Tensoring with $\mathcal{L}$ induces a bijection between isomorphism classes of simple $U(g_0)$-modules with annihilator $J$ and isomorphism classes of simple $U(g)$-supermodules with annihilator $\mathcal{I}$.
The main conjecture

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$L(\lambda)$ — a simple highest weight module with $I = \text{Ann}_{U(g)}(L(\lambda))$

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$J := \text{Ann}_{U(g_{\bar{0}})}(L^{g_{\bar{0}}}(\mu))$

$L := L(L^{g_{\bar{0}}}(\mu), L(\lambda))$

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$J := \text{Ann}_{U(g_{\hat{o}})}(L^{\hat{g}\hat{o}}(\mu))$

$L := L(L^{\hat{g}\hat{o}}(\mu), L(\lambda))$

Main conjecture. Tensoring with $L$ induces a bijection between isomorphism classes of simple $U(g_{\hat{o}})$-modules with annihilator $J$ and isomorphism classes of simple $U(g)$-supermodules with annihilator $I$. 
The main conjecture

$L$ — simple $g$-supermodule

$\mathcal{I} := \text{Ann}_{U(g)}(L)$

$L(\lambda)$ — a simple highest weight module with $\mathcal{I} = \text{Ann}_{U(g)}(L(\lambda))$

$L^{g_{\bar{0}}}(\mu)$ — a simple $U(g_{\bar{0}})$-submodule of $L(\lambda)$

$J := \text{Ann}_{U(g_{\bar{0}})}(L^{g_{\bar{0}}}(\mu))$

$\mathcal{L} := \mathcal{L}(L^{g_{\bar{0}}}(\mu), L(\lambda))$

Main conjecture. Tensoring with $\mathcal{L}$ induces a bijection between isomorphism classes of simple $U(g_{\bar{0}})$-modules with annihilator $J$ and isomorphism classes of simple $U(g)$-supermodules with annihilator $\mathcal{I}$. 
The main conjecture

$L$ — simple $g$-supermodule

$I := \text{Ann}_{U(g)}(L)$

$L(\lambda)$ — a simple highest weight module with $I = \text{Ann}_{U(g)}(L(\lambda))$

$L^{g\bar{\sigma}}(\mu)$ — a simple $U(g_{\bar{\sigma}})$-submodule of $L(\lambda)$

$J := \text{Ann}_{U(g_{\bar{\sigma}})}(L^{g\bar{\sigma}}(\mu))$

$L := L^{g\bar{\sigma}}(\mu), L(\lambda))$

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\[
\mathcal{I} := \text{Ann}_{U(g)}(L)
\]

\[ L(\lambda) \] — a simple highest weight module with \( \mathcal{I} = \text{Ann}_{U(g)}(L(\lambda)) \)

\[ L^{g_\bar{0}}(\mu) \] — a simple \( U(g_\bar{0}) \)-submodule of \( L(\lambda) \)

\[
\mathcal{J} := \text{Ann}_{U(g_\bar{0})}(L^{g_\bar{0}}(\mu))
\]

\[
\mathcal{L} := \mathcal{L}(L^{g_\bar{0}}(\mu), L(\lambda))
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**Main conjecture.** Tensoring with \( \mathcal{L} \) induces a bijection between isomorphism classes of simple \( U(g_\bar{0}) \)-modules with annihilator \( \mathcal{J} \) and isomorphism classes of simple \( U(g) \)-supermodules with annihilator \( \mathcal{I} \).
The main conjecture

$L$ — simple $\mathfrak{g}$-supermodule

$\mathcal{I} := \text{Ann}_{U(\mathfrak{g})}(L)$

$L(\lambda)$ — a simple highest weight module with $\mathcal{I} = \text{Ann}_{U(\mathfrak{g})}(L(\lambda))$

$L^{\mathfrak{g}\bar{0}}(\mu)$ — a simple $U(\mathfrak{g}_{\bar{0}})$-submodule of $L(\lambda)$

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**Main conjecture.** Tensoring with $L$ induces a bijection between isomorphism classes of simple $U(\mathfrak{g}_{\bar{0}})$-modules with annihilator $J$ and isomorphism classes of simple $U(\mathfrak{g})$-supermodules with annihilator $I$. 
The main conjecture

\[ L \] — simple \( g \)-supermodule

\[ I := \text{Ann}_{U(\mathfrak{g})}(L) \]

\[ L(\lambda) \] — a simple highest weight module with \( I = \text{Ann}_{U(\mathfrak{g})}(L(\lambda)) \)

\[ L^g(\mu) \] — a simple \( U(\mathfrak{g}_0) \)-submodule of \( L(\lambda) \)

\[ J := \text{Ann}_{U(\mathfrak{g}_0)}(L^g(\mu)) \]

\[ \mathcal{L} := \mathcal{L}(L^g(\mu), L(\lambda)) \]

**Main conjecture.** Tensoring with \( \mathcal{L} \) induces a bijection between isomorphism classes of simple \( U(\mathfrak{g}_0) \)-modules with annihilator \( J \) and isomorphism classes of simple \( U(\mathfrak{g}) \)-supermodules with annihilator \( I \).
The $q(2)$-example

**Theorem.** (V. M. 2010) The main conjecture is true for $q(2)$.

Root system: $\{ \pm \alpha \}$

Alternatives: $\mu \in \{ \lambda, \lambda - \alpha \}$ (depending on regularity, typicality etc.)

**Bonus:** Describes the rough structure of any simple $U(q(2))$-supermodule as a $U(gl(2))$-module

**Very special feature:** Every simple $U(q(2))$-supermodule is of finite length as a $U(gl(2))$-module

**Rough structure:** (O. Khomenko, V. M. 2004) Multiplicities of simple subquotients with “minimal possible” annihilators occurring in the module
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Rough structure conjecture

\[ g \rightarrow \text{classical Lie superalgebra} \]

\[ L \rightarrow \text{simple } g\text{-supermodule} \]

\[ U(g) \text{ is finite over } U(g_0) \]

\[ U(g_0) \text{ is noetherian} \]

\[ \text{Res}^g_{g_0}(L) \text{ is noetherian} \]

\[ \text{Res}^g_{g_0}(L) \text{ does not have to be artinian (T. Stafford. 1985)} \]

**Rough structure conjecture.** The rough structures of \( L \) and \( L(\lambda) \) “coincide” in the sense that under the bijection given by the main conjecture the multiplicities are preserved.

**Note:** Absolutely unclear how to control the “fine” structure.
Rough structure conjecture

\( \mathfrak{g} \) — classical Lie superalgebra

\( L \) — simple \( \mathfrak{g} \)-supersmodule

\( U(\mathfrak{g}) \) is finite over \( U(\mathfrak{g}_0) \)

\( U(\mathfrak{g}_0) \) is noetherian

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Rough structure conjecture

$\mathfrak{g}$ — classical Lie superalgebra

$L$ — simple $\mathfrak{g}$-supermodule

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$U(\mathfrak{g}_0)$ is noetherian

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**Rough structure conjecture.** The rough structures of $L$ and $L(\lambda)$ “coincide” in the sense that under the bijection given by the main conjecture the multiplicities are preserved.

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Lemma. Let $L$ be a simple $g$-supermodule. Then there exists a simple $g_0$-module $N$ such that $L \subset \text{Ind}_{g_0}^g(N)$ or $L \subset \prod \text{Ind}_{g_0}^g(N)$.

Proof. $U(g)$ is finite over $U(g_0)$.

$U(g_0)$ is noetherian, $\text{Res}_{g_0}^g(L)$ is noetherian

Zorn’s lemma implies that $\text{Res}_{g_0}^g(L)$ has a simple quotient, say $N$.

$\text{Ind}_{g_0}^g \cong \prod \text{dim } g_1 \circ \text{Coind}_{g_0}^g$

Adjunction: $\text{Hom}_g(L, \text{Coind}_{g_0}^g(N)) = \text{Hom}_{g_0}(\text{Res}_{g_0}^g(L), N) \neq 0$.

Q.E.D.
**Lemma.** Let $L$ be a simple $g$-supermodule. Then there exists a simple $g_0$-module $N$ such that $L \subset \text{Ind}^g_{g_0}(N)$ or $L \subset \prod \text{Ind}^g_{g_0}(N)$.

**Proof.** $U(g)$ is finite over $U(g_0)$.

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Lemma. Let $L$ be a simple $g$-supermodule. Then there exists a simple $g_\bar{0}$-module $N$ such that $L \subset \text{Ind}_{g_\bar{0}}^g(N)$ or $L \subset \prod \text{Ind}_{g_\bar{0}}^g(N)$.

Proof. $U(g)$ is finite over $U(g_\bar{0})$.

$U(g_\bar{0})$ is noetherian, $\text{Res}_{g_\bar{0}}^g(L)$ is noetherian.

Zorn’s lemma implies that $\text{Res}_{g_\bar{0}}^g(L)$ has a simple quotient, say $N$.

$\text{Ind}_{g_\bar{0}}^g \cong \prod_{\text{dim } g_\bar{1}} \circ \text{Coind}_{g_\bar{0}}^g$

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Adjunction: $\text{Hom}_g(L, \text{Coind}^g_{g_0}(N)) = \text{Hom}_{g_0}(\text{Res}^g_{g_0}(L), N) \neq 0$.

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**Lemma.** Let $L$ be a simple $\mathfrak{g}$-supermodule. Then there exists a simple $\mathfrak{g}_0$-module $N$ such that $L \subset \text{Ind}_{\mathfrak{g}_0}^\mathfrak{g} (N)$ or $L \subset \prod \text{Ind}_{\mathfrak{g}_0}^\mathfrak{g} (N)$.

**Proof.** $U(\mathfrak{g})$ is finite over $U(\mathfrak{g}_0)$.

$U(\mathfrak{g}_0)$ is noetherian, $\text{Res}_{\mathfrak{g}_0}^\mathfrak{g} (L)$ is noetherian.

Zorn’s lemma implies that $\text{Res}_{\mathfrak{g}_0}^\mathfrak{g} (L)$ has a simple quotient, say $N$.

\[
\text{Ind}_{\mathfrak{g}_0}^\mathfrak{g} \simeq \prod \dim_{\mathfrak{g}_1} \circ \text{Coind}_{\mathfrak{g}_0}^\mathfrak{g}
\]

Adjunction: $\text{Hom}_{\mathfrak{g}} (L, \text{Coind}_{\mathfrak{g}_0}^\mathfrak{g} (N)) = \text{Hom}_{\mathfrak{g}_0} (\text{Res}_{\mathfrak{g}_0}^\mathfrak{g} (L), N) \neq 0$.

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Lemma. Let $L$ be a simple $g$-supermodule. Then there exists a simple $g_0$-module $N$ such that $L \subset \text{Ind}_{g_0}^g(N)$ or $L \subset \prod \text{Ind}_{g_0}^g(N)$.

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Q.E.D.
Simple supermodules are quotients of induced modules

**Dual statement:** Each simple supermodule is a quotient of an induced module.

**Question:** Is this true?

**Idea:** Same proof as above works?

**Need:** If $L$ is a simple $g$-supermodule, then $\text{Res}^g_{g_0}(L)$ has a simple submodule.

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Main result

**Theorem:** Let \( \alpha \) be a finite dimensional reductive Lie algebra, \( V \) a simple \( \alpha \)-module and \( E \) a simple finite dimensional \( \alpha \)-module. Then \( E \otimes V \) has a well-defined **socle**, that is there exists a unique submodule \( N \) of \( E \otimes V \) which has the following properties:

1. \( N \) has finite length;
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3. any non-zero submodule of \( E \otimes V \) intersects \( N \) in a non-zero way.

**Corollary 1:** Every simple \( g \) supermodule has a well-defined socle (as a \( g_{\bar{0}} \)-module).

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\( \alpha \) — be a finite dimensional reductive Lie algebra

\( \mathcal{M} \) — the full subcategory in \( \alpha\)-Mod consisting of modules on which the action of \( Z(\alpha) \) is locally finite

\( E \otimes - : \mathcal{M} \to \mathcal{M} \) — a projective functor (in the sense of I. Bernstein and S. Gelfand 1980)

Indecomposable projective functors are classified (I. Bernstein and S. Gelfand 1980)

The tensor category of projective functors is generated by:

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induction reduces the claim to one of the three types of projective functors described above

for equivalences of categories the claim is obvious

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**Main idea:** Exploit the 2-categorical structure on the tensor category (2-category) of projective functors

the endomorphism algebra of the translation \( \theta \) out of a wall is known (I. Bernstein and S. Gelfand 1980)

this endomorphism algebra is commutative, has simple socle, and \( Z(\alpha) \) surjects onto it (this is the algebra of certain invariants in a certain coinvariant algebra), it is related to the endomorphism algebra of a certain projective in the BGG category \( \mathcal{O} \)

by noetherianity, we have at least one simple quotient of \( \theta \, V \)

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\( \alpha \) — reductive finite dimensional Lie algebra of type \( A \)

\( V \) — simple \( \alpha \)-module

\( J := \text{Ann}_{U(\alpha)}(V) \)

\( \lambda \) — a weight such that \( J = \text{Ann}_{U(\alpha)}(L(\lambda)) \)

\( \lambda' \) — the most singular weight with comparable annihilator appearing in \( \mathcal{JH}(E \otimes L(\lambda)) \) where \( E \) is finite dimensional

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\( \lambda' \) — the most singular weight with comparable annihilator appearing in \( JH(E \otimes L(\lambda)) \) where \( E \) is finite dimensional

\( J' := \text{Ann}_{U(a)}(L(\lambda')) \)

\( V' \) — the corresponding simple (sub)quotient of \( E \otimes V \)

**Note:** \( V \) is a quotient of \( E^* \otimes V' \)
Rough structure of supermodules: setup

$\alpha$ — reductive finite dimensional Lie algebra of type $A$

$V$ — simple $\alpha$-module

$J := \operatorname{Ann}_{U(\alpha)}(V)$

$\lambda$ — a weight such that $J = \operatorname{Ann}_{U(\alpha)}(L(\lambda))$

$\lambda'$ — the most singular weight with comparable annihilator appearing in $\mathcal{JH}(E \otimes L(\lambda))$ where $E$ is finite dimensional

$J' := \operatorname{Ann}_{U(\alpha)}(L(\lambda'))$

$V'$ — the corresponding simple (sub)quotient of $E \otimes V$

**Note:** $V$ is a quotient of $E^* \otimes V'$
Coker\((E \otimes V')\) — full subcategory of \(a\)-mod consisting of modules with presentation \(X_1 \to X_0 \to M \to 0\) with \(X_1, X_0 \in \text{add}(E \otimes V')\) for some finite dimensional \(E\)

**Proposition.** \(V'\) is projective in \(\text{Coker}(E \otimes V')\) (compare with R. Irving and B. Shelton 1988)

**Theorem.** (V.M. and C. Stroppel 2008) \(\text{Coker}(E \otimes V')\) does not depend on \(V'\) (if \(J'\) is fixed), up to equivalence.

**Corollary.** The rough structure conjecture is true if \(g_0\) is of type \(A\).

**Consequently:** Enough to describe the rough structure for highest weight supermodules.
Rough structure of supermodules: description

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Consequently: Enough to describe the rough structure for highest weight supermodules.
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Coker$(E \otimes V')$ — full subcategory of $a$-mod consisting of modules with presentation $X_1 \to X_0 \to M \to 0$ with $X_1, X_0 \in \text{add}(E \otimes V')$ for some finite dimensional $E$

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**Consequently:** Enough to describe the rough structure for highest weight supermodules.
Rough structure of supermodules: the \( q(2) \) example

\( \alpha \) — the positive root

\( L(0) \) — trivial supermodule

\( L(\lambda)_0 \cong L(\lambda)_\overline{1} \) if \( \lambda \neq 0 \)

Atypical \( \lambda \neq 0 \): \( L(\lambda)_0 = L^g_0(\lambda) \)

Regular typical \( \lambda \neq 0 \): \( L(\lambda)_0 = L^g_0(\lambda) \oplus L^g_0(\lambda - \alpha) \)

Singular typical \( \lambda \neq 0 \): \( L(\lambda)_0 \) is indecomposable, \( L^g_0(\lambda - \alpha) \hookrightarrow L(\lambda)_0 \twoheadrightarrow L^g_0(\lambda - \alpha) \), this sequence has one-dimensional homology (i.e. the fine structure is different from the rough structure)

Note Taking e.g. a simple dense \( g \)-supermodule with the same annihilator as \( L(\lambda) \), the corresponding sequence will be exact, that is in this case the fine structure coincides with the rough structure.
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THANK YOU!!!