CHARACTERIZATION OF SIMPLE HIGHEST WEIGHT MODULES

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Abstract. We prove that for simple complex finite dimensional Lie algebras, affine Kac-Moody Lie algebras, the Virasoro algebra and the Heisenberg-Virasoro algebra, simple highest weight modules are characterized by the property that all positive root elements act on these modules locally nilpotently. We also show that this is not the case for higher rank Virasoro and for Heisenberg algebras.

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1. Introduction

Trying to classify all modules over some algebra A, one often faces the following recognition problem: given some (simple) module M one has to determine whether M belongs to the class of already known modules. A common situation is when the module M is not given explicitly but rather by some general construction which allows one to derive easily some rough properties of M but does not really allow to see subtle properties of specific elements. It is therefore useful to have simple general characterizations for known classes of A-modules.

If A is the universal enveloping algebra of a Lie algebra with triangular decompositions, then one of the most classical families of A-modules is formed by the so-called highest weight modules, see e.g. [10] for details and examples. The aim of the present note is to prove the following main result which characterizes simple highest weight modules over certain Lie algebras with triangular decomposition.

Theorem 1. *Let $\mathfrak{g}$ be one of the following complex Lie algebras with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ in the sense of* [10]*:*

(*a*) *a semi-simple finite dimensional Lie algebra;*
(*b*) *an affine Kac-Moody Lie algebra;*
(*c*) *the Virasoro Lie algebra;*
(*d*) *the Heisenberg-Virasoro Lie algebra.*

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Let V be a $\mathfrak{g}$-module (not necessarily weight) on which every root element of the algebra $\mathfrak{n}_+$ acts locally nilpotently. Then we have the following:

(*i*) *The module V contains a nonzero vector v such that $\mathfrak{n}_+ v = 0$.*
(*ii*) *If V is simple, then V is a highest weight module.*

Note that the fact that every root element of the algebra $\mathfrak{n}_+$ acts locally nilpotently on every highest weight module is obvious. The proof of Theorem 1(i) is essentially combinatorial and is given in Section 2. The proof of Theorem 1(ii) relies on irreducibility of generic Verma modules and occupies Section 3. In Section 4 we show that for higher rank Virasoro and Heisenberg Lie algebras the claim of Theorem 1 is not true and discuss impossibility of certain natural relaxations of the conditions of the Theorem in general.

2. Proof of Theorem 1(i)

We denote by $\mathbb{N}$ the set of positive integers. We prove Theorem 1(i) using a case-by-case analysis of the cases (a)–(d).

2.1. **Case of finite dimensional Lie algebras.** In this subsection, we assume that $\mathfrak{g}$ is as in Theorem 1(a). In this case $\mathfrak{n}_+$ is finite dimensional and nilpotent. Hence there is a filtration

$$\mathfrak{n}_+ = \mathfrak{n}_0 \supset \mathfrak{n}_1 \supset \cdots \supset \mathfrak{n}_{\dim \mathfrak{n}_+} = 0$$

such that each $\mathfrak{n}_i$ is an ideal of $\mathfrak{n}_{i-1}$ of codimension 1, stable with respect to the adjoint action of $\mathfrak{h}$. We will use the backward induction on i to show that V contains a nonzero element annihilated by $\mathfrak{n}_i$ for all $i = 0, 1, \dots, \dim \mathfrak{n}_+$. Note that $\mathfrak{n}_{\dim \mathfrak{n}_+}$ is one-dimensional and generated by a root vector by our assumption on the filtration. Hence $\mathfrak{n}_{\dim \mathfrak{n}_+}$ acts locally nilpotently on V giving us the basis of our induction.

To prove the induction step, let $i > 0$ and $v \in V$ be such that $v \neq 0$ and $\mathfrak{n}_i v = 0$. Let $X \in \mathfrak{n}_{i-1} \setminus \mathfrak{n}_i$ be a root element. Then, by our assumptions, there exists a nonnegative integer $m \in \mathbb{N}$ such that $w = X^m v \neq 0$ and $X^{m+1} v = 0$.

We claim that $\mathfrak{n}_i w = 0$, in fact, we will prove that $\mathfrak{n}_i X^j v = 0$ for all $j = 0, 1, \dots, m$ by induction on j. The basis $j = 0$ of this induction follows from our assumptions. For the induction step for every $Y \in \mathfrak{n}_i$ we compute:

$$Y \cdot X^{j+1} v = Y \cdot X \cdot X^j v = X \cdot Y \cdot X^j v + [Y, X] \cdot X^j v.$$

In the latter expression, the first term is zero directly by induction, and the second term is zero by induction since $[Y, X] \in \mathfrak{n}_i$ (as $\mathfrak{n}_i$ is an ideal of $\mathfrak{n}_{i+1}$). The claim follows.

2.2. **Case of affine Kac-Moody Lie algebras.** In this subsection, $\mathfrak{g}$ is an affine Kac-Moody Lie algebra. Let α be the indivisible positive imaginary root. Then $\mathfrak{g}_\alpha$ is finite dimensional, commutative, and acts locally nilpotently on V. Hence V contains a nonzero v such that $\mathfrak{g}_\alpha\, v = 0$.

Now let β be a positive real root which cannot be written as a sum of some other positive real root and a positive imaginary root (we call such β elementary). If $\mathfrak{g}_\beta\, v = 0$, then $\mathfrak{g}_{\beta+i\alpha}\, v = 0$ for all $i \in \mathbb{N}$. If $\mathfrak{g}_\beta\, v \neq 0$, then $\mathfrak{g}_\alpha^i \mathfrak{g}_\beta\, v = 0$ for all $i \gg 0$ by our assumptions. Using $\mathfrak{g}_\alpha\, v = 0$, we obtain

$$\mathfrak{g}_\alpha \mathfrak{g}_\beta\, v = \mathfrak{g}_\beta \mathfrak{g}_\alpha\, v + [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]\, v = \mathfrak{g}_{\beta+\alpha}\, v$$

and, similarly, $\mathfrak{g}_\alpha^i \mathfrak{g}_\beta\, v = \mathfrak{g}_{\beta+i\alpha}\, v$. This implies that $\mathfrak{g}_{\beta+i\alpha}\, v = 0$ for all $i \gg 0$ in both cases. Since the number of elementary positive real roots is finite, we have such a statement simultaneously and uniformly for all of them.

Note that for any elementary root β the root $-\beta + \alpha$ is also elementary. Then for every elementary β and for all $i \gg 0$ all elements in $[\mathfrak{g}_{\beta+i\alpha}, \mathfrak{g}_{-\beta+\alpha+(i-1)\alpha}]$ annihilate v. Such elements generate $\mathfrak{g}_{k\alpha}$ for all $k \gg 0$. The sum of all $\mathfrak{g}_{k\alpha}$, $k \gg 0$, and all $\mathfrak{g}_{\beta+i\alpha}$, $i \gg 0$, for all elementary roots β, gives an ideal $\mathfrak{n}$ of $\mathfrak{n}_+$ of finite codimension. We have $\mathfrak{n}\, v = 0$. The algebra $\mathfrak{n}_+/\mathfrak{n}$ is finite dimensional and nilpotent. The proof is now completed similarly to the case of finite dimensional $\mathfrak{g}$, see Subsection 2.1.

2.3. **Case of the Virasoro algebra.** In this subsection, $\mathfrak{g}$ is the Virasoro algebra with basis $\{e_i : i \in \mathbb{Z}\} \cup \{c\}$, where c is central and the rest of the Lie brackets is given for $i, j \in \mathbb{Z}$ by

$$[e_i, e_j] = (j - i)e_{i+j} + \delta_{i,-j}\frac{i^3 - i}{12}c. \tag{2.1}$$

The algebras $\mathfrak{n}_\pm$ are spanned (over $\mathbb{C}$) by the elements $e_{\pm i}$, $i \in \mathbb{N}$, respectively.

As e_1 acts on V locally nilpotently, there is a nonzero $v \in V$ such that $e_1\, v = 0$. If $e_2\, v = 0$, then $\mathfrak{n}_+\, v = 0$ as e_1 and e_2 generate $\mathfrak{n}_+$. Assume $w = e_2\, v \neq 0$. Then

$$e_1\, w = e_1 e_2\, v = e_2 e_1\, v + [e_1, e_2]\, v = e_3\, v.$$

Similarly one shows that, up to a nonzero scalar, the element $e_1^i\, w$ coincides with $e_{2+i}\, v$. As e_1 acts on V locally nilpotently, it follows that there exists a positive integer $n \geqslant 2$ such that $e_i\, v = 0$ for all $i > n$.

We now show that for every positive integer m there exists a nonzero $u \in V$ such that $e_i\, u = 0$ for all $i > m$ by a backward induction on m. If $m \geqslant n$, the claim follows from the previous paragraph. Let us prove the induction step. Assume that $0 \neq u \in V$ is such that $e_i\, v = 0$ for all $i > m$. If $e_m\, u = 0$, the induction step is proved. Otherwise, for the vector $0 \neq u' := e_m\, u$ and for any $i > m$ we have

$$e_i\, u' = e_i e_m\, u = e_m e_i\, u + [e_i, e_m]\, u = 0$$

since $e_i\, u = 0$ by the inductive assumption and $[e_i, e_m]\, u = 0$ by the inductive assumption as $[e_i, e_m]$ equals e_{i+m} up to a nonzero scalar and $i + m > m$. Now the induction step follows from the fact that the action of e_m on V is locally nilpotent.

2.4. **Case of the Heisenberg-Virasoro algebra.** In this subsection, $\mathfrak{g}$ is the Heisenberg-Virasoro algebra (cf. [1]) with basis $\{e_i, z_i : i \in \mathbb{Z}\} \cup \{c_1, c_2, c_3\}$, where the c_i's are central and the rest of the Lie brackets is given for $i, j \in \mathbb{Z}$ by

$$\begin{array}{rcl} [e_i, e_j] & = & (j - i)e_{i+j} + \delta_{i,-j}\frac{j^3-j}{12}c_1; \\ [e_i, z_j] & = & j z_{i+j} - \mathbf{i} j^2 \delta_{i,-j} c_2; \\ [z_i, z_j] & = & j \delta_{i,-j} c_3 \end{array}$$

(here $\mathbf{i}$ is the imaginary unit). The algebras $\mathfrak{n}_\pm$ are spanned (over $\mathbb{C}$) by the elements $e_{\pm i}$ and $z_{\pm i}$, where $i \in \mathbb{N}$, respectively.

From the already considered case of the Virasoro algebra we know that there is a nonzero $v \in V$ such that $e_i\, v = 0$ for all $i \in \mathbb{N}$. If $z_1\, v = 0$, then $\mathfrak{n}_+\, v = 0$ (as $\mathfrak{n}_+$ is generated by e_1, e_2 and z_1) and we are done.

If $z_1\, v \neq 0$, then $e_1\, v = 0$ implies (similarly to analogous arguments used several times above) that $e_1^i \cdot z_1\, v$ equals $z_{i+1}\, v$ up to a nonzero scalar. Since e_1 acts locally nilpotently, it follows that there exists $m \in \mathbb{N}$ such that $z_i\, v = 0$ for $i > m$. For $k \in \mathbb{N}$ let $\mathfrak{n}_i$ denote the linear span of all e_i, $i \in \mathbb{N}$, and all z_i, $i \geqslant k$. Then $\mathfrak{n}_+ = \mathfrak{n}_1$ and each $\mathfrak{n}_i$, $i > 1$, is an ideal of $\mathfrak{n}_{i-1}$ of codimension one. Since we already know that $\mathfrak{n}_m\, v = 0$, the proof is completed similarly to the case of finite dimensional $\mathfrak{g}$, see Subsection 2.1.

3. Proof of Theorem 1(ii)

3.1. **The idea of the proof.** By Theorem 1(i), the module V contains a nonzero element v such that $\mathfrak{n}_+\, v = 0$. Hence there is a short exact sequence

$$0 \to K \to M \overset{\varphi}{\to} V \to 0,$$

where $M = U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+$ is the universal Verma module (see [14, 6]). We identify M with $U(\mathfrak{n}_- \oplus \mathfrak{h}) \cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{h})$ as a $U(\mathfrak{n}_- \oplus \mathfrak{h})$-module and denote by $\mathbf{v}$ the canonical generator of M. This identification equips M with the structure of a right $U(\mathfrak{h})$-module, which commutes with the left $U(\mathfrak{g})$-module structure by the universal property of Verma modules. To prove Theorem 1 it is enough to show that V is a weight module. The latter is equivalent to the claim that the ideal $I := K \cap U(\mathfrak{h})$ of $U(\mathfrak{h})$ is maximal. This is what we are going to prove in the rest of this section. Note that the ideal I coincides with the annihilator in $U(\mathfrak{h})$ of the element $\varphi(\mathbf{v})$.

3.2. **The action of $U(\mathfrak{h})$.** Let Δ be the root system of $\mathfrak{g}$ and $\mathbb{Z}\Delta$ the additive subgroup of $\mathfrak{h}^*$ spanned by Δ. Fix a basis in each root subspace of $\mathfrak{g}$ and a corresponding PBW basis $\{u_i : i \in P\}$ of $U(\mathfrak{n}_-)$ (here P is just an indexing set). This PBW basis is a basis of M as a right (and as a left) $U(\mathfrak{h})$-module.

For $\lambda \in \mathfrak{h}^*$ denote by $\Phi_\lambda : U(\mathfrak{h}) \to U(\mathfrak{h})$ the automorphism given by $\Phi_\lambda(h) = h + \lambda(h)$ for all $h \in \mathfrak{h}$. We have $\Phi_\lambda^{-1} = \Phi_{-\lambda}$. A monomial u_i is said to have weight $\lambda \in \mathbb{Z}\Delta$ provided that $fu_i = u_i\Phi_\lambda(f)$ for all $f \in U(\mathfrak{h})$. For $\lambda \in \mathbb{Z}\Delta$ we denote by P_λ the (finite) set of all indexes $i \in P$ for which u_i has weight λ.

For a $\mathfrak{g}$-module N and an ideal $J \subset U(\mathfrak{h})$ set

$$N_J := \{v \in N : J\,v = 0\}.$$

Then, for any $\alpha \in \Delta$ and any nonzero root element X_α we have

$$X_\alpha N_J \subset N_{\Phi_\alpha^{-1} J}. \tag{3.1}$$

Denote by $\hat{J}$ the set of all ideals in $U(\mathfrak{h})$ of the form $\Phi_\lambda\, J$, $\lambda \in \mathbb{Z}\Delta$.

Lemma 2. *Let N be a $\mathfrak{g}$-module.*

(*a*) *The set $\{\mathrm{Ann}_{U(\mathfrak{h})}(v) : v \in N, v \neq 0\}$ contains an element J, maximal with respect to inclusions.*
(*b*) *The ideal J above is prime.*
(*c*) *If N is simple, then $N \cong \oplus_{J' \in \hat{J}} N_{J'}$.*

Proof. Claim (a) follows from the fact that $U(\mathfrak{h})$ is noetherian. Let $v \in N$ be such that $\mathrm{Ann}_{U(\mathfrak{h})}(v) = J$. Assume that J is not prime and let $x, y \in U(\mathfrak{h}) \setminus J$ be such that $xy \in J$. Then $w := y\,v$ is nonzero as $y \notin J$, moreover, $\mathrm{Ann}_{U(\mathfrak{h})}(w) \supset J$ as $U(\mathfrak{h})$ is commutative, and further $\mathrm{Ann}_{U(\mathfrak{h})}(w)$ also contains $x \notin J$. This contradicts maximality of J, which implies claim (b).

If N is simple, it is generated by any nonzero element. Therefore, as all Φ_λ, $\lambda \in \mathbb{Z}\Delta$, are automorphisms, it follows that all corresponding ideals $\Phi_\lambda\, J$ are maximal in $\{\mathrm{Ann}_{U(\mathfrak{h})}(v) : v \in N\}$. Now claim (c) follows from (3.1). □

3.3. **The action of the center.** For $u = \sum_i u_i f_i \in M$ the set of all $\lambda \in \mathfrak{h}^*$ for which there exists $i \in P_\lambda$ such that $f_i \neq 0$ is called the *support* of u and denoted $\mathrm{supp}(u)$. The element u is called *homogeneous* if $|\mathrm{supp}(u)| \leqslant 1$.

Recall the following classical result (see for example [9, Proof of Proposition 3.1], or [8, Theorem 4.2.1(i)] for full details, see also [3, 2.6.5] for an alternative argument in the case of finite dimensional Lie algebras):

Lemma 3. *Any endomorphism of a simple module over a countably generated associative $\mathbb{C}$-algebra is scalar.*

Denote by $\mathfrak{z} \subset \mathfrak{h}$ the center of $\mathfrak{g}$. Definition of $\mathfrak{g}$ equips $\mathfrak{z}$ with a standard basis (see [10, 1]). As multiplication with central elements always define endomorphisms of a module, from Lemma 3 it follows that $I \cap U(\mathfrak{z})$ is a maximal ideal in $U(\mathfrak{z})$.

Note that for any homogeneous element $u \in M$ we have

$$U(\mathfrak{h})u = uU(\mathfrak{h}). \tag{3.2}$$

Our aim is to show that the ideal K is generated by homogeneous elements. If we could show this, then (3.2) would imply that K is stable under the right multiplication with $U(\mathfrak{h})$ and hence the latter must induce endomorphisms of V. Now Lemma 3 would imply that I must be maximal and we would be done.

3.4. **Reduction to Verma modules.** By a similar argument as in the proof of Lemma 2(b) we may assume that I is prime. Assume that K is not generated by homogeneous elements and let K_h be the submodule of K generated by all homogeneous elements (note that $K_h \supset U(\mathfrak{g})I$). Let $u \in K \setminus K_h$ be an element such that $\mathrm{supp}(u)$ has the minimal possible cardinality (note that $|\mathrm{supp}(u)| > 1$). Then $u = \sum_i u_i f_i$ and for every $\lambda \in \mathrm{supp}(u)$ the element $u_\lambda := \sum_{i\in P_\lambda} u_i f_i \notin K_h$ by the minimality of $\mathrm{supp}(u)$. Without loss of generality we may also assume that $f_i \notin I$ whenever $f_i \neq 0$.

As usual, for $\mu, \nu \in \mathfrak{h}^*$ we write $\mu \preceq \nu$ if $\nu - \mu$ can be written as a linear combination of positive roots with nonnegative integer coefficients. Assume additionally that $\mathrm{supp}(u)$ contains a λ which is maximal (with respect to $\preceq$) in the set of all possible elements of the support for all possible elements from $K \setminus K_h$ with support of the minimal possible cardinality. Let $\mu \neq \lambda$ be another element of $\mathrm{supp}(u)$. By the maximality of λ, for every positive root α and for any root element X_α in $\mathfrak{g}_\alpha$ we get $X_\alpha u \in K_h$, in particular, $X_\alpha u_\lambda \in K_h$ for every $\lambda \in \mathrm{supp}(u)$.

We have $\varphi(\mathbf{v}) \in V_I$ by our assumption. The element u_λ is homogeneous and does not belong to K_h. Hence it does not belong to K either and thus $u_\lambda\varphi(\mathbf{v}) \neq 0$, which implies that $u_\lambda\varphi(\mathbf{v}) \in V_{\Phi_\lambda^{-1}(I)}$ by (3.1). On the other hand, $u\varphi(\mathbf{v}) = 0$, which implies that

$$u_\lambda\varphi(\mathbf{v}) = -\sum_{\nu\in\mathrm{supp}(u)\setminus\{\lambda\}} u_\nu\varphi(\mathbf{v}).$$

The annihilator in $U(\mathfrak{h})$ of the element on the right hand side equals

$$I' := \bigcap_{\nu\in\mathrm{supp}(u)\setminus\{\lambda\}} \Phi_\nu^{-1}(I).$$

Therefore $\Phi_\lambda^{-1}(I) = I'$ is a prime ideal. Since all Φ_ξ are automorphisms, the ideals $\Phi_\lambda^{-1}(I)$ and $\Phi_\nu^{-1}(I)$ have the same height and the same depth, which implies $\Phi_\lambda^{-1}(I) = \Phi_\nu^{-1}(I)$ for all $\nu \in \mathrm{supp}(u) \setminus \{\lambda\}$. In particular, it follows that the ideal I is invariant under $\Phi_{\lambda-\mu}$ for $\lambda \neq \mu$ as fixed above.

Let $\mathbf{m}$ be any maximal ideal of $U(\mathfrak{h})$ containing I. Assume that $\mathbf{m}$ is given by $\lambda_{\mathbf{m}} \in \mathfrak{h}^*$. Then $U(\mathfrak{g})\mathbf{m} \supset U(\mathfrak{g})I$. Consider the Verma module $M(\lambda_{\mathbf{m}}) := M/U(\mathfrak{g})\mathbf{m}$ with highest weight $\lambda_{\mathbf{m}}$. Then $K_h + U(\mathfrak{g})\mathbf{m}$ is a submodule of $M(\lambda_{\mathbf{m}})$ and the intersection of this submodule with $U(\mathfrak{h})$ equals $\mathbf{m}$. Hence the corresponding quotient Q is nonzero. We may even choose $\mathbf{m}$ such that the images of the nonzero f_i's in both $u_\lambda = \sum_{i\in P_\lambda} u_i f_i$ and $u_\mu = \sum_{i\in P_\mu} u_i f_i$ are nonzero, which yields that the images of both u_λ and u_μ in Q are nonzero. Then Q contains a nonzero primitive vector of weight $\lambda_{\mathbf{m}} + \lambda$ and a nonzero primitive vector of weight $\lambda_{\mathbf{m}} + \mu$. By [10, 2.11], existence of a primitive vector in Q implies existence of a primitive vector in $M(\lambda_{\mathbf{m}})$ of the same weight.

3.5. **Completion of the proof.** As we have seen above, the ideal I is invariant under the action of $\Phi_{\lambda-\mu}$. This implies that the ideal I is generated by its intersection with $U(\mathfrak{h}')$, where $\mathfrak{h}'$ consists of all $h \in \mathfrak{h}$ such that $(\lambda - \mu)(h) = 0$. Indeed, let $\overline{h}$ be a nonzero element from the complement of $\mathfrak{h}'$ in $\mathfrak{h}$, which can be written as an integral linear combination of coroots. If $f \in I$, then $f - \Phi_{\lambda-\mu}(f) \in I$ and the latter has a strictly smaller degree with respect to $\overline{h}$.

The last paragraph means that, when choosing $\mathbf{m}$ above, we are free to choose any eigenvalue of $\overline{h}$ (the only requirement is that it should not kill the images of the u_λ and u_μ in Q, but this restriction means that a finite number of nonzero polynomials in $\overline{h}$ should not vanish). In particular, we can choose this eigenvalue to be a complex number, which is transcendental over the finite extension of $\mathbb{Q}$ given by adjoining eigenvalues of our fixed (finite) basis in $\mathfrak{z}$. Then the usual structure theory of Verma modules (see [1, 5, 10]) says that reducibility and submodules of Verma modules are controlled by vanishing of the Shapovalov form (see [12, 1, 5] or [10, 2.8]), whose determinant is given in terms of certain polynomials over $\mathfrak{h}$ with rational coefficients. This means that in the case the eigenvalue of $\overline{h}$ is as chosen above, it is not possible for a Verma module to have nonzero eigenvectors of weights $\lambda_{\mathbf{m}} + \lambda$ and $\lambda_{\mathbf{m}} + \mu$ at the same time. The obtained contradiction completes the proof.

4. Some (counter)examples

4.1. **Higher rank Virasoro algebras.** Let $G \subset \mathbb{R}$ be a nontrivial additive subgroup, which is not isomorphic to $\mathbb{Z}$, and Vir_G be the correspondent higher rank Virasoro algebra as in [13, 11]. It has basis $\{e_i : i \in G\} \cup \{c\}$, where c is central and the rest of the Lie brackets is given by (2.1). The subalgebra $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are spanned by e_i with $i > 0$ and $i < 0$, respectively. Let $M(0)$ be the Verma module over Vir_G

whose simple top $L(0)$ is the trivial Vir_G-module. Denote by $K(0)$ the kernel of the canonical epimorphism $M(0) \to L(0)$ (see [7]). In [4] it was shown that $K(0)$ is simple. Clearly, every root vector of $\mathfrak{n}_+$ acts locally nilpotently on $K(0)$. At the same time $K(0)$ does not have any highest weight. Indeed, its support coincides with the set of all negative elements of G. As $G \not\cong \mathbb{Z}$, G contains negative elements of arbitrarily small absolute value. This means that Theorem 1 does not hold for Vir_G.

4.2. **Heisenberg Lie algebra.** The Heisenberg Lie algebra $\mathfrak{H}$ has basis $\{z_i : i \in \mathbb{Z}\}$ and the Lie bracket is given by

$$[z_i, z_j] = j\delta_{i,-j} z_0.$$

The subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are spanned by z_i with $i > 0$ and $i < 0$, respectively.

Consider the set

$$I = \{\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) : \varepsilon_i \in \mathbb{N}; \varepsilon_i = 2 \text{ for all } i \gg 0\}$$

and let V have basis $\{v_\varepsilon : \varepsilon \in I\}$. For $i \in \mathbb{Z}$ and $\varepsilon \in I$ set

$$z_i\, v_\varepsilon = \begin{cases} v_\varepsilon, & i = 0; \\ 0, & i > 0 \text{ and } \varepsilon_i = 1; \\ v_{(\varepsilon_1,\dots,\varepsilon_{i-1},\varepsilon_i - 1,\varepsilon_{i+1},\varepsilon_{i+2},\dots)}, & i > 0 \text{ and } \varepsilon_i > 1; \\ i\varepsilon_i v_{(\varepsilon_1,\dots,\varepsilon_{i-1},\varepsilon_i + 1,\varepsilon_{i+1},\varepsilon_{i+2},\dots)}, & i < 0. \end{cases} \tag{4.1}$$

It is easy to check that this makes V into an $\mathfrak{H}$-module. Further $z_i^{\varepsilon_i}\, v_\varepsilon = 0$ and hence z_i acts on V locally nilpotently.

We claim that V is simple. Assume that this is not the case and let $W \subset V$ be a maximal submodule. Let $k \in \mathbb{N}$ be minimal such that W contains a nontrivial linear combination of basis elements with exactly k nonzero summands. Let $u \in V$ be such a combination. Note that V is clearly generated by any v_ε and hence $k > 1$. Then there exists $\varepsilon, \varepsilon' \in I$ and $i \in \mathbb{N}$ such that $\varepsilon_i < \varepsilon'_i$ and both v_ε and $v_{\varepsilon'}$ appear in u with nonzero coefficients. This implies that $z_i^{\varepsilon_i}\, u \in W$, on the one hand, is nonzero, but, on the other hand, contains less then k nonzero summands, a contradiction.

At the same time, every nonzero vector of V generates an infinite-dimensional $\mathfrak{n}_+$-submodule (applying z_i for $i \gg 0$). Hence Theorem 1 does not hold for $\mathfrak{H}$.

Remark 4. The construction above admits a straightforward generalization: for $\mathbf{e} \in \mathbb{N}^{\mathbb{N}}$ consider the set

$$I_{\mathbf{e}} = \{\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) : \varepsilon_i \in \mathbb{N}; \varepsilon_i = \mathbf{e}_i \text{ for all } i \gg 0\}$$

and define the $\mathfrak{H}$-module structure on the vector spaces $V_{\mathbf{e}}$ with basis $\{v_\varepsilon : \varepsilon \in I_{\mathbf{e}}\}$ using (4.1). Similarly to the above one shows that the module $V_{\mathbf{e}}$ is simple. These modules generalize simple $\mathfrak{H}$-modules constructed in [2, Section 6].

4.3. **Relaxing the conditions.** One might wonder whether conditions of Theorem 1 could be relaxed in some natural way (for example, by requiring the local nilpotency only for the generators of $\mathfrak{n}_+$). For instance, in the case of affine Lie algebras there are two types of roots, real and imaginary, and the algebra $\mathfrak{n}_+$ is generated by the real roots. It is therefore tempting to ask whether it would be enough in Theorem 1 to require that the action of all real root vectors is locally nilpotent. Unfortunately, this would affect the statement as the adjoint representation gives rise to a simple non-highest weight module on which all real root vectors acts locally nilpotently. We do not know which class of simple modules is described by the conditions relaxed in this way.

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