ON CATEGORIES OF GELFAND-ZETLIN MODULES

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1. The origins

Although the theory of Gelfand-Zetlin modules can be developed for all serial complex simple finite-dimensional Lie algebras and their (non-standard) quantum analogues, in this paper we will discuss the most classical case of the Lie algebra $\mathfrak{g} = \mathfrak{g}l(n, \mathbb{C})$ and will give a short overview of known results in other cases in the end of the paper. We will denote by $e_{i,j}$, $1 \leq i, j \leq n$, the matrix units and will always abbreviate Gelfand-Zetlin by GZ.

This theory starts from the famous original paper [9] by Gelfand and Zetlin, in which, using a step by step reduction to the smaller subalgebras, the authors constructed a very special and nice basis in each simple finite-dimensional $\mathfrak{g}$-module. It is well-known that simple finite-dimensional $\mathfrak{g}$-modules are parametrized by the vectors $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ with complex coefficients, satisfying $m_i - m_{i+1} \in \mathbb{N}$. These vectors represent the (shifted) highest weight of the corresponding simple module with respect to the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consisting of diagonal matrices. We will denote the simple module, which corresponds to $\mathbf{m}$, by $V(\mathbf{m})$. To formulate the result of Gelfand and Zetlin we have to introduce the notion of tableau. By a *tableau*, $[l]$, we will mean a doubly-indexed complex vector $(l_{i,j})$, where $1 \leq i \leq n$ and $1 \leq j \leq i$.

Theorem 1. $V(\mathbf{m})$ *possesses a basis, indexed by all tableaux* $[l]$*, satisfying the following conditions:* $l_{n,j} = m_j$, $1 \leq j \leq n$, *and* $l_{i,j} \geq l_{i-1,j} > l_{i,j+1}$, $1 < i \leq n$, $1 \leq j < i$. *Moreover, the action of the generators of* $\mathfrak{g}$ *in this*

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basis is given by the following Gelfand-Zetlin *formulae:*

$$e_{i,i+1}[l] = -\sum_{j=1}^{i} \frac{\prod_{k=1}^{i+1}(l_{i,j}-l_{i+1,k})}{\prod_{k\neq i}(l_{i,j}-l_{i,k})}[l+\delta^{i,j}],$$

$$e_{i+1,i}[l] = \sum_{j=1}^{i} \frac{\prod_{k=1}^{i-1}(l_{i,j}-l_{i-1,k})}{\prod_{k\neq i}(l_{i,j}-l_{i,k})}[l-\delta^{i,j}],$$

$$e_{i,i}[l] = \left(\sum_{j=1}^{i} l_{i,j} - \sum_{j=1}^{i-1} l_{i,j}\right)[l].$$

2. Generic Gelfand-Zetlin modules

The idea to use Theorem 1 to construct new $\mathfrak{g}$-modules goes back to Drozd, Ovsienko and Futorny ([4, 5]). This was based on the observation that GZ-formulae contain only rational functions in parameters, so, if one takes a set of tableaux, closed under the shifts, coming from the action of generators, such that all functions in GZ-formulae will be well-defined, the resulting space should be a $\mathfrak{g}$-module. This can be formally presented in the following statement.

Theorem 2. *Let* $[t]$ *be a tableau satisfying* $t_{i,j} - t_{i,k} \notin \mathbb{Z}$ *for all* $1 \leq i < n$ *and* $1 \leq j \neq k \leq i$. *Denote by* $P([t])$ *the set of all tableaux* $[l]$ *satisfying* $l_{n,j} = t_{n,j}$, $1 \leq j \leq n$ *and* $l_{i,j} - t_{i,j} \in \mathbb{Z}$ *for all possible* i, j. *Let* $V([t])$ *denote a vectorspace, where* $P([t])$ *is a basis. Then GZ-formulae define on* $V([t])$ *the structure of a* $\mathfrak{g}$*-module of finite length.*

Idea of the proof of the first statement. To prove the first part of the theorem (that V([t]) is a $\mathfrak{g}$-module) it is sufficient to check that any relation in $U(\mathfrak{g})$ is satisfied on $V([l])$. In our fixed basis $P([t])$ this relation can be rewritten as a collection of rational functions in entries of tableaux, which have to be shown to be zero. The last is easy cause finite-dimensional modules give sufficiently many points, in which these functions take zero values. The last argument uses crucially Theorem 1. □

To prove the second part we need to recall one more property of the GZ-basis of $V(\mathbf{m})$, which will lead us to the notion of Gelfand-Zetlin subalgebra.

3. Gelfand-Zetlin subalgebra

As we have already mentioned, Theorem 1 was obtained using step by step reduction to the smaller subalgebras. Now we make this statement more precise. We consider a chain of subalgebras

$$\mathfrak{gl}(1,\mathbb{C}) \subset \mathfrak{gl}(2,\mathbb{C}) \subset \cdots \subset \mathfrak{gl}(n,\mathbb{C})$$

embedded with respect to the left upper corner. This chain induces the chain of the corresponding universal enveloping algebras

$$U(\mathfrak{gl}(1,\mathbb{C})) \subset U(\mathfrak{gl}(2,\mathbb{C})) \subset \cdots \subset U(\mathfrak{gl}(n,\mathbb{C})).$$

Denote by Z_k the center $Z(\mathfrak{gl}(k,\mathbb{C}))$ of the algebra $U(\mathfrak{gl}(k,\mathbb{C}))$, $1 \leq k \leq n$.

The idea to get the GZ-basis of $V(\mathbf{m})$ was the following: we take $V(\mathbf{m})$ and consider it as $\mathfrak{gl}(n-1,\mathbb{C})$-module. The last is completely reducible and we can consider all components as $\mathfrak{gl}(n-2,\mathbb{C})$-module, decompose them and proceed till $\mathfrak{gl}(1,\mathbb{C})$. Now we recall that simple finite-dimensional $\mathfrak{gl}(k,\mathbb{C})$-modules are completely determined by their central character. It is also important that, if we decompose a simple finite-dimensional $\mathfrak{gl}(k,\mathbb{C})$-module into a direct sum of simple $\mathfrak{gl}(k-1,\mathbb{C})$ submodules, all latter will occur with multiplicity 1. Altogether this mean that the resulting GZ basis will be an eigenbasis for all algebras Z_k, or, in other words, for the commutative subalgebra $\Gamma \subset U = U(\mathfrak{gl}(n,\mathbb{C}))$, generated by all Z_k. Moreover, the remark about the multiplicities implies that Γ in fact separates the elements of the GZ-basis of $V(\mathbf{m})$.

Drozd, Ovsienko and Futorny called Γ the Gelfand-Zetlin subalgebra of U. It is well-known that Γ is a polynomial algebra in $n(n+1)/2$ variables. It was observed by Zhelobenko ([22]), that there is a set of generators, $\gamma_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq i$, of Γ such that the eigenvalue of the action of $\gamma_{i,j}$ on a tableaux, $[l]$, occurring in $V(\mathbf{m})$, should be computed as the j-th symmetric polynomial in variables $(l_{i,1}, l_{i,2}, \ldots, l_{i,i})$. Using the arguments analogous to that, presented in Section 2, one gets that the same is true in all $V([t])$.

Idea of the proof of the second statement of Theorem 2. As we saw, the basis $P([t])$ of $V([t])$ is an eigenbasis for Γ. Moreover, it is easy to get that Γ in fact separates the elements of $P([t])$. Hence, any submodule of $V([t])$ has a basis, which is a subset of $P([t])$. Now if one draws a graph with elements of $P([t])$ as vertices and joins thous pairs, who mutually appear with non-zero coefficients in GZ-formulae, one gets a graph with a finite number of connected components (this number can be easily computed). This finishes the proof. □

Remark that a complete proof of Theorem 2 can be found in [16].

4. Category of Gelfand-Zetlin modules

The introduction of GZ-subalgebra caused a natural definition of an abstract notion of Gelfand-Zetlin modules, analogous to the notion of the weight module. This was also done by Drozd, Ovsienko and Futorny. They proposed to call a *Gelfand-Zetlin module* any $\mathfrak{g}$-module, V, which decomposes into a direct sum of finite-dimensional modules, when viewed as Γ-module. Then by the category, $\mathcal{GZ}$, of Gelfand-Zetlin modules it is natural to understand the full subcategory of the category of all $\mathfrak{g}$-modules, consisting of all GZ-modules. As examples of Gelfand-Zetlin modules one can take finite-dimensional modules, $\mathfrak{h}$-weight modules with finite-dimensional weight spaces (in particular, all highest weight modules) or generic Gelfand-Zetlin modules.

Now we recall that tableaux naturally parameterize (not bijectively!) simple finite-dimensional Γ-modules, moreover, non-isomorphic Γ-simples do not have non-trivial extensions. Hence, any GZ-module, V, comes together with its *Gelfand-Zetlin support*, $\mathrm{gzsupp}(V)$, i.e. the set of all tableaux parameterizing all simple Γ-modules, occurring in V. We have to note that the product $G = S_1 \times S_2 \times \dots \times S_n$ of symmetric groups naturally acts on the space of all tableaux permuting the components in the rows. Any fundamental domain of this action bijectively parameterizes Γ-simples and, by definition, $\mathrm{gzsupp}(V)$ is invariant under this action. Hence the orbits of G acting on $\mathrm{gzsupp}(V)$ bijectively parameterize Γ-simples appearing in V.

Call two tableaus, $[l]$ and $[t]$, equivalent provided $l_{n,j} = t_{n,j}$ and $l_{i,j} - t_{i,j} \in \mathbb{Z}$ for all i, j. Let $\mathfrak{D}$ denote the set of equivalence classes of tableaux. First basic result about the category of Gelfand-Zetlin modules was the following statement, due to Drozd, Ovsienko and Futorny ([6]).

Theorem 3. *The category $\mathcal{GZ}$ decomposes into a direct sum,*

$$\mathcal{GZ} = \oplus_{P \in \mathfrak{D}} \mathcal{GZ}_P,$$

of full subcategories, where the category $\mathcal{GZ}_P$ consists of all Gelfand-Zetlin modules V such that $\mathrm{gzsupp}(V) \subset G \circ P$.

Proof. Is not difficult if one reminds that GZ-formulae preserve the equivalence classes of tableaux. □

In fact, Drozd, Ovsienko and Futorny embedded this special case of $U - \Gamma$ relative situation in a wide framework of *Harish-Chandra subalgebras*, which is very convenient (and very general) for study of the whole category of Gelfand-Zetlin modules. It is not our aim to discuss this approach and we refer the reader to the original paper [5].

5. A few theorems of Ovsienko

As soon as one has formulated the notion of a GZ-module, there is a natural and basic question arising: Is it true that each character of Γ can be continued to a $\mathfrak{g}$-module. Equivalently: is it true that each $\mathcal{G}Z_P$ is not empty. It is easy to answer "yes" for $n = 1, 2$. For $n = 3$ the same was prooved in [4]. The general case was recently completed by Ovsienko ([21]), but the paper has not appeared yet.

Theorem 4. *Each $\mathcal{G}Z_P$ is not empty.*

Idea of the proof. The proof is hard and technical. In fact, the result appears as a biproduct to a special geometrical statement. One should look at the image of Γ in $\mathrm{gr}(U)$. This image of $\{\gamma_{i,j}\}$ defines a certain algebraic variety, which is the variety of the so-called *strongly nilpotent matrices* (i.e. matrices, all main minors of which are nilpotent). The statement will follow from abstract nonsense if one proves that the sequence $\{\gamma_{i,j}\}$ is regular. The last can be derived if one proves that the variety of strongly nilpotent matrices is a complete intersection, i.e. that all the irreducible components of it have the same dimension. The last is the most difficult and technical part of the proof and is the main result of the mentioned paper of Ovsienko. $\square$

From Theorem 4 it follows that for any tableau $[l]$ there exists a simple GZ-module, V, such that $[l] \in \mathrm{gzsupp}(V)$. Using the convenient technique of Harish-Chandra subalgebras, mentioned above, Ovsienko managed to give much more useful information about simple GZ-modules.

Theorem 5. *1. For each $[l]$ there exists only finitely many (up to isomorphism) simple GZ-modules V with $[l] \in \mathrm{gzsupp}(V)$.*

2. Let V be a simple finite-dimensional $\mathfrak{g}$-module and F be a simple finite-dimensional Γ-module. Then the multiplicity of F in V (the last is viewed as Γ-module) is finite.

I have also to note that [21] contains a complete proof of the statement that Γ is a maximal commutative subalgebra of $U(\mathfrak{g})$. This statement can be found (without proof!) in all classical monographs (e.g. [22]). The proof in [21] is the first complete I have seen.

6. Generalized Verma modules and Gelfand-Zetlin modules

It seems that the first time, when it was understood that generic Gelfand-Zetlin modules are very convenient for computations was the paper [18], where the authors investigated the question about the structure of the so-called generalized Verma modules. Consider the inclusion $\mathfrak{gl}(k, \mathbb{C}) \subset$

$\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{g}$ with respect to the left upper corner. Let $\mathfrak{P}$ denote the parabolic subalgebra of $\mathfrak{g}$, generated by $\mathfrak{gl}(k,\mathbb{C})$ and the standard Borel subalgebra op upper-triangular matrices. Take a simple $\mathfrak{gl}(k,\mathbb{C})$-module, V, set that the rest of the Cartan subalgebra acts on it via some character, say λ, and the rest of the Borel subalgebra annihilates it. Thus V becomes a $\mathfrak{P}$-module. The induced module $M(V,\lambda) = U \otimes_{U(\mathfrak{P})} V$ is called a *generalized Verma module*. It turned out that taking V to be a simple generic GZ-module, $V([t])$, the structure of $M(V([t]),\lambda)$ can be described in terms of the Weyl group acting on the space of parameters, as it was done for the classical Verma modules by Bernstein, I.Gelfand and S.Gelfand ([2]).

It is trivial that $M(V([t]),\lambda)$ is a GZ-module over $\mathfrak{g}$. One can also see that it is generated by the elements (annihilated by the nilpotent radical of $\mathfrak{P}$), corresponding to the tableaux $[l]$, satisfying the following condition: $l_{i,j} = l_{i-1,j}$, $k < i \leq n$. The Weyl group S_n acts naturally on the set of such tableaux, permuting the elements of the upper row (which also causes the corresponding changes in all rows with $i > k$). For a transposition, $(i,j) \in S_n$, $i < j$, write $(i,j)[l] \leq [l]$ provided $l_{n,i} - l_{n,j} \in \mathbb{Z}_+$ and close the relation $\leq$ transitively. The next statement is the main result of [18].

Theorem 6. *Let $[l]$ (resp. $[l']$) be the tableau of a canonical generator of $M(V([t]),\lambda)$ (resp. $M(V([t']),\lambda')$). Assume that $l_{i,j} = l'_{i,j}$ for all $i < k$ and all j. Then the following statements are equivalent:*

1. *$M(V([t]),\lambda) \subset M(V([t']),\lambda')$.*
2. *The unique irreducible quotient of $M(V([t]),\lambda)$ is a composition subquotient of $M(V([t']),\lambda')$.*
3. *$[l] \leq [l']$.*

The proof of this theorem, presented in [18] goes the general line of the original proof in [2], but uses some calculations with generic GZ-modules. In particular, one of the mains things one needs here is a more or less precise description of $M(V([t]),\lambda)$ as a $\mathfrak{gl}(k,\mathbb{C})$-module. This question easily reduces to the calculation of $F \otimes V([t])$, where F is a simple finite-dimensional $\mathfrak{gl}(k,\mathbb{C})$-module. If one recalls that simple generic GZ-modules correspond to certain characters of Γ and the last one is generated by a sequence of centers, one can use the famous Theorem of Kostant ([12]), which tells how one can compute the action of the center on $F \otimes V([t])$. In this way one easily derives all potential subquotients of $F \otimes V([t])$. This (and existence of some of them, which is easy) was enough for the goals of Theorem 6.

7. Categories of $\mathfrak{gl}(n, \mathbb{C})$-modules generated by a simple generic Gelfand-Zetlin module

The necessity to study $F \otimes V([t])$ deeper was understood in [8], where some categories of Lie algebra modules where constructed, which are based on the categories of modules behaving well under tensoring with finite dimensional modules. As the main example of the latter, a category, generated by a simple generic GZ-module, was presented. Let $V([t])$ be a simple generic GZ-module. Denote by $\mathcal{C}([t])$ the full subcategory, consisting of all subquotients of modules $F \otimes V([t])$, where F is simple finite-dimensional. It turned out that this category has relatively easy structure.

Theorem 7. *$\mathcal{C}([t])$ decomposes into a direct sum of full subcategories, each of which is equivalent to the module category of a finite-dimensional associative and local algebra. In particular, $\mathcal{C}([t])$ has enough projective objects.*

Idea of the proof. One of the main ingredients of the proof is the following lemma:

Lemma 1. *The module $F \otimes V([t])$ has length $\dim(F)$, all simple subquotients of it are simple generic GZ-modules and the multiplicity of $V([s])$ in $F \otimes V([t])$, where $s_{i,j} = t_{i,j}$, $i < n$, equals $\sum \dim(F_\mu)$, where the sum is taken over all μ such that the vector $(t_{n,j})_{j=1,\dots,n} + \mu$ coincides with a permutation of $(s_{n,j})_{j=1,\dots,n}$.*

Lemma 1 is proved by a direct calculation, using GZ-formulae and the Littelwood-Richardson rule. It also represents a “generic behaviour” of simple generic GZ-modules in contrast with finite-dimensional modules.

After Lemma 1 one can first describe all simple modules in $\mathcal{C}([t])$. These will be $V([s])$, with $s_{i,j} - t_{i,j} \in \mathbb{Z}$. Then it is easy to find among them a projective module and prove the existence of projectives using the exactness os $F \otimes \_$. Decomposition with respect to central characters completes the proof. □

In two subsequent papers ([13, 14]) it was noticed that the category $\mathcal{C}([t])$ closely connected to various categories of $\mathfrak{g}$-modules, independently appeared in different contexts. The results of these two papers can be collected in the following statement.

Theorem 8. *Assume that $t_{n,j} \in \mathbb{Z}$ for all j. Then the following categories of $\mathfrak{g}$-modules are equivalent:*

1. *The category $\mathcal{C}([t])$.*
2. *The category of complete (in the sense of Enright, [7]) weight extensions of highest weight modules with integral support.*

3. *A certain category of algebraic Harish-Chandra bimodules in the sense of Bernstein and S.Gelfand ([1]).*

Idea of the proof. The equivalence of the first and the second categories is the content of [13]. It is based on a precise construction of the equivalence functor, which is a generalization of the Mathieu's twist functor ([15]). The equivalence of the second and the third categories is proved in [14], using an intermediate equivalence of the second category with a category of injectively copresented modules in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ ([3]). □

8. Case of classical and quantum algebras and open problems

An analogue of Theorem 1 for orthogonal algebras (simple finite dimensional complex Lie algebras of type B_n and D_n) was obtained also by Gelfand and Zetlin in [10]. The corresponding generic modules were constructed in [17]. For symplectic Lie algebras (type C_n) an analogue of Theorem 1 is a recent result of Molev, [20]. For $U_q(\mathfrak{gl}_n)$ the classical result was obtained by Jimbo ([11]) and generic modules were constucted by Turowska and the author ([19]). For non-standard quantum deformations of orthogonal algebras the classical construction of Gelfand-Zetlin basis in finite-dimensional modules can be found in a series of recent papers by Klimyk and Jorgov, available at "xxx.lanl.gov", where one can also find information about corresponding results for root of unity case.

Finally, we want to give a list of some questions and open problems related to Gelfand-Zetlin modules:

1. Classify and give a precise construction of all simple GZ-modules.
2. Find a criterion, when a given character of Γ has only one extension to a simple $\mathfrak{g}$-module.
3. Let F be a simple finite dimensional $\mathfrak{gl}(n,\mathbb{C})$-module. Consider two Gelfand-Zetlin basis of it, with respect to the inclusions of subalgebras into left upper and into right lower corners. What will be the transformation matrix?
4. Let V be a simple Gelfand-Zetlin module and F be a finite-dimensional module. Does $V \otimes F$ have a finite length? Is it possible to compute composition subquotients and multiplicities of $V \otimes F$?
5. Are there any analogues of Gelfand-Zetlin construction for exceptional Lie algebras?
6. Extend all already known for $\mathfrak{gl}(n,\mathbb{C})$ results to the case of orthogonal and symplectic algebras. Also find in those cases solutions to the above problems.

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