Structure of α-Stratified Modules for Finite-Dimensional Lie Algebras I

V. Futorny and V. Mazorchuk
Faculty of Mechanics and Mathematics
Kiev University, Kiev 252617, Ukraine

Abstract

For complex simple finite-dimensional Lie Algebras we study the structure of generalized Verma modules which are torsion free with respect to some subalgebra isomorphic to $s\ell(2)$. We obtain a criterion of irreducibility and establish necessary and sufficient conditions for the existence of a non-trivial homomorphism between such modules generalizing the BGG Theorem for Verma modules.

§1. Introduction

Let $\mathcal{G}$ be a complex simple finite dimensional Lie Algebra ($\mathcal{G} \neq G_2$) with a fixed Cartan subalgebra $\mathcal{H}$ and root system Δ and let W be the Weyl group of Δ. For $\beta \in \Delta$ we denote by $\mathcal{G}_\beta$ the corresponding root subspace. Let $\alpha \in \Delta$ and $\mathcal{G}^\alpha \simeq s\ell(2)$ be a subalgebra generated by $\mathcal{G}_\alpha$ and $\mathcal{G}_{-\alpha}$.

In the present paper we study α-stratified generalized Verma modules $M_\alpha(\lambda, p)$ on which $\mathcal{G}^\alpha$ acts torsion-free, i.e. $\mathcal{G}_{\pm\alpha} \setminus \{0\}$ act injectively. Such modules have no highest weight unlike the classical generalized Verma modules [GL].

Modules $M_\alpha(\lambda, p)$ were studied in [F1] and more extensively in [CF] and [FP]. In the latter two papers category $\mathcal{O}^\alpha$, a certain generalization of category $\mathcal{O}$ containing $M_\alpha(\lambda, p)$ as its objects, was introduced and analyzed. In particular, it was shown in [FP] that BGG duality with $M_\alpha(\lambda, p)$ as intermediate modules holds in the category $\mathcal{O}^\alpha$.

Recall that a well-known theorem of I.N. Bernstein, I.M. Gelfand and S.I. Gelfand on the inclusions of Verma modules states that if $M(\lambda)$ and $M(\mu)$ are Verma modules with higest weights $\lambda - \rho$ and $\mu - \rho$ respectively, then $M(\lambda) \subset M(\mu)$ if and only if there exists reflections s_{β_1}, $s_{\beta_2}, \ldots,$ s_{β_r} in the Weyl group such that

$$\mu = s_{\beta_r} s_{\beta_{r-1}} \ldots s_{\beta_1}(\lambda) \leq s_{\beta_{r-1}} \ldots s_{\beta_1}(\lambda) \leq \ldots \leq s_{\beta_1}(\lambda) \leq \lambda,$$

where $\leq$ is the standard order on $\mathcal{H}^*$. The main goal of our paper is to generalize this theorem in the case of α-stratified modules $M_\alpha(\lambda, p)$, i.e. to establish a criteria under which $M_\alpha(\mu, q)$ will be a submodule of $M_\alpha(\lambda, p)$, and to describe all irreducible subquotients of $M_\alpha(\lambda, p)$. Earlier such criteria was obtained for $\mathcal{G} = s\ell(3)$ [F2] and more generally for $\mathcal{G} = s\ell(n)$ [M1, M2]. Following the ideas of [M1, M2] we introduce a group $W_\alpha \subset \mathrm{Aut}(\mathcal{H}^* \times \mathbf{C})$ which is the Weyl group of a certain root system dual to Δ and thus is isomorphic to W. We show that W_α plays the same role for modules $M_\alpha(\lambda, p)$ as the Weyl group W plays for Verma modules. In particular modules $M_\alpha(\mu, q)$ and $M_\alpha(\lambda, p)$, belonging to one block, have the same central character if and only if $(\mu, q) \in W_\alpha(\lambda, p)$. We also obtain a criteria of irreducibility for α-stratified modules $M_\alpha(\lambda, p)$.

Instead of the group W_α one could talk about the classical group W with a new action on the space of parameters $\mathcal{H}^* \times \mathbf{C}$. But we decided to stick with a new notation to avoid on one hand any confusion due to the fact that $\mathcal{H}^*$ is not invariant under the action of W_α, and to underline, on the other hand, the dependence of the action on α.

The proof of the main result (Theorem 7.6) follows closely the original proof of Bernstein-Gelfand-Gelfand [BGG, Theorem 2,3]. We chose this "archaic" way of dealing with Verma-like modules since in our case it did not require the development of a necessary technique for modern approach (i.e. Jantzen filtration, translation functors etc.).

In subsequent papers we will discuss the analogs of BGG resolution and the character formula for modules $M_\alpha(\lambda, p)$.

Now we briefly describe the structure of the paper. In §2 we introduce our main notations and definitions while in §3 we recall the basic properties of the generalized Verma modules $M_\alpha(\lambda, p)$, $\alpha \in \Delta$, $\lambda \in \mathcal{H}^*$, $p \in \mathbf{C}$. In §4 we consider the modules $M_\alpha(\lambda, p)$ for algebra of rank two (except G_2) and obtain for them an analog of the BGG Theorem and criteria of irreducibility. The theory of α-stratified modules for rank two Lie algebras plays the same role for generalized Verma modules with no highest weight as $s\ell(2)$ theory plays for Verma modules. These results are crucial for the definition of the group W_α and for the proofs of Theorems 6.5 and 7.6. We introduce in §5 a group W_α acting on the space of parameters $\Omega = \mathcal{H}^* \times \mathbf{C}$ and prove that $W_\alpha \simeq W$ (Proposition 5.4). We also show that if two pairs (λ, p), $(\mu, q) \in \Omega$ belong to the same orbit of W_α then corresponding modules $M_\alpha(\lambda, p)$ and $M_\alpha(\mu, q)$ have equal central characters (Proposition 5.5). In §6 we prove the generalized Harish-Chandra Theorem (Theorem 6.3) establishing an isomorphism between the centre of the universal enveloping algebra of $\mathcal{G}$ and the W_α-invariants of some algebra. Using this result we show that modules $M_\alpha(\lambda, p)$ and $M_\alpha(\mu, q)$ from one block have the same central character if and only if $(\mu, q) \in W_\alpha(\lambda, p)$ (Corollary 6.4, Theorem 6.5). In §7 we prove our main result, Theorem 7.6, describing all submodules of type $M_\alpha(\mu, q)$ and all irreducible subquotients for α-stratified modules $M_\alpha(\lambda, p)$. The proof of this Theorem is analogous to the one of Bernstein-Gelfand-Gelfand for Verma modules [BGG, Theorem 2,3; D, Theorem 7.6.23].

Finally we obtain from Theorem 7.6 the criterion of irreducibility for α-stratified modules $M_\alpha(\lambda, p)$ (Theorem 7.7).

§2. Notations and Preliminary Results

Let $\mathbf{C}$ denote the complex numbers, $\mathbf{Z}$ denote all integers, $\mathbf{N}$ denote all positive integers and $\mathbf{Z}_+ = \mathbf{N} \cup \{0\}$.

We fix $\alpha \in \Delta$ for the remainder of the paper. Let π be a basis of Δ containing α, $\Delta^+ = \Delta^+(\pi)$ be the set of positive roots with respect to π and $\rho = \frac{1}{2} \sum_{\gamma \in \Delta^+} \gamma$. For λ, $\mu \in \mathcal{H}^*$ we will say that $\lambda \geq \mu$ if $\lambda - \mu = \sum_{\beta \in \pi} k_\beta \beta$, $k_\beta \in \mathbf{Z}_+$. Denote by $< \cdot, \cdot >$ the standard form on $\mathcal{H}^*$ and by $| \cdot |$ the corresponding norm in $\mathcal{H}^*_{\mathbf{R}}$. If $\beta \in \Delta^+$ then $s_\beta \in W$ will denote a corresponding reflection in $\mathcal{H}^*$: $s_\beta(\lambda) = \lambda - \dfrac{2 < \lambda, \beta >}{< \beta, \beta >} \beta$.

Fix a basis $\{H_\beta, \beta \in \pi\}$ of $\mathcal{H}$ and non-zero elements X_γ in each subspace $\mathcal{G}_\gamma$, $\gamma \in \Delta$ such that $\beta(H_\beta) = 2$ and $[X_\beta, X_{-\beta}] = H_\beta, \beta \in \pi$.

Denote $N_\pm = \sum_{\gamma\in\Delta^+} \mathcal{G}_{\pm\gamma}$, $N^\alpha_\pm = \sum_{\gamma\in\Delta^+\setminus\{\alpha\}} \mathcal{G}_{\pm\gamma}$, $\mathcal{H}^\alpha = \{h \in \mathcal{H} | \alpha(h) = 0\}$. Then we have $\mathcal{G} = N_- \oplus \mathcal{H} \oplus N_+ = \mathcal{G}^\alpha \oplus N^\alpha_- \oplus \mathcal{H}^\alpha \oplus N^\alpha_+$. Also let $\mathcal{H}_\alpha = \mathcal{G}^\alpha \cap \mathcal{H}$ and thus $\mathcal{G}^\alpha = \mathcal{G}_\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{G}_{-\alpha}$.

For a Lie Algebra $\mathcal{A}$ with a fixed Cartan subalgebra $\mathcal{B}$ we will denote by $U(\mathcal{A})$ the universal enveloping algebra of $\mathcal{A}$, by $U_0(\mathcal{A})$ the centralizer of $\mathcal{B}$ in $U(\mathcal{A})$ and by $Z(\mathcal{A})$ the centre of $U(\mathcal{A})$.

Let $N^\alpha_0 = U(\mathcal{G})N^\alpha_+ \cap U_0(\mathcal{G})$. Then N^α_0 is two–sided ideal and $U_0(\mathcal{G}) = N^\alpha_0 \oplus U_0(\mathcal{G}^\alpha) \otimes U(\mathcal{H}^\alpha)$. The projection $\phi_\alpha : U_0(\mathcal{G}) \to U_0(\mathcal{G}^\alpha) \otimes U(\mathcal{H}^\alpha)$ is called the Harish-Chandra α-homomorphism [DFO].

Let δ be an automorphism of $S(\mathcal{H})$ such that $\delta(f)(\lambda) = f(\lambda - \rho)$ for any $\lambda \in \mathcal{H}^*$ and for any polynomial function $f \in S(\mathcal{H})$. Also define an automorphism δ_α of $S(\mathcal{H}_\alpha)$ by $\delta_\alpha(f)(\lambda) = f(\lambda - \frac{1}{2}\alpha)$ and let $\delta^\alpha = \delta|_{S(\mathcal{H}^\alpha)}$. Denote by i the restriction of $(1 \otimes \delta^\alpha) \circ \phi_\alpha$ on $Z(\mathcal{G})$. Then $i(Z(\mathcal{G})) \subset Z(\mathcal{G}^\alpha) \otimes S(\mathcal{H}^\alpha)$. We also have the following commutative diagram:

$$\begin{array}{ccccccc}
U_0(\mathcal{G}^\alpha) & \otimes & S(\mathcal{H}^\alpha) & \xrightarrow{(\delta_\alpha \circ \phi^\alpha) \otimes 1} & S(\mathcal{H}_\alpha) & \otimes & S(\mathcal{H}^\alpha) \simeq S(\mathcal{H}) \\
 & \uparrow & & & & \uparrow & \\
Z(\mathcal{G}^\alpha) & \otimes & S(\mathcal{H}^\alpha) & \xrightarrow{\psi_\alpha} & & S(\mathcal{H})^{W(\alpha)} &
\end{array}$$

where ϕ^α is the Harish-Chandra homomorphism for $\mathcal{G}^\alpha$, $W(\alpha) \simeq S_2$ is the Weyl group of $\mathcal{G}^\alpha$ and ψ_α is a canonical isomorphism [Lemma 6, DFO].

Note that composition $\psi_\alpha \circ i$ gives the classical Harish-Chandra isomorphism $Z(\mathcal{G}) \simeq S(\mathcal{H})^W[D]$.

Consider the linear space $\Omega = \mathcal{H}^* \times \mathbf{C}$. For (λ, p) and (μ, q) in Ω we say that $(\lambda, p) > (\mu, q)$ if $\lambda - \mu = \sum_{\beta\in\pi\setminus\{\alpha\}} n_\beta \beta$, $n_\beta \in \mathbf{Z}_+$ and $\sum_{\beta\in\pi\setminus\{\alpha\}} n_\beta \neq 0$. We will denote by $\Omega_\mathbf{R}$ the real subspace of Ω, thus $\Omega = \Omega_\mathbf{R} + i\Omega_\mathbf{R}$. If $x \in \Omega$ then Re x (resp. Im x) will be its real (resp. imaginary) component.

Let $r \in \mathbf{C}$. Consider the linear space $B_r = \sum_{\beta\in\pi\setminus\{\alpha\}} \mathbf{C}\beta + r\alpha$ with a fixed point $r\alpha$, a $\mathbf{Z}$-module $\tilde{B}_r = B_r \oplus \mathbf{Z}\alpha$ and let $\Omega_r = B_r \times \mathbf{C}$, $\tilde{\Omega}_r = \tilde{B}_r \times \mathbf{C}$.

For a $\mathcal{G}$-module V with a Jordan-Hölder series $\mathcal{JH}(V)$ will denote the set of all irreducible subquotients of V. A $\mathcal{G}$-module V is called weight if

$$V = \bigoplus_{\lambda \in \mathcal{H}^*} V_\lambda$$

where $V_\lambda = \{v \in V | hv = \lambda(h)v \text{ for all } h \in \mathcal{H}\}$. If $V_\lambda \neq 0$ then λ is called a weight of V. A weight λ is called extreme if $V_{\lambda+\beta} = 0$ or $V_{\lambda-\beta} = 0$ for any $\beta \in \Delta^+$ and highest weight if $V_{\lambda+\beta} = 0$ for all $\beta \in \Delta^+$. A weight $\mathcal{G}$-module V is called α-stratified if X_α and $X_{-\alpha}$ act injectively on V.

Let V be a weight $\mathcal{G}$-module. A non-zero element $v \in V$ is called α-primitive (with respect to $\mathcal{G}$) if $v \in V_\lambda$ for some $\lambda \in \mathcal{H}^*$ and $N_+^\alpha v = 0$.

Denote by K_α the full subcategory of the category of $\mathcal{G}$-modules consisting of weight modules on which $X_{-\alpha}$ acts injectively. In particular, α-stratified modules are the objects of K_α. Let $r \in \mathbf{C}$, $\tilde{r} = r + \mathbf{Z}$ and $K_{\alpha,\tilde{r}}$ be the full subcategory of K_α consisting of such modules V that $V_\lambda \neq 0$ implies $\lambda + \rho \in \tilde{B}_r$. Then we immediately obtain the following.

Proposition 2.1

$$K_\alpha = \bigoplus_{\tilde{r} \in \mathbf{C}/\mathbf{Z}} K_{\alpha,\tilde{r}} \quad .$$

Thus if $V \in K_{\alpha,\tilde{r}}$, $V_1 \in K_{\alpha,\tilde{r}_1}$ and $\tilde{r} \neq \tilde{r}_1$ then $Hom_{\mathcal{G}}(V, V_1) = Hom_{\mathcal{G}}(V_1, V) = 0$ and one can work within each subcategory $K_{\alpha,\tilde{r}}$ separately.

It is known that $c = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ generates $Z(\mathcal{G}^\alpha)$. Let $a, b \in \mathbf{C}$. Any such pair defines a unique indecomposable weight $\mathcal{G}^\alpha$-module $N(a, b)$ on which $X_{-\alpha}$ acts injectively and where a is an eigenvalue of H_α and b is an eigenvalue of c. The module $N(a, b)$ has a $\mathbf{Z}$-basis $\{v_i\}$ such that $X_{-\alpha}v_i = v_{i-1}$, $H_\alpha v_i = (a + 2i)v_i$ and $X_\alpha v_i = \frac{1}{4}(b - (a + 2i + 1)^2)v_{i+1}$.

Lemma 2.2. The following statements are equivalent.

(i) $N(a, b)$ is irreducible

(ii) $N(a, b)$ is torsion free

(iii) $b \neq (a + 2\ell + 1)^2$ for all $e \in \mathbf{Z}$.

Proof. Follows immediately from the construction of $N(a,b)$.

$\square$

Set $\Omega^s = \{(\lambda,p) \in \Omega | p \neq \pm(\lambda(H_\alpha)+2\ell)$ for all $\ell \in \mathbf{Z}\}$, $\Omega_r^s = \Omega_r \cap \Omega^s$, $\tilde{\Omega}_r^s = \tilde{\Omega}_r \cap \Omega^s$. Hence, if $(\lambda,p) \in \Omega^s$ then $N((\lambda-\rho)(H_\alpha),p^2)$ is irreducible and torsion free.

§3. Modules $M_\alpha(\lambda,p)$.

In this paragraph we discuss the construction and properties of certain universal objects in K_α generated by α-primitive elements. We use [CF] as our main reference.

Since $\mathcal{H} = \mathcal{H}_\alpha \oplus \mathcal{H}^\alpha$, any element $\lambda \in \mathcal{H}^*$ can be written uniquely as $\lambda = \lambda_\alpha + \lambda^\alpha$ where $\lambda_\alpha \in \mathcal{H}_\alpha^*$ and $\lambda^\alpha \in (\mathcal{H}^\alpha)^*$. Let $a,b \in \mathbf{C}$ and $\lambda \in \mathcal{H}^*$ such that $(\lambda-\rho)(H_\alpha) = (\lambda_\alpha-\rho)(H_\alpha) = a$. Define an $\mathcal{H}$-module structure on $N(a,b)$ by letting $hv = \lambda^\alpha(h)v$ for any $h \in \mathcal{H}^\alpha$ and any $v \in N(a,b)$. Thus $N(a,b)$ becomes an $\mathcal{G}^\alpha + \mathcal{H}$-module. Moreover we can consider $N(a,b)$ as $D = \mathcal{H} + \mathcal{G}^\alpha \oplus N^\alpha$-module with trivial action of N_+^α.

Define a $\mathcal{G}$-module

$$M_\alpha(\lambda,b) = U(\mathcal{G}) \underset{U(D)}{\otimes} N(a,b)$$

associated with α, λ, b. We will call it generalized Verma modules in spite of the fact that unlike the classical generalized Verma modules [GL], $M_\alpha(\lambda,b)$ has no highest weight. On the other hand it may have some primitive elements. Also note that modules $M_\alpha(\lambda,b)$ are slightly different from the one considered in [CF] where it is assumed that $N(a,b)$ is irreducible. By Lemma 2.1 both constructions coincide if $b \neq (\lambda(H_\alpha)+2\ell)$ for all $\ell \in \mathbf{Z}$.

Set $M(\lambda,b) = M_\alpha(\lambda,b)$.

Proposition 3.1.

(i) $M(\lambda,b) \simeq U(N_-^\alpha) \otimes_{\mathbf{C}} N(a,b)]$ as a vector space and in particular it is a weight N_-^α-free module with finite-dimensional weight spaces.

(ii) $M(\lambda, b)$ has a unique maximal submodule.

(iii) $M(\lambda, b) \simeq M(\lambda + k\alpha, b)$ for all $k \in \mathbf{Z}$.

(iv) $M(\lambda, b) \in K_\alpha$.

(v) $M(\lambda, b)$ is α-stratified if and only if $b \neq (\lambda(H_\alpha) + 2\ell)^2$ for all $\ell \in \mathbf{Z}$.

Proof. (i) - (iii) follow directly from the construction on $M(\lambda, b)$; (iv) follows from [CF, Theorem 2.1] and (v) follows from Lemma 2.2 and [CF, Theorem 2.1].

□

It will be more convenient to use a slightly different parametrization of generalized Verma modules replacing $M(\lambda, b)$ by $M(\lambda, p)$ where $p^2 = b$. Thus we always have $M(\lambda, p) = M(\lambda, -p)$.

Corollary 3.2. $M(\lambda, p)$ is α-stratified if and only if $(\lambda, p) \in \Omega^s$.

Proof. Follows from Proposition 3.1, (iv).

□

It follows from [CF, Corollary 1.11] that module $M(\lambda, p)$ admits a central character $\theta_{(\lambda,p)} \in Z^*(\mathcal{G})$, i.e. $zv = \theta_{(\lambda,p)}(z)v$ for any $z \in Z(\mathcal{G})$ and $v \in M(\lambda, p)$. It also follows from [CF, Theorem 2.8, (i)] that $M(\lambda, p)$ has a Jordan-Hölder series.

Denote by $L(\lambda, p)$ a unique irreducible quotient of $M(\lambda, p)$. Note, that the subspace $L(\lambda, p)_{\lambda-\rho}$ can be trivial. Using Proposition 3.1, (iii) we can always avoid such a situation and assume that $L(\lambda, p)_{\lambda-\rho} \neq 0$.

Proposition 3.3 $L(\lambda, p) \simeq L(\lambda + k\alpha, p)$ for all $k \in \mathbf{Z}$.

Proof. Follows from Proposition 3.1, (iii) and Lemma 2.2.

□

The following Proposition shows the universality of α-stratified modules $M_\alpha(\lambda, p)$.

Proposition 3.4.

(i) If V is an α-stratified $\mathcal{G}$-module generated by an α-primitive element $v \in V_\lambda - \rho$ such that $cv = bv$ then V is a homomorphic image of $M(\lambda, \pm\sqrt{b})$. If in addition V is irreducible then $V \simeq L(\lambda, \pm\sqrt{b})$.

(ii) Every submodule and every subquotient of an α-stratified module $M(\lambda, p)$ is α-stratified.

Proof. Since V is α-stratified, then $U(\mathcal{G}^\alpha)v \simeq N((\lambda - \rho)(H_\alpha), p^2)$ and thus (i) follows from the construction of $M(\lambda, p)$. Statement (ii) follows from [CF, Proposition 1.3].

□

Proposition 3.5. If $M(\lambda, p)_{\mu-\rho}$ contains a non-zero α-primitive element v such that $cv = q^2 v$ then $M(\mu, q) \subset M(\lambda, p)$.

Proof. Using Proposition 3.1, (iii) one can construct a $\mathcal{G}^\alpha$-module $V \simeq N((\mu - \rho)(H_\alpha), q^2)$ such that $V \ni v$. Since v is α-primitive we immediately obtain that $N_+^\alpha X_{-\alpha}^m v = 0$ for all $m \in \mathbf{Z}_+$. Let $v = X_{-\alpha} v'$ and $\beta \in \pi \setminus \{\alpha\}$. Then $0 = X_\beta v = X_{-\alpha} X_\beta v'$ and $X_\beta v' = 0$ by Proposition 3.1, (iv). Analogously, $X_\gamma v' = 0$ for any $\gamma \in \Delta^+ \setminus \{\alpha\}$ and thus $N_+^\alpha v' = 0$. Applying the same arguments to every basis vector of V we conclude that $N_+^\alpha V = 0$. Hence, $U(\mathcal{G})V \simeq M(\mu, q)$ by Proposition 3.1, (i) and $M(\mu, q) \subset M(\lambda, p)$.

□

For $\lambda \in \mathcal{H}^*$, $M(\lambda)$ will denote a Verma module with highest weight $\lambda - \rho$ [D].

We will need the following two lemmas.

Lemma 3.6. Let $\lambda, \mu \in \mathcal{H}^*$. If $M(\mu) \subset M(\lambda)$ then $M(\mu, \mu(H_\alpha)) \subset M(\lambda, \lambda(H_\alpha))$.

Proof. Since $M(\lambda, \lambda(H_\alpha)) \supset M(\lambda) \supset M(\mu)$ there exists a non-zero α-primitive element $v \in M(\lambda, \lambda(H_\alpha))_{\mu-\rho}$ such that $cv = (\mu(H_\alpha))^2 v$. Thus $M(\mu, \mu(H_\alpha)) \subset M(\lambda, \lambda(H_\alpha))$ by Proposition 3.5.

□

Let $\beta \in \pi \setminus \{\alpha\}$ and $\alpha + \beta \in \Delta$. Then a subalgebra $\mathcal{G}^{\alpha,\beta} \subset \mathcal{G}$ generated by $X_{\pm\alpha}$ and $X_{\pm\beta}$ is simple Lie Algebra of rank two. Clearly, the subspace $M_{\alpha,\beta}(\lambda, p) = \sum_{k,m \in \mathbf{Z}_+} M(\lambda, p)_{\lambda - \rho + m\alpha - k\beta}$ of $M(\lambda, p)$ is

$\mathcal{H} + \mathcal{G}^{\alpha,\beta}$-module. Moreover it follows from the construction of $M(\alpha, p)$ that as $\mathcal{G}^{\alpha,\beta}$-module $M_{\alpha,\beta}(\lambda, p)$ isomorphic to a generalized Verma module for $\mathcal{G}^{\alpha,\beta}$.

Lemma 3.7. Let $\beta \in \pi \setminus \{\alpha\}$, $\lambda \in \mathcal{H}^*$, $p \in \mathbf{C}$.

(i) If $\alpha + \beta \notin \Delta$ and $s_\beta \lambda < \lambda$ then

$$M(s_\beta \lambda, p) \subset M(\lambda, p)$$

(ii) If $\alpha + \beta \in \Delta$, $\mu \in \mathcal{H}^*$, $q \in \mathbf{C}$ and there exists a non-zero α-primitive element (with respect to $\mathcal{G}^{\alpha,\beta}$) $v \in M_{\alpha,\beta}(\lambda, p)_{\mu-\rho}$ such that $cv = q^2 v$ then

$$M(\mu, q) \subset M(\lambda, p).$$

Proof. Let $0 \neq u \in M(\lambda, p)_{\lambda-\rho}$. Since $s_\beta \lambda < \lambda$ then $\lambda(H_\beta) = m \in \mathbf{N}$ and thus $u' = X_{-\beta}^m v \in M(\lambda, p)_{\lambda-\rho-m\beta}$ is α-primitive element. Moreover, $cv' = p^2 v'$ since $\alpha + \beta \notin \Delta$. Applying Proposition 3.5 we conclude that $M(s_\beta \lambda, p) \subset M(\lambda, p)$ that completes the proof of (i). Since v is α-primitive element (with respect to $\mathcal{G}^{\alpha,\beta}$) one can easily see that it is also α-primitive (with respect to $\mathcal{G}$). Thus $M(\mu, q) \subset M(\lambda, p)$ by Proposition 3.5 which proves (ii) and completes the proof of Lemma.

$\square$

Theorem 3.8 [CF, Theorems 3.7 and 3.9]. If (λ, p) and $(\mu, q) \in \Omega^s$ then each non-zero element of $H = Hom_{\mathcal{G}}(M(\mu, q), M(\lambda, p))$ is injective and $\dim H \leq 1$.

In §7 we will establish the conditions under which $\dim H = 1$.

§4. Generalized Verma Modules for Simple Lie Algebras of Rank Two except G_2.

Let $\mathcal{G}$ be a simple Lie Algebra of rank two and $\mathcal{G} \neq G_2$. Here we study generalized Verma $\mathcal{G}$-modules.

The results of this paragraph are essential for the definition of the group W_α in §5. Also the developed theory of representations of rank two algebras plays the same role in the general setup for generalized

Verma modules as the $sl(2)$-theory plays for classical Verma modules. For instance, all results of paragraph 6 heavily depend on the Theorems 4.1,4.3,4.5.

First we consider the case of $\mathcal{G} = A_2$. Let $\pi = \{\alpha, \beta\}$ be a basis of root system Δ for A_2. Then $\Delta = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$.

For $(\lambda, p) \in \Omega$ denote $n^{\pm}(\lambda, p) = \frac{1}{2}(\lambda(H_\alpha + 2H_\beta) \pm p)$.

The following theorem describes the structure of the module $M(\lambda, p)$. / Theorem 4.1 [F2]. Let $\mathcal{G} = A_2$ and $(\lambda, p), (\mu, q) \in \Omega$.

(i) $M(\mu, q) \subset M(\lambda, p)$ if and only if $\mu = \lambda - n\beta + k\alpha$, $n, k \in \mathbf{Z}$, $n \geq 0$ and one of the following conditions holds:

a) $n = 0$ and $q = \pm p$;

b) $n \in \{n^{\pm}(\lambda, p)\}$ and $q^2 = (p \mp n^{\pm}(\lambda, p))^2$.

(ii) If $n_1 = n^-(\lambda, p) \in \mathbf{N}$ and $n_2 = n^+(\lambda, p) \in \mathbf{N}$ then

$$M(\lambda, p) \supset M(\lambda - n_1\beta, p + n_1) \supset M(\lambda - n_2\beta, p - n_2)$$

(iii) Let $(\lambda, p) \in \Omega^S$. The module $M(\lambda, p)$ is irreducible if and only if $n^{\pm}(\lambda, p) \notin \mathbf{N}$.

$\square$

Next we consider the case of $\mathcal{G} = B_2$. Let $\pi = \{\alpha, \beta\}$ be a basis of root system Δ for B_2 and α a long root. Then $\Delta = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(\alpha+2\beta)\}$. We will use the following realization of $\mathcal{G}$ by 5×5 matrices: $X_\alpha = e_{21} + e_{54}$, $X_\beta = e_{32} + e_{43}$, $X_{-\alpha} = e_{12} + e_{45}$, $X_{-\beta} = 2(e_{23} + e_{34})$, $X_{\alpha+\beta} = e_{53} - e_{31}$, $X_{-\alpha-\beta} = 2(e_{35} - e_{13})$, $X_{\alpha+2\beta} = e_{52} + e_{41}$, $X_{-\alpha-2\beta} = 4(e_{25} + e_{14})$, $H_\alpha = e_{22} - e_{11} + e_{55} - e_{44}$, $H_2 = 2(e_{44} - e_{22})$ where $e_{k\ell} = (a_{ij}) \in M_5(\mathbf{C})$ with $a_{k\ell} = 1$ and $a_{ij} = 0$ if $i \neq k$ or $j \neq \ell$. All commuting relations between the basis elements can easily be computed.

Consider the Casimir operator B [K, p.22]. Since B belongs to the centre of $U(\mathcal{G})$, B acts as a scalar on $M(\lambda, p)$ for any $(\lambda, p) \in \Omega$.

Proposition 4.2. Let $\mathcal{G} = B_2$, α a long root and (λ, p), $(\mu, q) \in \Omega$. If $M(\mu, q) \subset M(\lambda, p)$ then $\mu = \lambda - n\beta + k\alpha$, $k, n \in \mathbf{Z}$, $n \geq 0$ and $q^2 = p^2 + 2n\lambda(H_\alpha + H_\beta) - n^2$.

Proof. If $M(\mu, q) \subset M(\lambda, p)$ then obviously $\mu = \lambda - n\beta + k\alpha$ for some k, $n \in \mathbf{Z}$, $n \geq 0$. Using [K, Corollary 2.6] and comparing the eigenvalues of B on $M(\lambda, p)$ and $M(\lambda - n\beta, q) \simeq M(\mu, q)$ we obtain that $q^2 = p^2 + 2n\lambda(H_\alpha + H_\beta) - n^2$ which completes the proof.

$\square$

For $(\lambda, p) \in \Omega$ denote $m^{\pm}(\lambda, p) = \lambda(H_\alpha + H_\beta) \pm p$ and $m(\lambda, p) = m^+(\lambda, p) + m^-(\lambda, p) = 2\lambda(H_\alpha + H_\beta)$.

Theorem 4.3. Let $\mathcal{G} = B_2$, α a long root and (λ, p), $(\mu, q) \in \Omega$. Then $M(\mu, q) \subset M(\lambda, p)$ if and only if $\mu = \lambda - m\beta + k\alpha$, $k, m \in \mathbf{Z}$, $m \geq 0$ and one of the following conditions holds:

(i) $m = 0$ and $q = \pm p$

(ii) $m \in \{m^{\pm}(\lambda, p)\}$ and $q^2 = (p \mp m^{\pm}(\lambda, p))^2$.

(iii) $m = m(\lambda, p)$, $\frac{1}{2}m \in \mathbf{N}$ and $q = \pm p$.

(iv) $m = m(\lambda, p)$, $\frac{1}{2}m \notin \mathbf{N}$, $m^{\pm}(\lambda, p) \in \mathbf{N}$ and $q = \pm p$.

Theorem 4.3 immediately implies the following criteria of irreducibility for $M(\lambda, p)$.

Corollary 4.4. Let $\mathcal{G} = B_2$, α a long root and $(\lambda, p) \in \Omega^S$. The module $M(\lambda, p)$ is irreducible if and only if $m^{\pm}(\lambda, p) \notin \mathbf{N}$ and $\frac{1}{2}m(\lambda, p) \notin \mathbf{N}$.

$\square$

Corollary 4.5. Let $\mathcal{G} = B_2$, α a long root and $(\lambda, p) \in \Omega$. Assume that $m_1 = m^-(\lambda, p) \in \mathbf{N}$, $m_2 = m^+(\lambda, p) \in \mathbf{N}$ and $m_2 > m_1$. Then $M(\lambda, p) \supset M(\lambda - m_1\beta, p + m_1) \supset M(\lambda - m_2\beta, p - m_2) \supset M(\lambda - (m_1 + m_2)\beta, p)$ if $p \in \mathbf{N}$ and

$$M(\lambda, p) \begin{array}{ccc} \supset & M(\lambda - m_1\beta, p + m_1) & \supset \\ & & \\ \supset & M(\lambda - m_2\beta, p - m_2) & \supset \end{array} M(\lambda - (m_1 + m_2)\beta, p) \quad \text{if } p \notin \mathbf{N}.$$

$\square$

Proof of Theorem 4.3. Let $0 \neq v \in M(\lambda,p)_{\lambda-\rho}$ and $\lambda(H_\alpha) = x$, $\lambda(H_\beta) = y$. Since $M(\mu,q) \simeq M(\mu - k\alpha, q)$ we will assume that $\mu = \lambda - m\beta$ for some $m \in \mathbf{N} \cup \{0\}$. Since $M(\mu,q) \subset M(\lambda,p)$ then $q = q_m$ where $q_m^2 = p^2 + m(2\lambda(H_\alpha + H_\beta) - m)$ by Propsition 4.2.

Step 1. The statement of the Theorem is obvious if $m = 0$.

Step 2. Assume that $m = 1$. Thus $\mu = \lambda - \beta$ and $q^2 = p^2 + 2(x+y) - 1$.

The elements $v_1 = X_{-\alpha-\beta}X_{+\alpha}v$ an $v_2 = X_{-\beta}v$ form a basis of the weight subspace $M(\lambda,p)_{\lambda-\rho-\beta}$. Let $a, b, \in \mathbf{C}$ and $u = av_1 + bv_2$ be an α-primitive element, i.e. $X_\beta u = X_{\alpha+\beta}u = 0$. Then the coefficients a, b must satisfy the following equations.

$$\begin{cases} (p^2 - x^2)a + 2(y-1)b = 0 \\ (2x + y - 1)a + 2b = 0 \end{cases}$$

The system has a non-trivial solution if and only if $x + y + p = 1$ or $x + y - p = 1$. Hence, $M(\lambda - \beta, q) \subset M(\lambda,p)$ if and only if $m^+(\lambda,p) = 1$ or $m^-(\lambda,p) = 1$.

Step 3. Let $m > 1$. Then the elements

$$u_{kj} = X_{-\beta}^k X_{-\alpha-2\beta}^j X_{-\alpha-\beta}^{m-k-2j} X_\alpha^{m-k-j} v,$$

$$k = 0,1,...,m; \quad j = 0,1,..., \left[\frac{m-k}{2}\right]$$

form a basis of the weight subspace $M(\lambda,p)_{\mu-\rho}$.

Let $0 \neq u = \sum_{k,j} a_{kj}u_{kj}$ be an α-primitive element, $a_{kj} \in \mathbf{C}$. Then $X_\beta u = X_{\alpha+\beta}u = X_{\alpha+2\beta}u = 0$. One can show that $X_\beta u = 0$ if and only if the coefficients a_{kj} satisfy the following system of equations

$$b_{ij}a_{ij} + d_{i+1j}a_{i+1j} + 2(j+1)a_{ij+1} = 0 \tag{1}$$

where b_{ij} are quadratic expressions in p and x and d_{ij} are linear expressions in y.

Also $X_{\alpha+\beta}u = 0$ if and only if the coefficients a_{kj} satisfy the following system of equations

$$c_{ij}a_{ij} + 2(i+1)a_{i+1j} + t_{ij}a_{i+1j-1} + g_{ij}a_{i-1j+1} = 0 \tag{2}$$

where c_{ij} are linear expressions in x and y and $t_{ij},\ g_{ij} \in \mathbf{Z}$. It follows immediately that if $a_{00} = 0$ then systems (1) – (2) have only a trivial solution. Hence, we can assume that $a_{00} = 1$. Then for $c = j = 0$ we have from (1)

$$\frac{1}{2}m^2(p^2 - (x+2m-2)^2) + (y-2m+1)a_{10} + 2a_{01} = 0 \tag{3}$$

and from (2)

$$m(2x+y+m-2) + 2a_{10} = 0 \tag{4}$$

Also we have from (2) for $i = 1,\ j = 0$ that

$$(m-1)(2x+y+m-5)a_{10} - 2a_{01} + 4a_{20} = 0 \tag{5}$$

Consider the equation $X_{\alpha+2\beta}u = 0$. We have

$$\begin{aligned} X_{\alpha+2\beta}u_{00} &= -\tfrac{1}{2}m(m-1)(p^2-(x+2m-2)^2)X^{m-2}_{-\alpha-\beta}X^{m-1}_{\alpha}v. \\ X_{\alpha+2\beta}u_{10} &= 2(m-1)(2x+y+m-3)X^{m-2}_{-\alpha-\beta}X^{m-1}_{\alpha}v + \ldots \\ X_{\alpha+2\beta}u_{01} &= 4(x+y-m)X^{m-2}_{-\alpha-\beta}X^{m-1}_{\alpha}v + \ldots \\ X_{\alpha+2\beta}u_{20} &= 4X^{m-2}_{-\alpha-\beta}X^{m-1}_{\alpha}v + \ldots \end{aligned}$$

and

$$X_{\alpha+2\beta}u = (-\frac{1}{2}m(m-1)(p^2-(x+2m-2)^2)a_{00} + 2(m-1)(2x+y+m-3)a_{10} +$$

$$+4(x+y-m)a_{01} + 4a_{20})X^{m-2}_{-\alpha-\beta}X^{m-1}_{\alpha}v + \ldots$$

Thus

$$-\frac{1}{2}m(m-1)(p^2-(x+2m-2)^2) + 2(m-1)(2x+y+m-3)a_{10} + 4(x+y-m)a_{01} + 4a_{20} = 0 \tag{6}$$

The system (3) - (6) has a solution if and only if $p,\ x,\ y,\ m$ satisfy the following condition

$$h_m(x,y,p) = (m-2(x+y))(m-(x+y+p))(m-(x+y-p)) = 0 \tag{7}$$

Therefore, if $M(\lambda,p)_{\lambda-m\beta-\rho},\ m \geq 2$ contains a non-trivial α-primitive element then $m \in \{m^{\pm}(\lambda,p), m(\lambda,p)\}$.

Step 4. Suppose that $m \in \{m^{\pm}(\lambda,p)\},\ m \geq 2$. We will show that in this case $M(\lambda,p)_{\lambda-m\beta-\rho}$ always contains a non-trivial α-primitive element and hence $M(\lambda-m\beta,q_m) \subset M(\lambda,p)$. Indeed, the consistency of the system (1)-(2) is equivalent to the condition that m, x, y, p satisfy certain polynomial equations. Let $\tilde{h}_m(x,y,p)=0$ be one of them. We will show that

$$(m^-(\lambda,p)-m)(m^+(\lambda,p)-m) \text{ divides } \tilde{h}_m(x,y,p),$$

$$\text{i.e.} \quad \tilde{h}_m^{\pm}(x,p)=\tilde{h}_m(x,-x\mp p+m,p)=0.$$

Consider $\tilde{h}_m^{\pm}$ as polynomial in x and p. Assume that $p\in \mathbf{N}$, $x_k=p-2k$, $k\in \mathbf{Z}_+$ and consider the Verma module $M(\mu)$ where $\mu(H_\alpha)=x_0$, $\mu(H_\beta)=-2p+m$ if $m=m^+(\lambda,p)$ and $\mu(H_\beta)=m$ if $m=m^-(\lambda,p)$. It follows from [D, Theorem 7.6.23] that $M(\mu-m(\alpha+\beta))\subset M(\mu)$ if $m=m^+(\lambda,p)$ and $M(\mu-m\beta)\subset M(\mu)$ if $m=m^-(\lambda,p)$. Thus by Lemma 3.6, $M(\mu-m(\alpha+\beta),q_m)\subset M(\mu-m\alpha,p)$ if $m=m^+(\lambda,p)$ and $M(\mu-m\beta,q_m)\subset M(\mu,p)$ if $m=m^-(\lambda,p)$. We conclude that $M(\mu-k\alpha-m\beta,q_m)\subset M(\mu-k\alpha,p)$ for any $k\in\mathbf{Z}$ by Proposition 3.1, (iii). Therefore, $\tilde{h}_m^{\pm}(x_k,p)=\tilde{h}_m^{\pm}(p-2k,p)=0$ for any $p\in\mathbf{N}$ and $k\in\mathbf{Z}$ and thus $\tilde{h}_m^{\pm}=0$. This completes the proof of (ii).

Step 5. Suppose that $m=m(\lambda,p)$, $m\geq 2$ and $\frac{1}{2}m\in\mathbf{N}$. Let again $\tilde{h}_m(x,y,p)=0$ be any polynomial equation on m,x,y,p that follows from the consistency of the system (1) - (2). Consider a polynomial function $\hat{h}_m(x,p)=\tilde{h}_m(x,\frac{1}{2}m-x,p)$ in x and p. We will show that $\hat{h}_m(x,p)=0$ which implies the existence of a non-trivial α-primitive element in $M(\lambda,p)_{\lambda-m\beta-\rho}$. Let $p\in\mathbf{C}$, $x_k=p-2k$, $k\in\mathbf{Z}$ and $\mu\in\mathcal{H}^*$ such that $\mu(\mathcal{H}_\alpha)=x_0$, $\mu(H_\beta)=\frac{1}{2}m-x_0$. Consider the Verma module $M(\mu)$ with highest weight $\mu-\rho$. Since $\mu(H_\alpha+H_\beta)=\frac{1}{2}m\in\mathbf{N}$ it follows from [D, Theorem 7.6.23] that $M(\mu-m\beta-\frac{1}{2}m\alpha)\subset M(\mu)$ and thus $M(\mu-m\beta-\frac{1}{2}m\alpha,q_m)\subset M(\mu,p)$ by Lemma 3.6. Therefore, $M(\mu-m\beta-k\alpha,q_m)\subset M(\mu-k\alpha,p)$ for any $k\in\mathbf{Z}$. Thus $\hat{h}_m(x_k,p)=0$ for any $p\in\mathbf{C}$ and $k\in\mathbf{Z}$ which implies that $\hat{h}_m=0$ and hence $M(\lambda-m\beta,q_m)\subset M(\lambda,p)$. This completes the proof of (iii).

Step 6. Suppose that $m=m(\lambda,p)$, $m\geq 2$ and $\frac{1}{2}m\notin\mathbf{N}$. Let $m=2k+1$. Denote $f_\ell(p,x,y)=p^2-(x+y-\ell)^2$, $\ell=1,3,...,2k+1$. If $f_\ell(p,x,y)=0$ for some $\ell=1,3,...,2k-1$ then it follows from Step 4 that

$$M(\lambda-m\beta,q_m)\subset M(\lambda-\ell\beta,q_\ell)\subset M(\lambda,\ p).$$

Also if $f_{2k+1}(p,x,y)=0$ then

$$M(\lambda - m\beta, q_m) \subset M(\lambda, p).$$

Consider the following subsystem of (1)

$$b_{0j}a_{0j} + d_{1j}a_{1j} + 2(j+1)a_{0j+1} = 0 \tag{8}$$

and the following subsystem of (2)

$$c_{0j}a_{0j} + 2a_{1j} + t_{0j}a_{1j-1} = 0 \tag{9}$$

where $a_{00}=1$ and $j=0,...,k$.

Solving the system (8) - (9) we obtain that p,x,y must satisfy a polynomial equation $g(p,x,y) = p^{2k+2} + ... = 0$. Thus $g(p,x,y)$ is a polynomial of degree $2k+2$ with respect to p. It follows from the discussion above that the zeros of polynomial function $f(p,x,y) = \prod_{\ell=1,3,...2k+1} f_\ell(p,x,y)$ is a subset of the zeros of $g(p,x,y)$. Since in addition the degree of $f(p,x,y)$ with respect to p equals $2k+2$ we conclude that $f(p,x,y) = g(p,x,y)$ and hence

$$M(\lambda - m\beta, q_m) \subset M(\lambda, p)$$

if and only if $f_\ell(p,x,y)=0$ for some $\ell = 1,3,...,2k+1$. This completes the proof of (iv) and the whole Theorem.

$\square$

Suppose now that $\mathcal{G} = B_2$, $\pi = \{\alpha, \beta\}$ and α is a short root. We will use the same realization of $\mathcal{G}$ simply interchanging α and β. The Casimir operator B acts scalarly on $M(\lambda,p)$ and the corresponding scalar can be obtained by using Corollary 2.6 in [K].

Proposition 4.4. Let $\mathcal{G} = B_2$, α a short root and (λ,p), $(\mu,q) \in \Omega$. If $M(\mu,q) \subset M(\lambda,p)$ then $\mu = \lambda - n\beta + k\alpha$, k, $n \in \mathbf{Z}$, $n \geq 0$ and $q^2 = p^2 + 4n\lambda(H_\alpha + 2H_\beta) - 4n^2$.

Proof. The proof is analogous to the proof of Proposition 4.2.

$\square$

For $(\lambda, p) \in \Omega$ denote $k^{\pm}(\lambda, p) = \frac{1}{2}(\lambda(H_\alpha + 2H_\beta) \pm p)$ and $k(\lambda, p) = k^+(\lambda, p) + k^-(\lambda, p) = \lambda(H_\alpha + 2H_\beta)$.

Theorem 4.5. Let $\mathcal{G} = B_2$, α a short root and (λ, p), $(\mu, q) \in \Omega$. Then $M(\mu, q) \subset M(\lambda, p)$ if and only if $\mu = \lambda - n\beta + k\alpha$, $n, k \in \mathbf{Z}$, $n \geq 0$ and one of the following conditions holds:

a) $n = 0$ and $q = \pm p$;

b) $n \in \{k^{\pm}(\lambda, p), k(\lambda, p)\}$ and $q^2 = p^2 + 4n\lambda(H_\alpha + 2H_\beta) - 4n^2$.

Theorem 4.5 implies the following criterion of irreducibility.

Corollary 4.6. Let $\mathcal{G} = B_2$, α a short root and $(\lambda, p) \in \Omega^s$. Module $M(\lambda, p)$ is irreducible if and only if $\{k^{\pm}(\lambda, p), k(\lambda, p)\} \cap \mathbf{N} = \emptyset$.

$\square$

Corollary 4.7. Let $\mathcal{G} = B_2$, α a short root and $(\lambda, p) \in \Omega$. Assume that $k_1 = k^-(\lambda, p) \in \mathbf{N}$, $k_2 = k^+(\lambda, p) \in \mathbf{N}$ and $k_2 > k_1$. Then

$$M(\lambda, p) \supset M(\lambda - k_1\beta, p + 2k_1) \supset M(\lambda - k_2\beta, p + 2k_1) \supset M(\lambda - (k_1 + k_2)\beta, p).$$

$\square$

Proof of Theorem 4.5. Let $0 \neq v \in M(\lambda, p)_{\lambda - \rho}$ and $\lambda(H_\alpha) = x$, $\lambda(H_\beta) = y$. Since $M(\mu, q) \subset M(\lambda, p)$ it follows from Proposition 4.4 that $\mu = \lambda - n\beta + k\alpha$ for some $n, k \in \mathbf{Z}$, $n \geq 0$, and $q = q_n$ where $q_n^2 = p^2 + 4n(x + 2y) - 4n^2$. Using the fact that $M(\mu, q) \simeq M(\mu - k\alpha, q)$ we may assume that $\mu = \lambda - n\beta$, $n \geq 0$. The case $n = 0$ is trivial. Assume that $n > 0$. Then the elements $u_{kj} = X_{-\beta}^k X_{-\beta-2\alpha}^j X_{-\alpha-\beta}^{n-k-j} X_\alpha^{n-k+j}$, $j = 0, ..., n-k$; $k = 0, ..., n$, form a basis of the weight space $M(\lambda, p)_{\lambda-\rho-n\beta}$. Suppose that $M(\lambda, p)_{\lambda-\rho-n\beta}$, contains a non-trivial α-primitive element $u = \sum_{k=0}^{n} \sum_{j=0}^{n-k} a_{kj} u_{kj}$. Then $X_\beta u = X_{\alpha+\beta} u = X_{\beta+2\alpha} u = 0$. An equation $X_\beta u = 0$ is equivalent to the following linear system on a_{kj}:

$$\begin{aligned} &(n-k-j)[p^2 - (x + 2n - 2k + 2j - 2)^2]a_{kj} + 2(n-k-j+1)(n-k-j)a_{kj-1} - \\ &-(k+1)(y - 2m + k + 1)a_{k+1j} = 0 \end{aligned} \tag{10}$$

An equation $X_{\alpha+\beta}u = 0$ is equivalent to the following linear system:

$$\begin{aligned}&(n-k-j)(x+2y-n-j-k-2)a_{kj} - 2(k+1)(j+1)a_{k+1j+1} - (k+1)a_{k+1j} - \\ &-\tfrac{1}{2}(j+1)[p^2 - (x+2n-2k+2j)^2]a_{kj+1} = 0\end{aligned} \tag{11}$$

Finally, an equation $X_{\beta+2\alpha}u = 0$ induces the following linear system on a_{kj}:

$$2(n-k-j)a_{kj} + 2(n-k-j+1)(n-k-j)a_{k-1j} - 4(j+1)(x+y+j)a_{kj+1} = 0. \tag{12}$$

One can easily check that if the system of equations has a non-trivial solution then p, x, y must satisfy the following condition:

$$h_n(p,x,y) = (2n-x-2y-p)(2n-x-2y+p)(n-x-2y) = 0 \tag{13}$$

Therefore, if $M(\lambda,p)_{\lambda-\rho-n\beta}$ contains a non-zero α-primitive element then $n \in \{k^{\pm}(\lambda,p), k(\lambda,p)\}$. Now the same arguments as in the proof of Steps 4 and 5 of Theorem 4.3 show that the converse is also true. Indeed, assume for example that $k(\lambda,p) = n \in \mathbf{N}$. Consider an arbitrary $p_0 \in \mathbf{N}$ and $\nu_\ell \in \mathcal{H}^*$ such that $\nu_\ell(H_\alpha) = p_0 - 2\ell$, $\nu_\ell(H_\beta) = \frac{1}{2}n - \frac{1}{2}p_0 + \ell$, $\ell \in \mathbf{Z}$. Then $k(\nu_\ell, p_0) = n$ for any $\ell \in \mathbf{Z}$. Let $M(\nu_0)$ be a Verma module with highest weight $\nu_0 - \rho$. Since $s_{\alpha+\beta}(\nu_0) = \nu_0 - n(\alpha+\beta)$ it follows from [D,Theorem 7.6.23] that $M(\nu_0 - n(\alpha+\beta)) \subset M(\nu_0)$ and thus $M(\nu_n - n\beta, p_0) \subset M(\nu_n, p_0)$ by Lemma 3.6. Therefore $M(\nu_\ell - n\beta, p_0) \subset M(\nu_\ell, p_0)$ for any $\ell \in \mathbf{Z}$. Since p_0 was arbitrary we conclude that whenever $k(\lambda,p) = n \in \mathbf{N}$ then $M(\lambda - n\beta, q_n) \subset M(\lambda, p)$. One can consider similarly the case when $k^{\pm}(\lambda,p) \in \mathbf{N}$. This completes the proof of the Theorem.

□

Remark. The conditions on the parameters for one generalized Verma module to be a submodule of another obtained in Theorems 4.1,4.3,4.5 can be reformulated (see Theorem 7.6) in terms of the action of a certain group anologously to the classical resalt of Bernstein, Gelfand, Gelfand for Verma modules.

§5. Generalized Weyl Group W_α

In this section we define the group W_α that acts on the space of parameters Ω and plays the same role for modules $M(\lambda, p)$ as the Weyl group plays for Verma modules.

Consider the following partition of π: $\pi = \pi_1 \cup \pi_2 \cup \pi_3 \cup \pi_4$ where $\pi_1 = \{\gamma \in \pi | \alpha + \gamma \in \Delta,\ |\alpha| = |\gamma|\}$, $\pi_2 = \{\gamma \in \pi | \alpha + \gamma \in \Delta,\ |\alpha| < |\gamma|\}$, $\pi_3 = \{\gamma \in \pi | \alpha + \gamma \in \Delta,\ |\alpha| > |\gamma|\}$, $\pi_4 = \{\gamma \in \pi | \alpha + \gamma \notin \Delta\}$. For $(\lambda, p) \in \Omega$ and $\beta \in \pi \setminus \pi_4$ denote

$$n_\beta^\pm(\lambda, p) = \begin{cases} \frac{1}{2}(\lambda(H_\alpha + 2H_\beta) \pm p), & \beta \in \pi_1 \cup \pi_2 \\ \lambda(H_\alpha + H_\beta) \pm p, & \beta \in \pi_3 \end{cases}$$

and define 3 pairs $(\lambda_\beta, p_\beta^i) \in \Omega$, $i = 1, 2, 3$ where $\lambda_\beta = \lambda - n_\beta^-(\lambda, p)\beta$, $p_\beta^1 = n_\beta^+(\lambda, p)$, $p_\beta^2 = p + 2n_\beta^-(\lambda, p)$, $p_\beta^3 = p + n_\beta^-(\lambda, p)$.

For each $\beta \in \pi$ consider $\ell_\beta \in GL(\Omega)$ such that

$$\ell_\beta(\lambda, p) = \begin{cases} (\lambda, -p), & \beta = \alpha \\ (s_\beta \lambda, p), & \beta \in \pi_4 \setminus \{\alpha\} \\ (\lambda_\beta, p_\beta^i), & \beta \in \pi_i, \qquad i = 1, 2, 3. \end{cases} \qquad (*)$$

Remark. One can see that for rank two algebras the formulae (*) coinsides with the conditions on the parameters obtained in section 4. Indeed, in this case $\pi_4 = \emptyset$, Theorem 4.1 corresponds to (*) for $\beta \in \pi_1$, Theorem 4.3 corresponds to (*) for $\beta \in \pi_3$, Theorem 4.5 corresponds to (*) for $\beta \in \pi_2$.

Define the generalized Weyl group $W_\alpha = < \ell_\beta, \beta \in \pi >$.

Lemma 5.1. For any $r \in \mathbf{C}$, Ω_r and $\tilde{\Omega}_r$ are invariant under the action of W_α.

Proof. Follows immediately from the definition of W_α.

□

Let Δ° be a root system dual to Δ, $\eta' : \Delta \to \Delta^\circ$ be a canonical bijection and $\pi^\circ = \eta'(\pi)$. Construct a map $\eta_r^\circ : \Delta^\circ \to \Omega_r$ as follows. For $\beta \in \pi$ let

$$\eta_r^\circ(\beta^\circ) = \begin{cases} (r\alpha, |\alpha^\circ|^2), & \beta = \alpha \\ \left(\frac{|\beta^\circ|^2}{2}\beta + r\alpha, -\frac{1}{2}|\beta^\circ|^2\right), & \alpha + \beta \in \Delta,\ |\alpha| \geq |\beta| \\ (\frac{1}{2}\beta + r\alpha, -1), & \alpha + \beta \in \Delta,\ |\alpha| < |\beta| \\ (\frac{|\beta^\circ|^2}{2}\beta + r\alpha, 0), & \alpha + \beta \notin \Delta,\ \alpha \neq \beta. \end{cases} \qquad (**)$$

Thus (**) defines a map from π° to Ω_r that can be extended to whole Δ° by linearity. Define

$\eta_r = \eta_r^\circ \circ \eta' : \Delta \to \Omega_r$. We will denote $\eta = \eta_r$, $\pi_{\alpha,r} = \eta(\pi)$, $\Delta_{\alpha,r} = \eta(\Delta)$, $\Delta^+_{\alpha,r} = \eta(\Delta^+)$. Clearly $\pi_{\alpha,r}$ forms a basis of Ω_r.

Examples.

1. Let $\mathcal{G} = A_2$, $\pi = \{\alpha, \beta\}$. Then $\pi_{\alpha,0} = \{(0,1), (\frac{1}{2}\beta, -\frac{1}{2})\}$, $\Delta^+_{\alpha,0} = \pi_{\alpha,0} \cup \{(\frac{1}{2}\beta,\ \frac{1}{2})\}$.

2. Let $\mathcal{G} = B_2$, $\pi = \{\alpha, \beta\}$, $|\alpha| > |\beta|$. Then $\pi_{\alpha,0} = \{(0,1), (\beta, -1)\}$, $\Delta^+_{\alpha,0} = \pi_{\alpha,0} \cup \{(\beta, 0), (\beta, 1)\}$.

3. Let $\mathcal{G} = B_2$, $\pi = \{\alpha, \beta\}$, $|\alpha| < |\beta|$. Then $\pi_{\alpha,0} = \{(0,2), (\frac{1}{2}\beta, -1)\}$, $\Delta^+_{\alpha,0} = \pi_{\alpha,0} \cup \{\frac{1}{2}\beta, 1), (\beta, 0)\}$.

Define a bilinear form $(\cdot,\cdot)_r : \Omega_r \times \Omega \to \mathbf{C}$ as follows. For $\beta \in \pi_{\alpha,r}, (\lambda, p) \in \Omega$ let

$$(\beta, (\lambda, p))_r = \begin{cases} p, & \eta^{-1}(\beta) = \alpha \\ \lambda(H_{\eta^{-1}(\beta)}), & \eta^{-1}(\beta) \in \pi_4 \setminus \{\alpha\} \\ n^-_{\eta^{-1}(\beta)}(\lambda, p), & \eta^{-1}(\beta) \in \pi \setminus \pi_4 \end{cases} \qquad (***)$$

We can extend (***) to the form on a whole space by linearity.

Proposition 5.2. The form $(\cdot,\cdot)$ is non-degenerated on Ω_r.

Proof. If for some $(\lambda, p) \in \Omega_r$, $(\beta, (\lambda, p))_r = 0$ for all $\beta \in \pi_{\alpha,r}$ then we immediately obtain that $p = 0$ and $\lambda(H_{\eta^{-1}(\beta)}) = 0$ for all $\beta \in \pi_{\alpha,r}$. Thus $\lambda = r\alpha$ which completes the proof.

□

We will denote by $|\cdot|_r$ the norm in $(\Omega_r)_{\mathbf{R}}$ induced by $(\cdot,\cdot)_r$.

Using (**) and (***) one can easily verify the following lemma.

Lemma 5.3. $\Delta_{\alpha,r}$ is a root system in $(\Omega_r,\ (\cdot,\cdot)_r)$ of the same type as Δ° and $\pi_{\alpha,r}$ is its basis.

□

It follows immediately from Lemma 5.3 that the Weyl group of $\Delta_{\alpha,r}$ is isomorphic to W. For $\beta \in \Delta^+_{\alpha,r}$ we will denote by σ_β the corresponding reflection in Ω_r:

$$\sigma_\beta(\lambda, p) = (\lambda, p) - \frac{2(\beta, (\lambda, p))_r}{(\beta, \beta)_r}\beta \text{ for any } (\lambda, p) \in \Omega_r.$$

Naturally, we can also view σ_β as an element of Aut $(\tilde{\Omega}_r)$.

Proposition 5.4. $W_\alpha \simeq W$.

Proof. Let $\beta \in \pi_{\alpha,r}$, $(\lambda, p) \in \Omega_r$. Then one can check that $\sigma_\beta(\lambda, p) = \ell_{\eta^{-1}(\beta)}(\lambda, p)$. Since σ_β, $\beta \in \pi_{\alpha,r}$ generate the group isomorphic to W and $\ell_{\eta^{-1}(\beta)}$, $\beta \in \pi_{\alpha,r}$ generate W_α the statement of Proposition follows.

□

Proposition 5.5. Let (λ, p), $(\lambda', p') \in \Omega_r$ and $(\lambda', p') \in W_\alpha(\lambda, p)$. Then $\theta_{(\lambda,p)} = \theta_{(\lambda',p')}$.

Proof. Clearly, $\theta_{l_\alpha(\lambda,p)} = \theta_{(\lambda,-p)} = \theta_{(\lambda,p)}$. Let $\beta \in \pi \setminus \{\alpha\}$ and $0 \neq z \in Z(\mathcal{G})$. For $t \in \mathbf{C}$ denote by $\tau_t : \mathbf{C}[c] \to \mathbf{C}$ the evaluation map, $\tau_t(c) = t^2$, and consider a polynomial function $f_z \in S(\Omega^*)$ such that $f_z(\mu, q) = ((\tau_q \otimes 1) \circ (1 \otimes \delta^\alpha) \circ \phi_\alpha(z))(\mu) = \theta_{(\mu,q)}(z)$ for any $(\mu, q) \in \Omega$. Clearly, $f_z \circ \ell_\beta(\lambda, p) = \theta_{\ell_\beta(\lambda,p)}$. Let $g_{z,\beta} = f_z \circ \ell_\beta - f_z$. If $\beta \in \pi_4 \setminus \{\alpha\}$ then for any $(\mu, q) \in \Omega_r$ such that $\mu(H_\beta) \in \mathbf{N}$ we obtain $M(\ell_\beta(\mu, q)) \subset M(\mu, q)$ by Lemma 3.7, (i). Thus for any such pair (μ, q), $\theta_{\ell_\beta(\mu,q)}(z) = \theta_{(\mu,q)}(z)$ which implies $g_{z,\beta} = 0$. Let now $\beta \in \pi \setminus \pi_4$. Then for any pair $(\mu, q) \in \Omega_r$ such that $n_\beta^-(\mu, q) \in \mathbf{N}$, the Theorems 4.1, 4.3 and 4.5 and Lemma 3.7, (ii) imply that

$$M(\ell_\beta(\mu, q)) \subset M(\mu, q)$$

and hence $g_{z,\beta} = 0$ again. Since $g_{z,\beta} = 0$ for any $z \in Z(\mathcal{G})$ we conclude that $\theta_{(\lambda,p)} = \theta_{\ell_\beta(\lambda,p)}$. Since the group W_α is generated by ℓ_β, $\beta \in \pi$ the statement of Proposition follows.

□

We make the following simple observation that follows from (***).

Lemma 5.6. Let $(\lambda, p) \in \Omega$. Then

$$\left\{(\beta, (\lambda, p))_r,\ \beta \in \Delta^+_{\alpha,r} \setminus \{\eta(\alpha)\}\right\} = \left\{(\beta, (\lambda, -p))_r,\ \beta \in \Delta^+_{\alpha,r} \setminus \{\eta(\alpha)\}\right\}.$$

□

§6. Generalized Harish-Chandra Theorem

In this section we establish an analog of the Harish-Chandra theorem for the centre $Z(\mathcal{G})$. Consider a map $\chi : Z(\mathcal{G}^\alpha) \otimes S(\mathcal{H}^\alpha) \to \mathbf{C}[t] \otimes S(\mathcal{H}^\alpha)$, such that $\chi(c) = t$, and algebra $\Lambda = \mathbf{C}[\sqrt{t}] \otimes S(\mathcal{H}^\alpha)$ that acts on Ω_r by polynomial functions. Denote by $\Lambda^{W_\alpha} \subset \Lambda$ the polynomial functions invariant under W_α. Then $\Lambda^{W_\alpha} \subset \mathbf{C}[t] \otimes S(\mathcal{H}^\alpha)$, since $l_\alpha \in W_\alpha$.

Lemma 6.1. $(\chi \circ i)(Z(\mathcal{G})) \subset \Lambda^{W_\alpha}$.

Proof. Let $0 \neq z \in Z(\mathcal{G})$, $w \in W_\alpha$, $(\lambda, p) \in \Omega_r$. Then by Proposition 5.5 $w(\chi \circ i)(z)(\lambda, p) = (\chi \circ i)(z)(w^{-1}(\lambda, p)) = \theta_{w^{-1}(\lambda,p)}(z) = \theta_{(\lambda,p)}(z) = (\chi \circ i)(z)(\lambda, p)$. Hence $w(\chi \circ i)(z) = (\chi \circ i)(z)$.

□

Lemma 6.2. $(\psi_\alpha \circ \chi^{-1})(\Lambda^{W_\alpha}) \subset S(\mathcal{H})^W$.

Proof. Let $\psi : W \tilde{\to} W_\alpha$ be the isomorphism established in Proposition 5.4. One can easily check that for any $w \in W_\alpha$ and $f \in \Lambda^{W_\alpha}$: $w(\psi_\alpha \circ \chi^{-1})(f) = (\psi_\alpha \circ \chi^{-1})(\psi(w)f)$ and thus Lemma follows.

□

Theorem 6.3 (Generalized Harish-Chandra Theorem)

$\chi \circ i : Z(\mathcal{G}) \to \Lambda^{W_\alpha}$ is an isomorphism that does not depend on the choice of π containing α.

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} Z(\mathcal{G}) & \xrightarrow{\chi \circ i} & \Lambda^{W_\alpha} & \xrightarrow{j} & S(\mathcal{H})^W \\ & & \downarrow & & \downarrow \\ & & \mathbf{C}[t] \otimes S(\mathcal{H}^\alpha) & \xrightarrow{\psi_\alpha \circ \chi^{-1}} & S(\mathcal{H})^{W(\alpha)} \end{array}$$

where j is the restriction of $\psi_\alpha \circ \chi^{-1}$ on Λ^{W_α}. Since $j \circ (\chi \circ i)$ is an isomorphism then $\mathrm{Ker}(\chi \circ i) = 0$ and $j(\Lambda^{W_\alpha}) = S(\mathcal{H})^W$. On the other hand $\psi_\alpha \circ \chi^{-1}$ is an isomorphism too. Thus, $\ker j = 0$ which implies that $\chi \circ i$ is epimorphism and hence isomorphism. This isomorphism does not depend on the choice of π containing α by [D, Th.7.4.5].

□

Corollary 6.4. Let (λ,p), $(\lambda',p')\in\Omega_r$. If $\theta_{(\lambda,p)}=\theta_{(\lambda',p')}$ then $(\lambda',p')\in W_\alpha(\lambda,p)$.

Proof. Let O_1 and O_2 be two different orbits of W_α in Ω_r. Then there exists a polynomial function $f\in \mathbf{C}[t]\otimes S(\mathcal{H}^\alpha)$ such that $f(O_1)=0$, $f(O_2)=1$. Consider $g=\dfrac{1}{|W_\alpha|}\sum_{w\in W_\alpha} w\cdot f\in\Lambda^{W_\alpha}$ and $z=(\chi\circ i)^{-1}(g)\in Z(\mathcal{G})$. Then $g(O_1)=0$, $g(O_2)=1$ and hence z takes different values on O_1 and O_2.

□

Combining the results of Proposition 5.5 and Corollary 6.4 we obtain

Theorem 6.5. Let (λ,p), $(\lambda',p')\in\Omega_r$. The following statements are equivalent

(i) $\theta_{(\lambda,p)}=\theta_{(\lambda',p')}$

(ii) $(\lambda',p')\in W_\alpha(\lambda,p)$.

§7. Submodule structure of $M(\lambda,p)$

In this section we establish a criterion for a module $M(\mu,q)$ to be a submodule of $M(\lambda,p)$ (Theorem 7.6) and hence obtain an analog for α-stratified modules of the classical BGG Theorem for Verma modules. The proofs follow the general lines of the proofs of corresponding results for Verma modules [BGG,D]. First we find some sufficient conditions for the inclusions of generalized Verma modules (Proposition 7.5) following [BGG, Theorem 3], [D, Lemma 7.6.13]. We prove the necessity of those conditions in Theorem 7.6 using the idea of tensoring with finite-dimensional modules. We can apply the proof of Theorem 2 in [BGG] thanks to the fact that the groups $(W,\mathcal{H}^*)$ and $(W_\alpha,\tilde{\Omega}_r)$ are similar in the category of transormation groups. We start with the following essential result.

Proposition 7.1 For any $(\lambda,p)\in\tilde{\Omega}_r$ the set $A=\{(\mu,q)\in\tilde{\Omega}_r|M(\mu-\lambda,q-p)\subset M(\mu,q)\}$ is closed in the Zariski topology in $\tilde{\Omega}_r$.

Proof. Let $(\mu,q)\in A$ and $\ell\in\mathbf{N}$. Then there exists $k=k(\ell)\in\mathbf{N}$ such that $M(\mu,q)_{\mu-\lambda-\rho+r\alpha}=U(N_-)X_{-\alpha}^{\ell}M(\mu,q)_{\mu+k\alpha-\rho}$.

Choose the following basis in $U(\mathcal{G})$:

$$X_{-\beta_1}^{k_1}\dots X_{-\beta_s}^{k_s}X_{-\alpha}^{t}H_{\alpha_1}^{\ell_1}\dots H_{\alpha_n}^{\ell_n}X_{\alpha}^{n}X_{\beta_1}^{n_1}\dots X_{\beta_s}^{n_s},\ \beta_i\in\Delta^+\setminus\{\alpha\},\ i=1,\dots,s.$$

Denote by π' the minimal subset of $\Delta^+\setminus\{\alpha\}$ containing $\pi\setminus\{\alpha\}$ that generates $\Delta^+\setminus\{\alpha\}$.

Let $0\neq u\in U(N_-)X_{-\alpha}^{\ell}$, $\beta\in\pi'$. Then $X_\beta u=u_0+\sum_{i=1}^{n}u_iH_{\alpha_i}+\sum_{i=1}^{r}u_{n+i}X_\alpha^i+v$, where $u_i\in U(N_-)$, $i=0,\dots,n+r$ and each basis element in v contains at least one X_γ with $\gamma\in\Delta^+\setminus\{\alpha\}$. For ℓ big enough we can assume the u_{n+i} contains $X_{-\alpha}^i$ for all $i=1,\dots,r$. Then $u_{n+i}X_\alpha^i=u'_{n+i}X_{-\alpha}^iX_\alpha^i=u'_{n+i}\zeta_i(c,H_\alpha)$ where $\zeta_i(c,H_\alpha)$ is some polynomial in c and H_α. Since $U(\mathcal{G})$ is a free $Z(\mathcal{G})$-module [D,Th.8.2.4], such u_i, $i=0,\dots,n$ and u'_{n+j}, $j=1,\dots,r$ are defined uniquely. Let

$$f_{(\mu,q)}^{\beta}(u)=u_0+\sum_{i=1}^{n}(\mu-\rho)(H_i)u_i+\sum_{j=1}^{r}u'_{n+j}\zeta_i(q^2,(\mu-\rho)(H_\alpha)).$$

Also there exist uniquely defined $u_0^{''}$, $u_1^{''}$, $u_2^{''}\in U(N_-)$ such that

$$(c-(q-p)^2)u=u_0^{''}+u_1^{''}H_\alpha+u_2^{''}(c-q^2).$$

We let $f_{(\mu,q)}^{c}(u)=u_0^{''}+u_1^{''}(\mu-\rho)(H_\alpha)$.

Denote $t=|\pi'|$ and define the linear map $g_{(\mu,q)}:U(N_-)X_{-\alpha}^{\ell}\to(U(N_-)^{t+1}$ for which

$$g_{(\mu,q)}(u)=(f_{(\mu,q)}^{\gamma_1}(u),\dots,f_{(\mu,q)}^{\gamma_t}(u),\ f_{(\mu,q)}^{c}(u)),\ \gamma_i\in\pi',\ i=1,\dots,t.$$

Denote

$$X=\{u\in U(N_-)X_{-\alpha}^{\ell}|[h,u]=-(\lambda+(k-r)\alpha)\ (h)u,\ \text{for all } h\in\mathcal{H}\}.$$

Obviously, $\dim X<\infty$.

Let $v_{\mu,k}\neq 0$ be an α-primitive element in $M(\mu,q)_{\mu+k\alpha-\rho}$. Then the condition $M((\mu,q)-(\lambda,p))\subset M(\mu,q)$ is equivalent to the existence of a non-zero element $v\in M(\mu,q)_{\mu-\lambda-\rho+r\alpha'}$ such that $N_+^{\alpha}v=0$ and $(c-(q-p)^2)v=0$. Thus it is equivalent to the existence of $u\in X\setminus\{0\}$ such that $N_+^{\alpha}uv_{\mu,k}=(c-(q-p)^2)uv_{\mu,k}=0$ and hence $f_{(\mu,q)}^{\beta}(u)v_{\mu,k}=f_{(\mu,q)}^{c}(u)v_{\mu,k}=0$ for all $\beta\in\pi'$. Since module $M(\mu,q)$

is $U(N_-)$-free we can rewrite the last condition in the form that there exists $u \in X \setminus \{0\}$, such that $f^{\beta}_{(\mu,q)}(u) = f^{c}_{(\mu,q)}(u) = 0$ for all $\beta \in \pi'$ which is equivalent to the fact that $\mathrm{rank}(g_{(\mu,q)}|X) < \dim X$. This means that some determinants, whose elements are the polynomial functions on μ and q, equal zero. The proposition is proved.

□

Lemma 7.2. Let $r,\ a \in \mathbf{C}$, $(\lambda, p) \in \Omega_r$, $\beta \in \Delta^+_{\alpha,r}$ and $\beta_a = \beta + (a\alpha, 0) \in \Delta^+_{\alpha,r+a}$.

(i) $(\beta, (\lambda, p))_r = (\beta, (\lambda + a\alpha, p))_r$.

(ii) $\sigma_{\beta_a}(\lambda + a\alpha, p) = \sigma_\beta(\lambda, p) + (a\alpha, 0)$.

(iii) If $p = \lambda(H_\alpha)$ then $\sigma_\beta(\lambda, p) = (\mu, \mu(H_\alpha))$ where $\mu = s_{\eta^{-1}(\beta)}(\lambda)$.

Proof. The statements (i) and (iii) can be checked by direct computations, (ii) follows from (i).

□

Lemma 7.3. Let $\beta \in \Delta^+ \setminus \{\alpha\}$, $(\lambda, \lambda(H_\alpha)) \in \tilde{\Omega}_r$, $\lambda(H_\beta) \in \mathbf{N}$ and $n \in \mathbf{Z}_+$. Then

$$M(\sigma_{\eta(\beta)}(\lambda - n\alpha, \lambda(H_\alpha))) \subset M(\lambda, \lambda(H_\alpha))$$

Proof. Let $\mu = s_\beta \lambda$. Then $\mu < \lambda$ and $M(\mu - n\alpha, \mu(H_\alpha)) \subset M(\lambda, \lambda(H_\alpha))$ by Proposition 3.1, (iii) and Lemma 3.6. But $\sigma_{\eta(\beta)}(\lambda - n\alpha, \lambda(H_\alpha)) = (s_\beta\lambda, (s_\beta\lambda)(H_\alpha)) - (n\alpha, 0) = (\mu - n\alpha, \mu(H_\alpha))$ by Lemma 7.2. This completes the proof.

□

Proposition 7.4. Let $\beta \in \Delta^+_{\alpha,r} \setminus \{\eta(\alpha)\}$, $(\lambda, p) \in \tilde{\Omega}_r$. If $(\beta, (\lambda, p))_r \in \mathbf{N}$ then $M(\sigma_\beta(\lambda, p)) \subset M(\lambda, p)$.

Proof. Consider the following two sets $A_\beta = \{(\mu, q) \in \tilde{\Omega}_r | M(\sigma_\beta(\mu, q)) \subset M(\mu, q)\}$ and $B_{k,\beta} = \{(\mu, q) \in \tilde{\Omega}_r | (\beta, (\mu, q))_r = k\}$, $k \in \mathbf{N}$. To prove our statement we will show that $B_{k,\beta}$ is a subset of A_β. One can easily check that for $n \in \mathbf{Z}_+$, $\gamma \in \pi_{\alpha,r}$ and $(\mu, \mu(H_\alpha)) \in \tilde{\Omega}_r$

$$(\gamma, (\mu - n\alpha, \mu(H_\alpha)))_r = \mu(H_{\eta^{-1}(\gamma)}).$$

Using the fact that η is a composition of the canonical bijection η' with the linear isomorphism $\eta^\circ_{\alpha,r}$ we conclude that $(\gamma,(\mu-n\alpha,\mu(H_\alpha)))_r=\mu(H_\eta^{-1}(\gamma))$ for any $n\in\mathbf{Z}_+$ and $\gamma\in\Delta^+_{\alpha,r}$. Thus, $(\mu-n\alpha,\mu(H_\alpha))\in B_{k,\beta}$ if and only if $\mu\in\tilde{B}_r$ and $\mu(H_{\eta^{-1}(\beta)})=k$. For any such μ we have $s_{\eta^{-1}(\beta)}(\mu)<\mu$. Therefore,

$$M(\sigma_\beta(\mu-n\alpha,\mu(H_\alpha)))\subset M(\mu,\mu(H_\alpha))\simeq M(\mu-n\alpha,\mu(H_\alpha))$$

by Lemma 7.3. Hence,

$$C_{k,\beta}=\left\{(\mu-n\alpha,\mu(H_\alpha))\in\tilde{\Omega}_r|\mu(H_{\eta^{-1}(\beta)})=k,\quad n\in\mathbf{Z}_+\right\}\subset B_{k,\beta}\cap A_\beta.$$

But $C_{k,\beta}$ is dense in $B_{k,\beta}$ in the Zariski topology. Thus, $B_{k,\beta}\subset A_\beta$ by Proposition 7.1 and the statement is proved.

□

Remark. If $\beta=\eta(\alpha)$ then $M(\sigma_\beta(\lambda,p))=M(\lambda,-p)=M(\lambda,p)$.

Definition. Let (λ,p), $(\mu,q)\in\Omega_r$ and $\beta\in\Delta^+_{\alpha,r}$.

(i) We will write $(\lambda,p)\xrightarrow{\beta}(\mu,q)$ if $(\mu,q)=\sigma_\beta(\lambda,p)$ and for $\beta\neq\eta(\alpha)$, $(\beta,(\lambda,p))_r\in\mathbf{N}$.

(ii) We say that a sequence $\beta,,...,\beta_k$ of elements of $\Delta^+_{\alpha,r}$ satisfies condition (A) for a pair $\{(\lambda,p),(\mu,q)\}$ and write $(\mu,q)<<(\lambda,p)$ if

$$(\lambda,p)\xrightarrow{\beta_1}\sigma_{\beta_1}(\lambda,p)\xrightarrow{\beta_2}\sigma_{\beta_2}\sigma_{\beta_1}(\lambda,p)\to\ldots\xrightarrow{\beta_k}\sigma_{\beta_k}\ldots\sigma_{\beta_1}(\lambda,p)=(\mu,q)$$

or $\lambda=\mu,\ \ p=q$.

Example. Let $\mathcal{G}=B_2$, $\pi=\{\alpha,\beta\}$, $|\alpha|>|\beta|$, $\Delta^+_{\alpha,0}=\{(0,1),(\beta,-1),(\beta,0),(\beta,1)\}$. Consider $(\lambda,p)\in\Omega_0$ such that $\lambda(H_\alpha+H_\beta)=k+\frac{1}{2}$, $k\in\mathbf{Z}_+$ and $k+\frac{1}{2}\pm p\notin\mathbf{N}$. Then $((\beta,0),(\lambda,p))_0=p+n^-_\beta(\lambda,p)=\lambda(H_\alpha+H_\beta)=k+\frac{1}{2}\notin\mathbf{N}$, $((\beta,-1),(\lambda,p))_0=n^-_\beta(\lambda,p)=k+\frac{1}{2}-p\notin\mathbf{N}$, $((\beta,1),\ (\lambda,p))_0=n^+_\beta(\lambda,p)=k+\frac{1}{2}+p\notin\mathbf{N}$. Thus, $\sigma_{(\beta,0)}(\lambda,p)=(\lambda-(2k+1)\beta,p)\not\ll(\lambda,p)$.

Proposition 7.5. Let $(\lambda,p),(\mu,q)\in\tilde{\Omega}_r$. If $(\mu,q)\ll(\lambda,p)$ then $M(\mu,q)\subset M(\lambda,p)$.

Proof. Let $(\mu, p) \ll (\lambda, p)$ and $(\mu, q) \neq (\lambda, p)$. Then there exists $\beta, ..., \beta_m \in \Delta^+_{\alpha,r}$ such that

$$(\lambda, p) \xrightarrow{\beta_1} \sigma_{\beta_1}(\lambda, p) \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_m} \sigma_{\beta_m} \ldots \sigma_{\beta_1}(\lambda, p) = (\mu, q).$$

Without loss of generality we may assume that $\beta_i \neq \eta(\alpha)$ for all i (see Remark after Proposition 7.4). Hence, $(\beta_i, \sigma_{\beta_{i-1}} \ldots \sigma_{\beta_1}(\lambda, p))_r \in \mathbf{N}$ for all $i = 2, ..., m$ and $(\beta_1, (\lambda, p))_r \in \mathbf{N}$ and thus Proposition 7.4 implies that

$$M(\mu, q) = M(\sigma_{\beta_m} \ldots \sigma_{\beta_1}(\lambda, p)) \subset M(\sigma_{\beta_{m-1}} \ldots \sigma_{\beta_1}(\lambda, p)) \subset \ldots \subset M(\sigma_{\beta_1}(\lambda, p)) \subset M(\lambda, p).$$

□

In particular Proposition 7.5 gives sufficient conditions for the existence of a submodule of type $M(\mu, q)$ in an α-stratified module $M(\lambda, p)$. We will show the necessity of these conditions. Our main result is the following theorem.

<u>Theorem 7.6</u>. Let (λ, p) and $(\mu, q) \in \tilde{\Omega}^s_r$. The following statements are equivalent.

(i) $M(\mu, q) \subset M(\lambda, p)$;

(ii) $L(\mu, q) \in \mathcal{JH}(M(\lambda, p))$;

(iii) There exists $k \in \mathbf{Z}$ such that $(\mu + k\alpha, q) \ll (\lambda, p)$.

Using Theorem 7.6 we obtain the following criterion for the irreducibility of the α-stratified modules $M(\lambda, p)$.

<u>Theorem 7.7</u>. Let $(\lambda, p) \in \tilde{\Omega}^s_r$. The module $M(\lambda, p)$ is irreducible if and only if $(\beta, (\lambda, p))_r \notin \mathbf{N}$ for all $\beta \in \Delta^+_{\alpha,r} \setminus \{\eta(\alpha)\}$.

Proof. If $(\beta, (\lambda, p))_r \in \mathbf{N}$ for some $\beta \in \Delta^+_{\alpha,r} \setminus \{\eta(\alpha)\}$ then $M(\sigma_\beta(\lambda, p)) \subset M(\lambda, p)$ by Proposition 7.5 and $M(\lambda, p)$ is reducible. Conversely, assume that $(\beta, (\lambda, p))_r \notin \mathbf{N}$ for all $\beta \in \Delta^+_{\alpha,r} \setminus \{\eta(\alpha)\}$. Then for all such β we have that $(\beta, (\lambda, -p))_r \notin \mathbf{N}$ by Lemma 5.6 and thus $(\lambda, \pm p) \not\to \sigma_\beta(\lambda, \pm p)$ which implies the irreducibility of $M(\lambda, p)$ by Theorem 7.6.

□

To prove the Theorem 7.6 we will need several Lemmas.

Lemma 7.8. [M2] Let V be an irreducible finite-dimensional $s\ell(2)$-module with highest weight $\tilde{p}$ and N be an irreducible weight $s\ell(2)$-module such that $(c - p^2)N = 0$. Then the eigenvalues of c on $V \otimes N$ are: $(p - \tilde{p})^2$, $(p - \tilde{p} + 2)^2, \ldots, (p + \tilde{p})^2$.

□

Lemma 7.9. Let F be a finite-dimensional $\mathcal{G}$-module and $(\lambda, p) \in \Omega$. Then there exists a filtration

$$0 \subset V_0 \subset V_1 \subset \ldots \subset V_k = M(\lambda, p) \otimes F$$

such that $V_i/V_{i-1} \simeq M(\lambda_i, p_i)$, where $\lambda_i - \lambda$ is a weight of F, $\quad i = 1, \ldots, k$.

Proof. Follows from Lemma 7.8 (cf. [BGG, Lemma 5] and [M2, Lemma 4.7]).

□

Denote by $\tilde{P}$ the set of all different highest weights of $s\ell(2)$-submodules in a finite-dimensional module F. Then in Lemma 7.9 $p_i^2 = (p - \tilde{p} + m)^2$ for some $\tilde{p} \in \tilde{P}$ and some $m \in \{0, 2, \ldots 2\tilde{p}\}$ by Lemma 7.8. We will assume that $p_i \in \{p - \tilde{p} + m | \tilde{p} \in \tilde{P}, \quad m = 0, 2, \ldots, 2\tilde{p}\}$ for all $i = 1, ..., k$.

Set $P(F) = \{(\lambda_i, p_i) - (\lambda, p), \quad i = 1, \ldots, k\}$. It is clear that $P(F)$ does not depend on (λ, p). It follows from Lemma 7.8 that if $x \in \Omega^s$ and $y \in P(F)$ for some finite-dimensional $\mathcal{G}$-module F then $x + y \in \Omega^s$.

Let $T \subset \tilde{\Omega}_r$ and $(\lambda, p) \in T$. We will say that (λ, p) is maximal in T if there is no $(\mu, q) \in T$ such that $(\mu, q) > (\lambda, p)$.

Lemma 7.10. Let $M \in K_\alpha$ and let

$$0 = M_0 \subset M_1 \subset \ldots \subset M_k = M$$

be a filtration such that $M_i/M_{i-1} \simeq M(\lambda_i, p_i), \quad i = 1, \dots k$. Suppose that $(\lambda, p) \in \{(\lambda_i, p_i), \quad i = 1, \dots k\}$ is maximal in $W_\alpha(\lambda, p) \cap \{(\lambda_i, p_i), \quad i = 1, \dots, k\}$. Then there is a submodule $N \subset M$ isomorphic to $M(\lambda, p)$.

Proof. If follows immediately that $M_{\lambda-\rho}$ contains a non-zero α-primitive element v such that $(c - p^2)v = 0$. Then $U(\mathcal{G})v \simeq M(\lambda, p)$ and Lemma is proved.

□

It follows from Lemma 7.9 or more generally from [Fe, Theorem 4.21] that if $(\lambda, p) \in \Omega$ and F is an irreducible finite-dimensional $\mathcal{G}$-module then the module $M(\lambda, p) \otimes F$ has a Jordan-Hölder series.

Lemma 7.11. Let $x \in \Omega$, F be a finite-dimensional $\mathcal{G}$-module and $y \in P(F)$. If $x + y$ is maximal in the set $W_\alpha(x + y) \cap (x + P(F) + \mathbf{Z}\alpha)$ then

$$L(x + y) \in \mathcal{JH}(L(x) \otimes F).$$

Proof. The proof is analogous to the proof of Lemma 7.6.16 in [D].

□

Lemma 7.12. Let $x, \xi \in \Omega$, $L(x) \in \mathcal{JH}(M(\xi)), F$ is a finite-dimensional $\mathcal{G}$-module, $y \in P(F)$ and $x + y$ is maximal in $W_\alpha(x + y) \cap (x + P(F) + \mathbf{Z}\alpha)$. Then there exists $\zeta \in P(F)$ such that

$$L(x + y) \in \mathcal{JH}(M(\xi + \zeta)).$$

Proof. It follows immediately from Lemmas 7.9 and 7.11 that

$$L(x + y) \in \mathcal{JH}(L(x) \otimes F) \subset \mathcal{JH}(M(\xi) \otimes F) \subset \cup_{\tau \in P(F)} \mathcal{JH}(M(\xi + \tau))$$

(cf. [D, Lemma 7.6.17]).

□

Let $r \in \mathbf{C}$ and $\Omega_{\mathbf{Z}} = \{(\lambda, p) \in \Omega | (\beta, (\lambda, p))_r \in \mathbf{Z} \text{ for all } \beta \in \pi_{\alpha,r}\}$. Obviously, $\Omega_{\mathbf{Z}}$ does not depend on r. One can also check that $(a\alpha, 0) + P(F) \subset \Omega_{\mathbf{Z}}$ for any finite-dimensional $\mathcal{G}$-module F and any $a \in \mathbf{C}$.

Denote $\Omega_{r,\mathbf{Z}} = \Omega_{\mathbf{Z}} \cap \tilde{\Omega}_r$.

Lemma 7.13. If $(\lambda, p) \in \Omega_{r,\mathbf{Z}}$ and $\beta \in \pi_{\alpha,r}$ then $\sigma_\beta(\lambda, p) \in \Omega_{r,\mathbf{Z}}$.

Proof. For any $\gamma \in \pi_{\alpha,r}$ we obtain using Lemma 7.2, (i) that

$$(\gamma, \sigma_\beta(\lambda, p))_r = (\gamma, (\lambda, p))_r - \frac{2(\beta, (\lambda, p))_r}{(\beta, \beta)_r}(\gamma, \beta)_r$$

which is an integer since $(\gamma, (\lambda, p))_r \in \mathbf{Z}$, $(\beta, (\lambda, p))_r \in \mathbf{Z}$ and $\dfrac{2(\gamma, \beta)_r}{(\beta, \beta)_r} \in \mathbf{Z}$. This completes the proof.

$\square$

For $r \in \mathbf{C}$ consider a linear space Ω_r with fixed point $(r\alpha, 0)$. Let $E_r = \cup_{\beta \in \Delta_{\alpha,r}} \{x \in \Omega_r | (\beta, Rex)_r = 0\} \subset \Omega_r$. We define the Weyl chambers in Ω_r as connected components in $\Omega_r \setminus E_r$. Let C be a Weyl chamber in Ω_r and $a \in \mathbf{C}$. Then $C_a = C + ((a - r)\alpha, 0)$ is a Weyl chamber in Ω_a. Set $\tilde{C} = \cup_{a \in \mathbf{C}} C_a$. We will call $\tilde{C}$ the Weyl chamber in Ω corresponding to C. Let C, C' be the Weyl chambers in Ω_r and $\tilde{C}$, $\tilde{C}'$ be the corresponding Weyl chambers in Ω. It follows from Lemma 7.2, (ii) that if $\sigma_\beta C = C'$ for some $\beta \in \Delta^+_{\alpha,r}$ then $\sigma_{\beta_{a-r}} C_a = C'_a$ for all $a \in \mathbf{C}$ where $\beta_{a-r} = \beta + ((a - r)\alpha, 0)$. We will simply write $\sigma_\beta \tilde{C} = \tilde{C}'$. Denote by $\bar{C}_a$ the closure of chamber C_a and set $\bar{C} = \cup_{a \in \mathbf{C}} \bar{C}_a$.

Lemma 7.14. Let C and C' be the Weyl chambers in Ω_r, $x \in C$, $x' \in C'$, $y \in \bar{C}$, $y' \in \bar{C}'$, $y - x \in \Omega_{r,\mathbf{Z}}$, $y' - x' \in \Omega_{r,\mathbf{Z}}$, $x' \in W_\alpha x$ and $y' \in W_\alpha y$. If $x' \ll x$ then $y' \ll y$.

Proof. Since $x' \ll x$ there exist $\beta_1, \ldots \beta_k \in \Delta^+_{\alpha,r}$ such that $x \xrightarrow{\beta_1} \sigma_{\beta_1} x \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_k} \sigma_{\beta_k} \ldots \sigma_{\beta_1} x = x'$. Thus if $\beta_i \neq \eta(\alpha)$ we have $(\beta_i, \sigma_{\beta_{i-1}} \ldots \sigma_{\beta_1} x)_r \in \mathbf{N}$. Denote $y_i = \sigma_{\beta_i} \ldots \sigma_{\beta_1} y$, $i = 1, \ldots, k$, We have by Lemma 7.2, (i) that if $\beta_i \neq \eta(\alpha)$ then $(\beta_i, y_{i-1})_r = (\beta_i, \sigma_{\beta_{i-1}} \ldots \sigma_{\beta_1}(y - x))_r + (\beta_i, \sigma_{\beta_{i-1}} \ldots \sigma_{\beta_1} x)_r$ and hence $(\beta_i, y_{i-1})_r \in \mathbf{N} \cup \{0\}$ since x and y belong to the same chamber and $y - x \in \Omega_{r,\mathbf{Z}}$. If $(\beta_i, y_{i-1})_r = 0$ then $y_i = y_{i-1}$. Let $i_1, \ldots, i_m \in \{1, \ldots, k\}$, $i_j < i_{j+1}$, $j = 1, \ldots, m$ be indexes that $(\beta_{i_j}, y_{i_{j-1}})_r \neq 0$ for all $j = 1, \ldots, m$. Then $y \xrightarrow{\beta_{i_1}} y_{i_1} \xrightarrow{\beta_{i_2}} \ldots \xrightarrow{\beta_{i_m}} y_{i_m} = y_k$. But $y_k \in \bar{C}' \cap W_\alpha y'$

and thus $\Re y_k = \Re y'$. Now using the fact that $y - x \in \Omega_{r,\mathbf{Z}}$ and $y' - x' \in \Omega_{r,\mathbf{Z}}$ we obtain that $y' - y_k \in \Omega_{r,\mathbf{Z}}$ and hence $\Im y' = \Im y_k$. One can conclude now that $y' = y_k$ and thus $y' \ll y$. Lemma is proved.

$\square$

Lemma 7.15 [BGG, Lemma 8]. Let C and C' be two neighbour chambers in Ω_r, $x \in C$, $y \in C'$ and $\beta \in \Delta^+_{\alpha,r}$ such that $(\beta, \Re\, x)_r < 0$, $(\beta, \Re\, y)_r > 0$. Then $\sigma_\beta C = C'$.

Lemma 7.16. Let $(\lambda, p) \in \Omega_{\mathbf{Z}}$. Then there exist $a \in \mathbf{C}$ and finite-dimensional irreducible $\mathcal{G}$-module F such that $(\lambda + a\alpha, p) \in P(F)$ and $\lambda' = \lambda + a\alpha$ is an extreme weight of F. In particular, if $(\lambda, p) \in \Omega_{\mathbf{Z}_+}$ then λ' is a highest weight of F.

Proof. Suppose that $(\lambda, p) \in \Omega_{\mathbf{Z}_+}$, i.e. $(\beta, (\lambda, p))_r \in \mathbf{Z}_+$ for all $\beta \in \pi_{\alpha,r}$. Denote $a = \frac{1}{2}p - \frac{1}{2}\lambda(H_\alpha)$ and $\lambda' = \lambda + a\alpha$. Then $\lambda'(H_\gamma) = (\eta(\gamma), (\lambda, p))_r \in \mathbf{Z}_+$ for all $\gamma \in \pi$ and in particular $\lambda'((H_\alpha) = p$. Consider a finite-dimensional irreducible module F with highest weight λ'. Then $(\lambda', p) \in P(F)$ and Lemma follows. The general case can be treated analogously.

$\square$

Proof of the Theorem 7.6. Our proof is analogous to the proof of BGG Theorem for Verma modules [BGG, Theorem 2; D, Theorem 7.6.23].

(i) $\Longrightarrow$ (ii). This implication is obvious.

(iii) $\Longrightarrow$ (i). Assume that $(\mu + k\alpha, q) \ll (\lambda, p)$ for some $k \in \mathbf{Z}$. Then $M(\mu + k\alpha, q) \subset M(\lambda, p)$ by Proposition 7.5 and hence $M(\mu, q) \subset M(\lambda, p)$ by Proposition 3.1, (iii).

(ii) $\Longrightarrow$ (iii). For $x \in \Omega_r^s$ denote by $Y_r(x)$ the following statement: for any $y \in \Omega_r^s$ such that $L(x) \in \mathcal{JH}(M(y))$ there exist $\beta_1, \ldots, \beta_m \in \Delta^+_{\alpha,r}$ satisfying

$$x = \sigma_{\beta_m} ... \sigma_{\beta_1} y \ll \sigma_{\beta_{m-1}} ... \sigma_{\beta_1} y \ll ... \ll \sigma_{\beta_1} y \ll y.$$

To show the implication (ii) $\Longrightarrow$ (iii) it is enough to prove $Y_r(x)$ for all $x \in \Omega_r^s$ by Proposition 3.3.

Let $\rho_r = \frac{1}{2} \sum_{\beta \in \Delta^+_{\alpha,r}} \beta$ and let $C^+ \subset \Omega^s_r$ be a Weyl chamber containing ρ_r. Obviously, $Y_r(x)$ holds for any $x \in \bar{C}^+$. One can reduce the general case to this one by the following three steps.

Step 1. Let $C \subset \Omega^s_r$ be a Weyl chamber, $x \in C$ and $\beta \in \Delta^+_{\alpha,r}$ such that $(\beta, \Re\, x)_r < 0$ and $\sigma_\beta C$ is a neighbour-chamber of C. Let F be a finite-dimensional $\mathcal{G}$-module such that $x + P(F) \subset \bar{C} \cup \sigma_\beta \bar{C}$ and let $y \in P(F)$. If $x + y \in \sigma_\beta \bar{C}_a$, $a \in \mathbf{C}$ then $Y_a(x+y)$ implies $Y_r(x)$.

Proof. Suppose that $\xi \in \Omega^s_r$ and $L(x) \in \mathcal{JH}(M(\xi))$. Since $x+y \in \sigma_\beta \bar{C}_a$ we have that $x+y$ is maximal in $W_\alpha(x+y) \cap \{x + P(F) + \mathbf{Z}\alpha\}$. Applying Lemma 7.12 we conclude that there exists $\zeta \in P(F) \cap \Omega_b$, $b \in \mathbf{C}$ such that $L(x+y) \in \mathcal{JH}(M(\xi+\zeta))$. Since $M(\xi+\zeta) \simeq M(\xi+\zeta')$, $\zeta' = \zeta + ((a-b-r)\alpha, 0)$, we obtain that $L(x+y) \in \mathcal{JH}(M(\xi+\zeta'))$. From $Y_a(x+y)$ we conclude that there exists a sequence of elements of $\Delta^+_{\alpha,a}$ that satisfies the condition (A) for a pair $\{\xi+\zeta', x+y\}$. Since the groups $(W, \mathcal{H}^*)$ and $(W_\alpha, \tilde{\Omega}_r)$ are similar as transdormation groups then using Theorem 6.5, Lemmas 7.13, 7.14 and 7.15, and applying the same arguments used in the proof of step 1 in [BGG, Theorem 2; D, Theorem 6.6.23] one can show the existence of a sequence of elements of $\Delta^+_{\alpha,a}$ satisfying the condition (A) for $\{\xi + ((a-r)\alpha, 0), x + ((a-r)\alpha, 0)\}$. Finally, Lemma 7.2 implies the existence of a sequence of elements of $\Delta^+_{\alpha,r}$ that satisfies the condition (A) for $\{\xi, x\}$, which completes the proof.

Step 2. Let $C \subset \Omega^s_r$ be a Weyl chamber, $x \in C$ and F be such finite-dimensional module that $x + P(F) \subset \tilde{C}$, the Weyl chamber in Ω corresponding to C. If $y \in P(F) \cap \Omega_a$, $a \in \mathbf{C}$ then $Y_{r+a}(x+y)$ implies $Y_r(x)$.

Proof. Analogous to the proof of Step 1.

The steps 1 and 2 imply the statement $Y_r(x)$ for all $x \in \Omega^s_r$ that are "far from the walls". Indeed, let d_r be the distance in $(\Omega^s_r)_{\mathbf{R}}$ and $x \in \Omega^s_r$ such that $d(\Re\, x, E_r) > 3|\rho_r|$. Then following [BGG] one can construct a sequence $x_0 = x, x_1, ..., x_k$ of elements of Ω^s_r such that $d(\Re\, x_i, E_r) > 2|\rho_r|$, $x_{i+1} - x_i \in \Omega_{\mathbf{Z}}$, $x_k \in C^+$ and one of the following two conditions holds:

(i) x_i and x_{i+1} belong to the neighbour chambers C and $\sigma_\beta C$ for $\beta \in \Delta^+_{\alpha,r}$, $(\beta, \Re\, x_i)_r < 0$ and $|x_{i+1} - x_i|$ is much less than the distance from x_i to any other Weyl chamber;

(ii) x_i and x_{i+1} belong to the same Weyl chamber and $|x_{i+1} - x_i| < 2|\rho_r|$.

We will say that $x_0, x_1, ..., x_k$ form an r-sequence. Let $x_i - x_{i-1} = (\lambda_i, p_i)$, $i = 1, ..., k$. Since $x_1 - x_0 \in \Omega_{\mathbf{Z}}$ it follows from Lemma 7.16 that there exists $a_1 \in \mathbf{C}$ such that $\lambda_1 + a_1\alpha$ is an extreme weight of a finite-dimensional irreducible $\mathcal{G}$-module F and $(\lambda_1 + a_1\alpha, p_1) \in P(F)$. Then using Step 1 or Step 2 we conclude that $Y_{r+a_1}(x_1 + (a_1\alpha, 0))$ implies $Y_r(x_0)$. Denote $x_i' = x_i + (a_1\alpha, 0)$, $i = 1, ..., k$. Then $x_1', x_2', ..., x_k'$ form an $(r+a_1)$-sequence. By induction on k we conclude that there exist $a_2, ..., a_k \in \mathbf{C}$ such that $Y_{r+a_1+...+a_k}(x_k+((a_1+...+a_k)\alpha, 0)) \Rightarrow ... \Rightarrow Y_{r+a_1+a_2}(x_2+((a_1+a_2)\alpha, 0)) \Rightarrow Y_{r+a_1}(x_1+(a_1\alpha, 0)) \Rightarrow Y_r(x_0)$. Since $x_k \in C^+$ the statement $Y_{r+a_1+...+a_k}(x_k + ((a_1 + ... + a_k)\alpha, 0))$ holds and $Y_r(x_0)$ follows.

Step 3. Let $x \in \Omega_r^s$, $\Re\, x \neq 0$. We will prove the statement $Y_r(x)$.

For $b > 0$ consider $D_{b,r} \subset (\Omega_r^s)_{\mathbf{R}}$ consisting of such elements t that the angle between t and $\Re\, x$ is less than b. We can choose b to be so small that a Weyl chamber $C \subset \Omega_r^s$ intersects $D_{b,r}$ if and only if $x \in \bar{C}$. Now choose $y \in \Omega_{0,\mathbf{Z}}$ such that $d(\Re\,(x+y), E_r) > 3|\rho_r|$ and $\Re\,(x+y)$ is maximal in $W_\alpha(\Re\,(x+y)) \cap D_{b,r}$. Let $y = (\lambda, p)$. It follows from Lemma 7.16 that there exist $a \in \mathbf{C}$ such that $\lambda + a\alpha$ is an extreme weight of a finite-dimensional irreducible module F and $(\lambda + a\alpha, p) \in P(F)$. Note that $d(\Re\,(x+y+(a\alpha, 0)), E_{r+a}) > 3|\rho_{r+a}|$ and $\Re\,(x+y+(a\alpha, 0))$ is maximal in $W_\alpha(\Re\,(x+y+(a\alpha, 0)) \cap D_{b,r+a}$. As in the proof of Step 3 of Theorem 2 in [BGG] one can show that $x+y+(a\alpha, 0)$ is maximal in $(W_\alpha(x+y+(a\alpha, 0))) \cap (x+P(F)+\mathbf{Z}\alpha)$ and thus we can apply Lemma 7.12. Suppose that $L(x) \in \mathcal{JH}(M(\xi))$, $\xi \in \Omega_r^s$. Then there exists $\zeta \in \Omega_a$ such that $L(x + y + (a\alpha, 0)) \in \mathcal{JH}(M(\xi + \zeta))$. Since $x + y + (a\alpha, 0)$ is "far from the walls" we conclude that $Y_{r+a}(x + y + (a\alpha, 0))$ holds and hence there exists a sequence of elements of $\Delta^+_{\alpha,r+a}$ that satisfies condition (A) for a pair $\{\xi + \zeta,\ x + y + (a\alpha, 0)\}$. Applying Lemma 7.14 we obtain that there exists a sequence of elements of $\Delta^+_{\alpha,r+a}$ satisfying condition (A) for a pair $\{\xi + (a\alpha, 0), x + (a\alpha, 0)\}$ and therefore by Lemma 7.2 there exists a sequence of elements of $\Delta^+_{\alpha,r}$ satisfying condition (A) for $\{x, y\}$. This proves the statement $Y_r(x)$ and completes the proof of the Theorem.

$\square$

References

[**BGG**] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, The structure of representations generated by vectors of highest weight, Funkt. Anal. i Prilozhen. 5 (1971), 1-9.

[**CF**] A.J. Coleman, V.M. Futorny, Stratified L-modules, J. Algebra 163 (1994), 219-234.

[**D**] J. Dixmier, "Algèbres enveloppantes", Gauthier–Villars, Paris, 1974; English translation, North-Holland, Amsterdam, 1977.

[**DFO**] Yu. A. Drozd, V.M. Futorny, S.A. Ovsienko, S-homomorphism of Harish-Chandra and $\mathcal{G}$-modules generated by semiprimitive elements, Ukrainian Math. J. 42 (1990), 1032-1037.

[**F1**] V.M. Futorny, A generalization of Verma modules and irreducible representations of the Lie Algebra $s\ell(3)$, Ukrainian Math. J. 38 (1986), 422-427.

[**F2**] V.M. Futorny, Weight $s\ell(3)$-modules generated by semiprimitive element, Ukrainian Math. J. 43 (1991), 281-285.

[**FP**] V. Futorny, D. Pollack, A new category of Lie Algebra modules with BGG reciprocity principle, Commun. in Algebra 22(1) (1994), 213-227.

[**GL**] H. Garland, J. Lepowsky, Lie algebra homology and the Macdonald-Kac formulas, Invent. Math. 34 (1976), 37-76.

[**K**] V.G. Kac, Infinite dimensional Lie algebras, Third edition, Cambridge University Press 1990.

[**M1**] V.S. Mazorchuk, α-stratified modules over Lie algebra $s\ell(n)$, Ukrainian Math. J. (45) (1993), 1215-1224.

[**M2**] V.S. Mazorchuk, On the structure of an α-stratified generalized Verma module over Lie Algebra $s\ell(n, \mathbf{C})$, Preprint 93-065, Universität Bielefeld.