Radical ∗-doubles of finite-dimensional algebras

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Abstract

We classify the ∗-representation types for the radical ∗-doubles of finite-dimensional associative algebras over the field of complex numbers.

1 Introduction

Let \( \mathbb{C} \) denote the field of complex numbers and \( \overline{\cdot} : \mathbb{C} \to \mathbb{C} \) denote the complex conjugation. All algebras we consider in the present paper are over \( \mathbb{C} \) and are assumed to have a unit element. All tensor products and dimensions are taken over \( \mathbb{C} \).

Recall that for two complex associative algebras \( A \) and \( B \) the map \( \varphi : A \to B \) is called an anti-homomorphism provided that \( \varphi(\lambda a + \mu b) = \overline{\lambda} \varphi(a) + \overline{\mu} \varphi(b) \) and \( \varphi(ab) = \varphi(b) \varphi(a) \) for all \( \lambda, \mu \in \mathbb{C} \) and \( a, b \in A \).

Let \( A \) be a finite-dimensional algebra with \( n \) generators. Consider two free associative \( \mathbb{C} \)-algebras \( A_n^{(\varepsilon)} \), \( \varepsilon = 1, 2 \), with respective generators \( x_1^{(\varepsilon)}, \ldots, x_n^{(\varepsilon)} \). Denote by \( \sigma : A_n^{(1)} \to A_n^{(2)} \) the unique anti-homomorphism satisfying \( \sigma(x_j^{(1)}) = x_j^{(2)} \) for all \( j = 1, \ldots, n \). Let \( I \) be an ideal of \( A_n^{(1)} \), such that \( A \cong A_n^{(1)}/I \). Then the set \( \sigma(I) \) is an ideal in \( A_n^{(2)} \) and we can consider the algebra \( A^* = A_n^{(2)}/\sigma(I) \). It is easy to see that \( A^* \) does not depend on the presentation of \( A \) up to an isomorphism.

Construct now a new algebra, \( A(*) \), which is the quotient of the free product \( A_n^{(1,2)} \) of \( A_n^{(1)} \) and \( A_n^{(2)} \) (i.e., the free algebra with \( 2n \) generators \( x_1^{(1)}, \ldots, x_n^{(1)}, x_1^{(2)}, \ldots, x_n^{(2)} \)), modulo the ideal \( J \), which is generated by \( I \) and \( \sigma(I) \). The algebra \( A(*) \) is identified with the free product (over \( \mathbb{C} \)) of \( A \) and \( A^* \) in a natural way. The algebra \( A_n^{(1,2)} \) possesses a natural ∗-structure, defined by \( \langle x_j^{(1)} \rangle^* = x_j^{(2)}, j = 1, \ldots, n \), and one sees that \( J \) is a ∗-ideal with respect to this structure. Hence, \( A(*) \) inherits a ∗-structure and the corresponding ∗-algebra is called the ∗-double of \( A \), see [MT]. It is easy to see that, up to a ∗-isomorphism, the algebra \( A(*) \) does not depend on the presentation of \( A \). The ∗-representation types of ∗-doubles of finite-dimensional algebras were classified in [MT]. It was shown that \( A(*) \) is ∗-finite if and only if \( A \cong \mathbb{C} \), \( A(*) \) is of type I if and only if \( \dim(A) \leq 2 \), and \( A(*) \) is ∗-wild (in the sense of [OS2]) in all other cases.

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In the present paper we study the $*$-representation types of a more subtle construction, which we call the radical $*$-doubling. The difference is that for the usual $*$-doubling we add independent $*$-adjoints to all elements of the original algebra, whereas for the radical $*$-doubling we add independent $*$-adjoints only to the elements from the Jacobson radical of the algebra, preserving the natural $*$-structure on a maximal semi-simple subalgebra. We classify the $*$-representation type for the radical $*$-doubles of all finite-dimensional algebras, and the answer we obtain is much more interesting than that of [MT]. The principal advantage of the new construction is that the $*$-representation type of the radical $*$-doubles happens to be a Morita invariant of the original algebra. The list of those $\mathcal{A}$, whose radical $*$-doubles are of type I, is also much more interesting and contains all semi-simple algebras and all finite-dimensional algebras, the length of indecomposable modules over which is bounded by 2. As a consequence we also obtain a tame-wild dichotomy for our problem (which is not automatic in the $*$-case in contrast with the usual finite-dimensional associative algebras, for which it was proved by Drozd, [Dr], in a very general setup). Some analogous problems were earlier considered in [Be1, Be2, Se], see also [OS2] and the references therein.

The paper is organized as follows: in the next section we present a rigorous definition of the radical $*$-double of a finite-dimensional algebra, in Section 3 we recall basic facts about the $*$-representation types, in Section 4 we formulate our main result, which classifies the $*$-representation types of the radical $*$-doubles of finite-dimensional algebras. The rest of the paper is devoted to the proof of the main result, which is spread over three sections. In Section 5 we collected several auxiliary lemmas classifying the $*$-representation types of the radical $*$-doubles of certain finite-dimensional algebras. In Section 6 we establish the Morita invariance of the $*$-representation types of the radical $*$-doubles. The latter study has led us to a very interesting question, which seems to be both quite natural and rather nontrivial: describe, up to unitary equivalence, all projections in the full matrix algebra $M_n(C^*(\mathcal{F}_2))$, where $\mathcal{F}_2$ is a free group with 2 generators. The answer to this question would substantially clarify the notion of $*$-wildness in the sense of [OS2], see Remark 4. Finally, the proof of our main result is completed in Section 7.

2 Radical $*$-doubling

In this section we give a thorough definition of the intuitive construction of the radical $*$-doubling, described in Section 1.

Let $\mathcal{A}$ be a finite-dimensional associative complex algebra, $S$ be a maximal semi-simple subalgebra of $\mathcal{A}$ and $\text{Rad} (\mathcal{A})$ be the Jacobson radical of $\mathcal{A}$. Then $\mathcal{A}$ decomposes, as a complex vector space, into a direct sum $\mathcal{A} = S \oplus \text{Rad} (\mathcal{A})$. Note that $\mathcal{A}$ is not isomorphic to the direct sum $S \oplus \text{Rad} (\mathcal{A})$ of associative algebras.

Being semi-simple, the algebra $S$ admits a decomposition into a direct sum of full matrix algebras $M_{n_i}(\mathbb{C})$, $n_i \in \mathbb{N}$, by the Wedderburn-Artin Theorem. Every $M_{n_i}(\mathbb{C})$ has a natural $*$-structure associated with the transposition of a matrix. In every $M_{n_i}(\mathbb{C})$ we can choose a standard basis, consisting of matrix units. Let $\{b_1, \ldots, b_s\}$ be a list of all
diagonal matrix units in all $M_{n_i}(\mathbb{C})$ (they are self-dual with respect to $*$, i.e. $b_i^* = b_i$, $i = 1, \ldots, s$), and $\{c_1, \ldots, c_1\}$ be a list of all upper triangular matrix units in all $M_{n_1}(\mathbb{C})$. Then $\{c_1^*, \ldots, c_1^*\}$ will be a list of all lower triangular matrix units in all $M_{n_1}(\mathbb{C})$. Note that the basis, constructed above, is closed with respect to $*$.

Fix some basis, $\{a_1, \ldots, a_k\}$, in $\text{Rad}(A)$, and let $\mathcal{B}$ denote the basis of $A$, formed as the union of the bases for $S$ and $\text{Rad}(A)$, which we have just fixed. For $x, y, z \in \mathcal{B}$ let $\alpha_{x,y}^z \in \mathbb{C}$ be the corresponding structural constant, i.e. for $x, y \in \mathcal{B}$ we have

$$xy = \sum_{z \in \mathcal{B}} \alpha_{x,y}^z z.$$ 

Denote by $A(\text{Rad} - *)$ the associative algebra, generated over $\mathbb{C}$ by the elements from $\mathcal{B} \cup \{a_1^*, \ldots, a_k^*\}$, subject to the following relations:

$$xy = \sum_{z \in \mathcal{B}} \alpha_{x,y}^z z,$$  

$x, y \in \mathcal{B};$

$$xy = \sum_{z \in \mathcal{B}} \alpha_{x,y}^z z^*,$$  

$x \in \mathcal{B} \setminus \{a_1, \ldots, a_k\}, y \in \{a_1^*, \ldots, a_k^*\};$

$$xy = \sum_{z \in \mathcal{B}} \alpha_{x,y}^z z^*,$$  

$x \in \{a_1^*, \ldots, a_k^*\}, y \in \mathcal{B} \setminus \{a_1, \ldots, a_k\}.$

We will call the algebra $A(\text{Rad} - *)$ the \textit{radical $*$-double} of the algebra $A$. It is straightforward that $A(\text{Rad} - *)$ inherits a natural $*$-structure from that on $\mathcal{B} \cup \{a_1^*, \ldots, a_k^*\}$. It is an easy (but quite lengthy) exercise to show that, up to a $*$-isomorphism, $A(\text{Rad} - *)$ does not depend neither on the presentation of $A$, nor on the choice of $S$, nor on the choice of $\{a_1, \ldots, a_k\}$. For example, if $A = \mathbb{C}[x]/(x^2)$, then $A(\text{Rad} - *) \cong A(*)$ is the quotient of the free algebra $\mathbb{C}\langle x, y \rangle$ modulo relations $x^2 = y^2 = 0$ with the involution given by $x^* = y$.

Both $A$ and $A^*$ are subalgebra of $A(\text{Rad} - *)$ in a natural way. However, in contrast with $A(*)$, $A(\text{Rad} - *)$ is no longer a free product of $A$ and $A^*$ over $\mathbb{C}$, but rather a free product of $A$ and $A^*$ over the "common subalgebra" $S$. Remark that $S$ can be arbitrary semi-simple finite-dimensional algebra. In particular, $S$ can be non-commutative. Neither is $S$ central in $A$ in general. However $A(\text{Rad} - *) \cong A(*)$ (as $*$-algebras) in the case when $A$ is local and basic.

3 Basic definitions and facts about the $*$-representation types

In this section we list some notation and definitions related to $*$-wild and $*$-tame algebras. In this exposition we follow [KS2, OS2]. All $*$-algebras considered here are unital with the unit $1$ and representations of $*$-algebras are unital $*$-homomorphisms into $B(H)$, the $*$-algebra of all linear bounded operators on a separable Hilbert space $H$. For a $*$-algebra, $\mathcal{A}$, we denote by $\text{Rep}(\mathcal{A})$ the category of all $*$-representation of $\mathcal{A}$. Given a $*$-algebra,
$\mathcal{A}$, of operators on $H$, denote by $\mathcal{A}'$ its commutant, i.e. $\mathcal{A}' = \{ C \in B(H) \mid [C, A] = 0 \text{ for every } A \in \mathcal{A} \}$.

**Definition 1.** Let $\mathcal{A}$ be a $*$-algebra. A pair, $(\bar{\mathcal{A}}; \varphi: \mathcal{A} \to \bar{\mathcal{A}})$, where $\bar{\mathcal{A}}$ is a $*$-algebra and $\varphi$ is a unital $*$-homomorphism, is called an *enveloping $*$-algebra* of the algebra $\mathcal{A}$ if for any $*$-representation $\pi: \mathcal{A} \to B(H)$ of $\mathcal{A}$ there exists a unique $*$-representation $\bar{\pi}: \bar{\mathcal{A}} \to B(H)$ such that the diagram

$$
\begin{array}{ccc}
\bar{\mathcal{A}} & \xrightarrow{\bar{\pi}} & B(H) \\
\pi \downarrow & & \downarrow \pi \\
\mathcal{A} & \xrightarrow{\pi} & B(H)
\end{array}
$$

is commutative, and any operator $X: H_1 \to H_2$ which intertwines representations $\pi_1: \mathcal{A} \to B(H_1)$ and $\pi_2: \mathcal{A} \to B(H_2)$ of $\mathcal{A}$ is also an intertwining operator for the representations $\bar{\pi}_1$ and $\bar{\pi}_2$ of the algebra $\bar{\mathcal{A}}$.

It is easy to see that $(\mathcal{A}; \text{Id}: \mathcal{A} \to \mathcal{A})$ is an enveloping $*$-algebra of $\mathcal{A}$.

Let $M_n(\mathcal{A})$ ($= M_n(\mathbb{C}) \otimes \mathcal{A}$) be the full matrix algebra over $\mathcal{A}$ with the natural $*$-structure. If $\mathcal{A}$ is a $C^*$-algebra then $M_n(\mathcal{A})$ carries also the structure of a $C^*$-algebra. Any representation $\pi: \mathcal{A} \to B(H)$ of $\mathcal{A}$ induces the representation $\pi_n: M_n(\mathcal{A}) \to B(H \oplus \ldots \oplus H)$ of the algebra $M_n(\mathcal{A})$. The representation $\pi_n$ determines the representation $\bar{\pi}_n$ of an enveloping algebra, $(\bar{M}_n(\mathcal{A}), \varphi)$, of $M_n(\mathcal{A})$ on the same Hilbert space. If $\psi$ is a unital $*$-homomorphism of a $*$-algebra $\mathcal{B}$ to the algebra $M_n(\mathcal{A})$ then $\bar{\pi}_n \circ \psi$ defines a representation of $\mathcal{B}$. So we can define a functor, $F_\psi: \text{Rep}(\mathcal{A}) \to \text{Rep}(\mathcal{B})$, in the following natural way:

- $F_\psi(\pi) = \bar{\pi}_n \circ \psi$, for every $\pi \in \text{Rep}(\mathcal{A})$,
- $F_\psi(c) = \text{diag}(c, \ldots, c)$ for a morphism, $c: \pi_1 \to \pi_2$, of representations $\pi_1$, $\pi_2$ of $\mathcal{A}$.

**Definition 2.** We say that a $*$-algebra, $\mathcal{B}$, *majorizes* a $*$-algebra, $\mathcal{A}$, denoted by $\mathcal{B} \succ \mathcal{A}$, if there exist $n \in \mathbb{N}$, an enveloping algebra, $\bar{M}_n(\mathcal{A})$, of the algebra $M_n(\mathcal{A})$, and a $*$-homomorphism, $\psi: \mathcal{B} \to M_n(\mathcal{A})$, such that the functor $F_\psi: \text{Rep}(\mathcal{A}) \to \text{Rep}(\mathcal{B})$ is full.

We say that $\mathcal{B}$ *strongly majorizes* $\mathcal{A}$ ($\mathcal{B} \succ^s \mathcal{A}$) if there exist $n \in \mathbb{N}$ and a $*$-homomorphism, $\psi: \mathcal{B} \to M_n(\mathcal{A})$, such that the functor $F_\psi: \text{Rep}(\mathcal{A}) \to \text{Rep}(\mathcal{B})$ is full.

Note that to define strong majorization we consider $M_n(\mathcal{A})$ as an enveloping $*$-algebra of $M_n(\mathcal{A})$.

Clearly, $\mathcal{B} \succ \mathcal{A} \Rightarrow \mathcal{B} \succ \mathcal{A}$. Note that both the majorization and the strong majorization are quasi-order relations: $\mathcal{C} \succ \mathcal{B}$ and $\mathcal{B} \succ \mathcal{A}$ imply $\mathcal{C} \succ \mathcal{A}$, and $\mathcal{C} \succ^s \mathcal{B}$ and $\mathcal{B} \succ^s \mathcal{A}$ imply $\mathcal{C} \succ^s \mathcal{A}$.

It follows easily from the definition that if $\mathcal{B} \succ \mathcal{A}$ then two representations $\pi_1$, $\pi_2$ of $\mathcal{A}$ are unitarily equivalent if and only if the representations $F_\psi(\pi_1)$, $F_\psi(\pi_2)$ of $\mathcal{B}$ are unitarily equivalent, a representation $\pi$ of $\mathcal{A}$ is irreducible if and only if the representation $F_\psi(\pi)$ is irreducible. Thus the problem of unitary classification of the representations of the $*$-algebra $\mathcal{B}$ contains, as a subproblem, the problem of unitary classification of the representations of the $*$-algebra $\mathcal{A}$.
Practically, in order to verify that the functor $F_{\psi}$ is full, it is sufficient to show that for each representation $\pi \in \text{Rep} \mathcal{A}$ on $H$ and $C \in B(H)$ the inclusion $C \in F_{\psi}(\pi)(B)'$ implies $C = \text{diag}(c, \ldots, c)$, where $c \in \pi(\mathcal{A})$.

Let $\mathbb{C}[\mathcal{F}_2]$ denote the group $*$-algebra of the free group $\mathcal{F}_2$ with two generators, $u, v,$ and involution defined on the generators in the usual way: $u^* = u^{-1}, v^* = v^{-1}$. Let $C^*(\mathcal{F}_2)$ be the full $C^*$-algebra of $\mathcal{F}_2$, i.e. the completion of $\mathbb{C}[\mathcal{F}_2]$ with respect to the norm

$$||a|| = \sup \{ \pi(a) : \pi \in \text{Rep} (\mathbb{C}[\mathcal{F}_2]) \}.$$

**Definition 3.** A $*$-algebra, $\mathcal{A}$, is called $*$-wild if $\mathcal{A} \succ C^*(\mathcal{F}_2)$. We say that $\mathcal{A}$ is strongly $*$-wild if $\mathcal{A}$ strongly majorizes the group $*$-algebra $\mathbb{C}[\mathcal{F}_2]$.

Clearly, any strongly $*$-wild algebra is $*$-wild. A motivation for such definition of $*$-wildness was a result proved in [KS1, KS2] saying that $C^*(\mathcal{F}_2)$ majorizes any finitely-generated $*$-algebra.

Since the majorization is a quasi-order, to prove that a $*$-algebra, $\mathcal{A}$, is $*$-wild it is enough to find some $*$-wild algebra which majorizes the algebra $\mathcal{A}$. One very important $*$-wild algebra, which we will frequently use in the paper, is the following. Let $\mathcal{S}_2 = \mathbb{C}(a_1, a_2 | a_1 = a_1^*, a_2 = a_2^*)$. Consider, for some fixed $0 < m < n$, the semi-norm $||a|| = ||a||_{m,n} = \sup \pi(a)$ on the $*$-algebra $\mathcal{S}_2$, where the supremum is taken over all representation $\pi$ of $\mathcal{S}_2$ such that $mI \leq \pi(a_i) \leq nI, i = 1, 2, I$ being the identity operator. Denote by

$$\mathcal{C} = \mathcal{C}_{m,n} = C^*(a_1, a_2 : m \leq a_i = a_i^* \leq n, i = 1, 2)$$

the $C^*$-algebra which is obtained by the completion of $\mathcal{S}_2/(a : ||a|| = 0)$ with respect to $|| \cdot ||$. Clearly, the elements $a_1$ and $a_2$ become invertible in $\mathcal{C}$ and positive in every bounded representation. The following statement was proved in [MT, Lemma 4], but the formulation there contained only the first part of the statement below.

**Lemma 1.** The $C^*$-algebra $\mathcal{C}$ is $*$-wild. Moreover, there exists a homomorphism $\psi : \mathcal{C} \to M_4(C^*(\mathcal{F}_2))$ such that $\psi(a_i) \in M_4(\mathbb{C}[\mathcal{F}_2])$ and the corresponding functor $F_{\psi}$ is full.

**Remark 1.** From Lemma 1 it follows that a finitely generated $*$-algebra $\mathcal{A}$ is strongly $*$-wild if $\mathcal{A}$ majorizes $\mathcal{C}$ and the corresponding homomorphism $\psi$ is such that the image $\psi(\mathcal{A})$ is contained in $M_4(\langle a_1, a_2 \rangle)$, where $\langle a_1, a_2 \rangle$ is the (not completed) $*$-subalgebra of $\mathcal{C}$.

**Definition 4.** A $*$-algebra is called $*$-finite if it has only finitely many irreducible representations up to unitary equivalence, and $*$-tame if it is of type I (see [Di, Chapter 9]) and not $*$-finite.

**Remark 2.** A finitely generated $*$-algebra, $\mathcal{A}$, is of type I if and only if for any irreducible representation $\pi$ of the algebra $\mathcal{A}$ on a Hilbert space, $\mathcal{H}_\pi$, the operator closure $\pi(\mathcal{A})$ contains a compact operator, and therefore contains all compact operators on $\mathcal{H}_\pi$ ([Di, Theorem 9.1,Corollary 4.1.10]). Clearly, if a $*$-algebra has only finite-dimensional irreducible representations, it is of type I.
4 Main Result

To formulate the main theorem we have to introduce the following notation: for every positive integer \(n\) we denote by \(A_n\) and \(\tilde{A}_n\) respectively the path algebras of the quivers

\[
A_n : \begin{array}{c}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \ldots & \rightarrow & \bullet \\
n & \quad & 2 & \quad & 3 & \quad & \ldots & \quad & n-2 & \rightarrow & n-1 & \rightarrow & n
\end{array}
\quad \text{Rad}^2(A_n) = 0,
\]

\[
\tilde{A}_n : \begin{array}{c}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \ldots & \rightarrow & \bullet \\
n & \quad & 2 & \quad & 3 & \quad & \ldots & \quad & n-3 & \rightarrow & n-2 & \rightarrow & n-1
\end{array}
\quad \text{Rad}^2(\tilde{A}_n) = 0
\]

modulo the relation that the radical square of the algebra is zero. In particular, the algebra \(A_1\) is isomorphic to \(\mathbb{C}\) and the algebra \(\tilde{A}_1\) is isomorphic to \(\mathbb{C}[x]/(x^2)\).

Theorem 1. \(I\) Let \(A\) be a finite-dimensional indecomposable associative complex algebra and \(A(\text{Rad} - \ast)\) be its radical \(\ast\)-double.

1 \(A(\text{Rad} - \ast)\) is \(\ast\)-finite if and only if \(A\) is simple if and only if \(A \cong M_n(\mathbb{C})\) for some \(n\) if and only if \(A\) is Morita equivalent to \(\mathbb{C}\).

2 \(A(\text{Rad} - \ast)\) is \(\ast\)-tame if and only if \(A\) is Morita equivalent to either \(A_n\) for some \(n > 1\) or to \(\tilde{A}_n\) for some \(n\).

3 \(A(\text{Rad} - \ast)\) is \(\ast\)-wild if and only if \(A\) is not Morita equivalent to any of \(A_n\) or \(\tilde{A}_n\) for all \(n\).

\(II\) Let \(A\) and \(B\) be two finite dimensional algebras.

1 \((A \oplus B)(\text{Rad} - \ast)\) is \(\ast\)-finite if and only if both \(A(\text{Rad} - \ast)\) and \(B(\text{Rad} - \ast)\) are \(\ast\)-finite.

2 \((A \oplus B)(\text{Rad} - \ast)\) is \(\ast\)-tame if and only if it is not \(\ast\)-finite and both \(A(\text{Rad} - \ast)\) and \(B(\text{Rad} - \ast)\) are either \(\ast\)-finite or \(\ast\)-tame.

3 \((A \oplus B)(\text{Rad} - \ast)\) is \(\ast\)-wild if and only if at least one of \(A(\text{Rad} - \ast)\) and \(B(\text{Rad} - \ast)\) is \(\ast\)-wild.

Remark 3. We remark that the algebras \(A_n\) and \(\tilde{A}_n\) are precisely those finite-dimensional indecomposable algebras, which do not have indecomposable representations of length (= the number of simple subquotients) greater than 2. This is very well-known and can be proved for example using the following argument. Let \(A\) be a basic algebra, which does not have indecomposable representations of dimension greater than 2. Consider the quiver of \(A\). First one shows that any vertex \(x\) of the quiver is a starting point of at most one arrow and is an ending point of at most one arrow, since otherwise the idempotent, representing \(x\), and elements of \(A\) representing two different arrows starting from (ending at) \(x\) define a 3-dimensional indecomposable representation of \(A\). This implies that the quiver of \(A\) is a disjoint union of quivers of \(A_n\) and \(\tilde{A}_n\). If \(\text{Rad}^2(A) \neq 0\) we get that there are two arrows,
in the quiver, whose product is non-zero. With the idempotent, representing the common vertex, we again get a 3-dimensional indecomposable representation of $A$. This implies that $A$ is a direct sum of algebras of type $A_n$ and $\tilde{A}_n$.

Hence Theorem 1 can be reformulated as follows.

**Corollary 1.** The radical $*$-double of an indecomposable associative complex finite-dimensional algebra $A$ is $*$-finite or $*$-tame (that is of type 1) if and only if $A$ does not have indecomposable representations of length greater than 2.

## 5 Preparatory lemmas: tame and wild collections

**Lemma 2.** The radical $*$-double of the algebra $A_n$ and $\tilde{A}_n$ is $*$-tame.

**Proof.** Let $e_1, \ldots, e_n$ be the orthogonal primitive idempotents of $A_n$ (or $\tilde{A}_n$) corresponding to vertices of the quiver $A_n$ ($\tilde{A}_n$ respectively) and let $x_{i,i+1}$ be the element of $A_n$ ($\tilde{A}_n$ respectively) which corresponds to the arrow $i \longrightarrow i+1$.

Let $\pi$ be a non-zero irreducible representation of $A_n(Rad - *)$. Denoting $p_i = \pi(e_i)$, $X_i = \pi(x_{i,i+1})$, we have that $p_i \neq 0$ for some $i$, $X_i : p_i H \rightarrow p_{i+1} H$, $i = 1, \ldots, n - 1$, and $X_i p_j H = 0$ if $j \neq i$. Choose the smallest $i$ such that $p_i \neq 0$. If $i \neq 1$ we have $X_j = 0$ for any $j = 1, \ldots, i - 1$. If $X_i = 0$, then any subspace $U \subset p_i H$ is invariant with respect to $\pi$ and therefore $\pi$ is one-dimensional. If $X_i \neq 0$, one can easily show that for any subspace $U \subset p_i H$ which is invariant with respect to $X_i^* X_i$, the direct sum $U \oplus X_i U$ is invariant with respect to $\pi$. Then using the fact that $\pi$ is irreducible we get that $U$ is necessarily one-dimensional and generated by an eigenvector of $X_i^* X_i$ and the representation $\pi$ is two-dimensional. This shows that $A_n(Rad - *)$ is $*$-tame.

Since radical $*$-double of $\tilde{A}_1$ coincides with its $*$-double and $\tilde{A}_1 \simeq \mathbb{C}[x]/(x^2)$ we have, by [MT], that $\tilde{A}_1(Rad - *)$ is $*$-tame. Consider now $A_n(Rad - *), n \geq 1$. Let $\pi$ be its non-zero irreducible representation. Keeping the above notation we have that $X_i X_{i-1}^* X_{i-1} X_i = 0 = X_i^* X_i X_{i+1} X_{i+1}^*$ and therefore either $\ker X_{i-1} X_{i-1}^* = \ker X_{i-1}^*$ or $\ker X_i^* X_i = \ker X_i$ is non-zero for some $i$. If $\ker X_{i-1} \neq \{0\}$ (resp. $\ker X_i \neq \{0\}$) then this kernel is invariant with respect to $X_i^* X_i$ (resp. $X_{i-1} X_{i-1}^*$) and for any subspace $U \subset \ker X_{i-1}^*$ (resp. $U \subset \ker X_i$), which is invariant with respect to $X_i^* X_i$ (resp. $X_{i-1} X_{i-1}^*$), we have that $U \oplus X_i U$ (resp. $U \oplus X_{i-1} U$) is invariant with respect to $\pi$. Since $\pi$ is irreducible, using the same arguments as above we conclude that $U$ is one-dimensional and the representation $\pi$ is one or two-dimensional. Therefore we have that $A_n(Rad - *)$ is $*$-tame for any $n$. \qed

**Lemma 3.** The radical $*$-double $\mathcal{A}$ of the quiver algebra $\bullet \xrightarrow{x} \bullet \bigcirc y$, with the relations $y^2 = xy = 0$ is strongly $*$-wild

**Proof.** We let $f$ to be the primitive idempotent, corresponding to the right point. The homomorphism $\psi : \mathcal{A} \rightarrow M_3(\mathbb{C})$, defined by

$$
\psi(f) = \begin{pmatrix}
e 0 & 0 & 0 \\
0 & e & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\psi(y) = \begin{pmatrix}
0 & a_1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
\psi(x) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & a_2 & 0
\end{pmatrix}.
$$
generates a full functor $F_\psi : \text{Rep}(\mathcal{C}) \to \text{Rep}(\mathcal{A})$. In fact, let $\pi$ be a representation of $\mathcal{C}$. To prove that $F_\psi$ is full it is enough to show that any operator $C = C^* = [c_{ij}]_{i,j=1}^3$, which intertwines the representation $\pi_3 \circ \psi$ of $\mathcal{A}$, is $\text{diag}(c, c, c)$, where $c$ intertwines the representation $\pi$ of $\mathcal{C}$. If $[C, \pi_3(\psi(f))] = 0$ then $C = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}$. Taking into account that $\pi(a_i), i = 1, 2$, are invertible one gets from $[C, \pi_3(\psi(y))] = 0$ that $c_{12} = c_{21} = 0$ and $c_{11}\pi(a_1) = \pi(a_1)c_{22}, c_{22}\pi(a_1) = \pi(a_1)c_{11}$. Since $c_{11}$ and $c_{22}$ are necessarily self-adjoint, we obtain from this that $c_{11}\pi(a_1)^2 = \pi(a_1)^2c_{11}$ and therefore, by the positivity of the operator $\pi(a_1)$, $c_{11}\pi(a_1) = \pi(a_1)c_{11}$. Thus we have $\pi(a_1)c_{22} = c_{11}\pi(a_1) = \pi(a_1)c_{11}$ and, using invertibility of $\pi(a_1)$, it yields $c_{11} = c_{22}$. Similarly, from $[C, \pi_3(\psi(x))] = 0$ we have $c_{22} = c_{33}$, giving $\mathcal{A} \triangleright \mathcal{C}$. Since $\psi(\mathcal{A}) \subset \langle a_1, a_2 \rangle \subset \mathcal{C}$, the $*$-algebra $\mathcal{A}$ is strongly $*$-wild by Remark 1. □

**Lemma 4.** The radical $*$-doubles of the following quiver algebras are strongly $*$-wild:

(a) $\bullet \overset{x}{\longrightarrow} \bullet \overset{y}{\longrightarrow} \bullet$, 
(b) $\bullet \overset{x}{\longrightarrow} \bullet \overset{y}{\longrightarrow} \bullet$, 
(c) $\bullet \overset{x}{\longrightarrow} \bullet \overset{y}{\longrightarrow} \bullet$, 
(d) $\bullet \overset{x}{\longrightarrow} \bullet$, with the relation $xy = 0$, 
(e) $\bullet \overset{x}{\longrightarrow} \bullet$, 
(f) $\bullet \overset{x}{\longrightarrow} \bullet \overset{y}{\longrightarrow} \bullet$, with the relation $y^2 = 0$, 
(g) $\bullet \overset{x}{\longrightarrow} \bullet \overset{y}{\longrightarrow} \bullet$, with the relation $y^2 = 0$.

**Proof.** We shall only give homomorphisms $\psi$ from the corresponding $*$-algebras, $\mathcal{A}$, to $M_n(\langle a_1, a_2 \rangle) \subset M_n(\mathcal{C})$ which generate full functors $F_\psi : \text{Rep}(\mathcal{C}) \to \text{Rep}(\mathcal{A})$. We denote by $f_1, f_2$ and $f_3$ the primitive idempotents for the quiver algebras, which correspond to the points, counted from the left.

(a) 

$$
\psi(f_1) = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi(f_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\psi(x) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix}.
$$
Similar for the case (b) and (c).

\[
\psi(f_1) = \begin{pmatrix} e & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 & 0 & 0 \end{pmatrix}, \quad \psi(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Similar for the case (e).

(g) The strongly \(*\)-wild algebra from Lemma 3 is a factor-algebra of \(\mathcal{A}\). Therefore \(\mathcal{A}\) is strongly \(*\)-wild.

(f) is similar to Lemma 3 and (g).

**Lemma 5.**

1. A direct sum of \(*\)-algebras is \(*\)-finite if and only if all summands are \(*\)-finite.

2. A direct sum of \(*\)-algebras is of type I if and only if all summands are of type I.

3. A direct sum of \(*\)-algebras is \(*\)-wild if and only if some of the summands is \(*\)-wild.

**Proof.** The first statement of the lemma is obvious. To prove the rest, it is certainly enough to consider the case \(\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2\), where \(\mathcal{A}_1, \mathcal{A}_2\) are \(*\)-algebras.

Let \(\pi\) be a representation of \(\mathcal{A}\) and let \(1_{\mathcal{A}_i}\) denote the unit in \(\mathcal{A}_i\). Then \(p = \pi(1_{\mathcal{A}_i} \oplus 0)\) is a self-adjoint projection commuting with any element of \(\mathcal{A}\). Therefore \(\pi = \pi_1 \oplus \pi_2\), where \(\pi_1(a) = \pi(a)p, \pi_2(a) = \pi(a)(1-p), a \in \mathcal{A}\). By [Di, 5.4.3], \(\pi\) is of type I if and only if both \(\pi_1\) and \(\pi_2\) are of type I. We have \(\pi_1(a_1 \oplus a_2) = \pi(a_1)\) and \(\pi_2(a_1 \oplus a_2) = \pi(a_2), a_1, a_2 \in \mathcal{A}_i\). If \(\mathcal{A}_i, i=1,2,\) are both of type I, the restrictions of \(\pi\) to each \(\mathcal{A}_i\) are representations of type I and therefore the representation \(\pi\) itself also is of type I.

Let \(\mathcal{A}\) be a type I algebra. Assuming that, say \(\mathcal{A}_1\) is not of type I, we have that there exists a representation \(\pi_1\) of \(\mathcal{A}_1\) such that the von-Neumann algebra generated by \(\pi_1(1)\) is not of type I. Now, setting \(\pi(a_1 \oplus a_2) = \pi_1(a_1), a_i \in \mathcal{A}_i\), we get a non-type I representations of \(\mathcal{A}\) giving a contradiction.

Assume that \(\mathcal{A}\) is \(*\)-wild. Let \(\varphi: \mathcal{A} \to M_n(C^*(\mathcal{F}_2))\) be a \(*\)-homomorphism generating the full functor \(F_\varphi: \text{Rep}(C^*(\mathcal{F}_2)) \to \text{Rep}(\mathcal{A})\). Then \(\pi(\varphi(A))' = M_n(C^*(\mathcal{F}_2))'\) for any representation \(\pi \in \text{Rep}(M_n(C^*(\mathcal{F}_2)))\) (see, for example, the proof of [OS2, Theorem 50]). By [OS2, Lemma 14], \(\pi(\varphi(A)) = M_n(C^*(\mathcal{F}_2))\), where bar indicates the closure in the \(*\)-algebra \(M_n(C^*(\mathcal{F}_2))\). It is well-known that \(M_n(C^*(\mathcal{F}_2))\) is an irreducible algebra. In fact, it is well-known that \(C^*(\mathcal{F}_2)\) has a faithful irreducible representation, \(\pi\), (see, for example [Da, Theorem VII.6.5]), and hence so does \(M_n(C^*(\mathcal{F}_2))\): \(id \otimes \pi\). On the other hand, \(\pi(\varphi(A)) = \varphi(A_1 \oplus 0) \oplus \varphi(0 \oplus A_2)\). Therefore, either \(\varphi(A_1 \oplus 0)\) or \(\varphi(0 \oplus A_2)\) is zero implying that either \(\mathcal{A}_1\) or \(\mathcal{A}_2\) is \(*\)-wild with the corresponding \(*\)-homomorphisms \(\varphi_i: \mathcal{A}_i \to M_n(C^*(\mathcal{F}_2))\) defined via \(\varphi_i(a_1) = \varphi(a_1 \oplus 0), a_1 \in \mathcal{A}_1\), and \(\varphi_2(a_2) = \varphi(0 \oplus a_2), a_2 \in \mathcal{A}_2\), respectively. The converse statement is trivial.

**6 Preparatory lemmas: Morita equivalence**

Let \(\mathcal{A}\) be a \(*\)-algebra and let \(1 = e_1 + e_2 + \ldots + e_n\) be a decomposition of the identity of the algebra \(\mathcal{A}\) into a sum of pairwise orthogonal projections. Set \(\mathcal{A}_{ij} = e_i \mathcal{A} e_j\) and consider
a vector space $\mathcal{B} = \bigoplus_{i=1}^n B_{ij}$ where $B_{ij} = \mathbb{C}^{m_i} \otimes \mathcal{A}_{ij} \otimes \mathbb{C}^{m_j}$, $\{m_i\}$ are positive integers. We write elements of $\mathcal{B}$ as matrices $(b_{ij})$, $b_{ij} \in B_{ij}$. $\mathcal{B}$ possesses an algebra structure: if $b_{ij} = f_i \otimes a_{ij} \otimes g_j$ and $c_{ij} = u_i \otimes d_{ij} \otimes w_j$ with $f_i, u_i \in \mathbb{C}^{m_i}$, $g_j, w_j \in \mathbb{C}^{m_j}$, $a_{ij}, d_{ij} \in \mathcal{A}_{ij}$, we define a product of $(b_{ij})$ and $(c_{ij})$ by

$$(b_{ij}) \cdot (c_{ij}) = (s_{ij}),$$

$$s_{ij} = \sum_k b_{ik} \cdot c_{kj},$$

where bar indicates the complex conjugation and $(g_k, \bar{a}_k)$ is the scalar product in $\mathbb{C}^{m_k}$ of $g_k, \bar{a}_k$.

$\mathcal{B}$ is a $\ast$-algebra with involution defined as follows:

$$(b_{ij})^\ast = (b_{ji}^\ast), \quad b_{ji}^\ast = \bar{g}_k \otimes a_{ji}^\ast \otimes \bar{f}_j.$$  

Taking a trivial decomposition of the identity ($n = 1$) we get $\mathcal{B} \cong M_{m_1}(\mathbb{C}) \otimes \mathcal{A}$ with an isomorphism $\varphi$ given by $\varphi(f_i \otimes a \otimes f_j) = e_{ij} \otimes a$, where $\{f_i\}$ is the standard basis in $\mathbb{C}^{m_1}$ and $e_{ij}$ are the matrix units in $M_{m_1}(\mathbb{C})$. In general, considering in $M_{m_1+m_2+\ldots+m_n}(\mathcal{A})$ the projection $p = \text{diag}(e_1 \otimes I_{m_1}, e_2 \otimes I_{m_2}, \ldots, e_n \otimes I_{m_n})$, where $I_{m_i}$ is the identity matrix in $M_{m_i}(\mathbb{C})$, we have $\mathcal{B} \cong pM_N(\mathcal{A})p$, where $N = m_1 + m_2 + \ldots + m_n$.

Let $\rho$ be a $\ast$-representation of $\mathcal{A}$ on $\mathcal{H}$ and set $q_i = \rho(e_i)$. $\rho$ generates a $\ast$-representation $\Pi(\rho)$ of $\mathcal{B}$ on $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$, where $\mathcal{H}_i = \mathbb{C}^{m_i} \otimes q_i \mathcal{H}$: if $h_i = v_i \otimes w_i$, with $v_i \in \mathbb{C}^{m_i}$ and $w_i \in q_i \mathcal{H}$ and $b_{ij} = f_i \otimes a_{ij} \otimes g_j$,

$$\Pi(\rho)((b_{ij}))(h_1, h_2, \ldots, h_n) = (u_1, u_2, \ldots, u_n), \quad u_i = \sum_{j=1}^n (g_j, \bar{v}_j)f_i \otimes \rho(a_{ij})w_j.$$  

The representation $\Pi(\rho)$ comes from the following representation $\tilde{\Pi}(\rho)$ of the $\ast$-algebra $pM_N(\mathcal{A})p$: The representation $\rho$ naturally induces the representation $\rho_N$ of $M_N(\mathcal{A})$. Let $\tilde{\mathcal{H}} = \bigoplus_{i=1}^n \mathbb{C}^{m_i} \otimes H_i$. Then $\mathcal{H} = \rho_N(\mathcal{A}) \otimes \tilde{\mathcal{H}}$. For $a \in M_N(\mathcal{A})$, we set $\tilde{\Pi}(\rho(p)a) = \rho_N(pa)|_{\tilde{\mathcal{H}}}$ as a representation on the Hilbert space $\tilde{\mathcal{H}}$.

**Lemma 6.** Any $\ast$-representation $\pi$ of $\mathcal{B}$ is unitarily equivalent to $\Pi(\rho)$ for some $\ast$-representation $\rho$ of $\mathcal{A}$.

**Proof.** Let $\pi$ be a $\ast$-representation of $\mathcal{B}$ on $\mathcal{H}$ and let $p_i$ be the projection onto the subspace $\pi(B_{ii})\mathcal{H}$. Since $b_{ii} \cdot b_{ij} = 0$ for $b_{ii} \in B_{ii}$, $b_{ij} \in B_{jj}$ and $i \neq j$, we have $p_ip_j = 0$ if $i \neq j$ and $\sum_{i=1}^n p_i = 1$ so that $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$, where $\mathcal{H}_i = p_i \mathcal{H}$. We also have that $\pi(B_{ij})\mathcal{H}_i = 0$ if $k \neq j$ and $\pi(B_{ij})\mathcal{H}_j \subset \mathcal{H}_i$. Since each $B_{ii}$ is an algebra isomorphic to $M_{m_i}(\mathbb{C}) \otimes \mathcal{A}_{ii}$ with the unit $I_{m_i} \otimes e_i$ and any representation of the later is unitarily equivalent to $i\mathbb{C} \otimes \rho_i$, where $\rho_i$ is a representation of $\mathcal{A}_{ii}$ on a Hilbert space $H_i$, we have that there exists a unitary operator $V : \bigoplus_{i=1}^n H_i \to H = \bigoplus_{i=1}^n \mathbb{C}^{m_i} \otimes \mathcal{H}_i$ such that for the representation $\pi' = V \pi V^{-1}$,

$$\pi'(b_{ii})|_{\mathbb{C}^{m_i} \otimes \mathcal{H}_i} = (i\mathbb{C} \otimes \rho_i)(\varphi_i(b_{ii})) = \Pi(\rho_i)(b_{ii}),$$

where $b_{ii} \in B_{ii}$ and $\varphi_i : B_{ii} \to M_{m_i}(\mathbb{C}) \otimes \mathcal{A}_{ii}$ is the isomorphism of the corresponding algebras defined above.
Let \( \{f^k_i\} \) be the standard basis in \( \mathbb{C}^{m_i} \). Then \( \text{Ran} \left( f^k_i \otimes p_i \otimes f^k_i \right) \subset f^k_i \otimes H_i \), and we have
\[
\pi'(f^k_i \otimes a_{ij} \otimes f^k_j) f^\ast_j \otimes w_j = \pi'(f^k_i \otimes p_i \otimes f^k_i) \pi'(f^k_i \otimes a_{ij} \otimes f^k_j) f^\ast_j \otimes w_j = \\
\pi'(f^k_i \otimes p_i \otimes f^k_i) \pi'(f^k_i \otimes a_{ij} \otimes f^k_j) (f^\ast_j, f^\ast_j) f^\ast_j \otimes w_j = (f^\ast_j, f^\ast_j) f^\ast_i \otimes \tilde{w}_i.
\]
for some \( \tilde{w}_i \in H_i \). Since \( f^\ast_i \otimes a_{ij} \otimes f^\ast_j = (f^\ast_i \otimes p_i \otimes f^\ast_i) \cdot (f^\ast_i \otimes a_{ij} \otimes f^\ast_j) \cdot (f^\ast_j \otimes p_j \otimes f^\ast_j) \) one easily checks that \( \pi'(f^\ast_i \otimes a_{ij} \otimes f^\ast_j) f^\ast_j \otimes w_j = (f^\ast_j, f^\ast_j) f^\ast_i \otimes \tilde{w}_i \) so that \( \tilde{w}_i \) depends only on \( w_j \) and \( a_{ij} \).

Let \( X_{a_{ij}} \) be the mapping from \( H_j \) to \( H_i \) which sends \( w_j \) to \( \tilde{w}_i \). It is easy to check that it is a bounded linear operator and that
\[
\pi'(f_i \otimes a_{ij} \otimes g_j) u_j \otimes w_j = (g_j, a_j) f_i \otimes X_{a_{ij}} w_j
\]
for arbitrary \( f_i \in \mathbb{C}^{m_i} \), \( g_j, u_j \in \mathbb{C}^{m_j} \), and \( w_j \in H_j \). We extend \( X_{a_{ij}} \) to the whole space \( H \) in the trivial way and denote the resulting mapping by the same letter. What is left to prove is that \( \rho(a) := \sum_{i,j=1}^n X_{p_{iapj}} a \in A \), is a \( * \)-representation of \( A \) on \( \bigoplus_{i=1}^n H_i \). Direct verification shows that \( X_{a_{ij} + b_{ij}} = X_{a_{ij}} + X_{b_{ij}} \) and \( X_{\lambda a_{ij}} = \lambda X_{a_{ij}} \) implying \( \rho(a + b) = \rho(a) + \rho(b) \) and \( \rho(\lambda a) = \lambda \rho(a) \).

\[
\rho(a) \rho(b) = \left( \sum_{i,j=1}^n X_{p_{iapj}} \right) \left( \sum_{i,j=1}^n X_{p_{ibpj}} \right) = \sum_{i,j=1}^n \left( \sum_{k=1}^n X_{p_{iapk} X_{p_{ibp} j}} \right)
\]
and, if \( h_k \in \mathbb{C}^{m_k} \) such that \( (h_k, \tilde{h}_k) = 1 \),
\[
\pi'(f_i \otimes p_{iapj} \otimes g_j)(u_j \otimes w_j) = \pi'(f_i \otimes \sum_{k=1}^n (p_{iapk} \cdot p_{kbpj}) \otimes g_j)(u_j \otimes w_j) = \\
\sum_{k=1}^n \pi'(f_i \otimes p_{iapk} \otimes h_k) \pi'(h_k \otimes p_{kbpj} \otimes g_j)(u_j \otimes w_j) = \\
\sum_{k=1}^n \pi'(f_i \otimes p_{iapk} \otimes h_k)((g_j, a_j) h_k \otimes X_{p_{kbpj}} w_j) = \\
(g_j, a_j) f_i \otimes \left( \sum_{k=1}^n X_{p_{iapk} X_{p_{kbpj}}} w_j \right).
\]

On the other hand, \( \pi'(f_i \otimes p_{iapj} \otimes g_j)(u_j \otimes w_j) = (g_j, a_j) f_i \otimes X_{p_{iapj}} w_j \), giving us
\[
X_{p_{iap}} = \sum_{k=1}^n X_{p_{iapk} X_{p_{kbpj}}}
\]
and
\[
\rho(ab) = \sum_{i,j=1}^n X_{p_{iapj}} = \sum_{i,j=1}^n \sum_{k=1}^n X_{p_{iapk} X_{p_{kbpj}}} = \rho(a) \rho(b).
\]
Since $\rho(a^*) = \sum_{i,j=1}^n X_{p_i,a^*p_j}$ and $\rho(a) = \sum_{i,j=1}^n X_{p_j,ap_i}$, to show that $\rho$ is a $*$-representation we have to prove that $X_{p_i,a^*p_j} = X_{p_j,ap_i}^*$. The verification of this is an easy task and we leave it to the reader. □

**Lemma 7.** Any idempotent in the algebra $M_n(\mathbb{C}[\mathcal{F}_2])$ is equivalent to an idempotent of the the form $q \otimes e$, where $q$ is an idempotent in $M_n(\mathbb{C})$ and $e$ is the unit in $\mathbb{C}[\mathcal{F}_2]$.

**Proof.** By [Co, Corollary 3], the algebra $\mathbb{C}[\mathcal{F}_2]$ is a free ideal ring. Hence the statement of the lemma follows from [Co, Lemma 2.5]. □

**Lemma 8.** 1. $\mathcal{A}$ is $*$-finite if and only if $\mathcal{B}$ is $*$-finite.

2. $\mathcal{A}$ is of type I if and only if $\mathcal{B}$ is of type I.

3. If $\mathcal{A}$ is strongly $*$-wild then $\mathcal{B}$ is $*$-wild.

**Proof.** The first statement follows from Lemma 6.

Assume $\mathcal{A}$ is of type I. Then so is the algebra $M_N(\mathcal{A})$. In fact, any $*$-representation of $M_N(\mathbb{C}) \otimes \mathcal{A}$ is unitarily equivalent to $\rho = id \otimes \pi$, where $\pi$ is a $*$-representation of $\mathcal{A}$. Therefore the von-Neumann algebra generated by $\rho(M_N(\mathbb{C}) \otimes \mathcal{A}) = M_N(\mathbb{C}) \bar{\otimes} \mathcal{N}$, where $\mathcal{N}$ is the von-Neumann algebra generated by $\pi(\mathcal{A})$. Since $\mathcal{N}$ and $M_N(\mathbb{C})$ are of type I so is $M_N(\mathbb{C}) \bar{\otimes} \mathcal{N}$ ([Ta, Theorem 2.30]). Like $\mathcal{A}$, $M_N(\mathcal{A})$ is a finitely generated algebra. Therefore, by Remark 2, if $\rho$ is an irreducible representation of $M_N(\mathcal{A})$ on $\mathcal{H}$, the closure $\rho(M_N(\mathcal{A}))$ contains a compact operator $K$. Let $p$ be the projection given the isomorphism $pM_N(\mathcal{A})p \simeq \mathcal{B}$. Clearly, for $P = \rho(p)$, the operator $PKP$ is compact as an operator from $B(\mathcal{H})$, where $\mathcal{H} = PH$. Thus for the representation $\Pi(\rho)$ of the algebra $pM_N(\mathcal{A})p$, the operator closure of the image $\Pi(\rho)(pM_N(\mathcal{A})p)$ contains a compact operator. Since, by Lemma 6, any representation of $pM_N(\mathcal{A})p$ is unitarily equivalent to $\Pi(\rho)$ for some $\rho \in \text{Rep} \, \mathcal{A}$, this implies that $pM_N(\mathcal{A})p$ and therefore $\mathcal{B}$ is of type I. We leave the converse statement (which we do not need) to the reader.

Assume now that $\mathcal{A}$ is a strongly $*$-wild algebra. To prove $*$-wildness of $\mathcal{B} \simeq pM_N(\mathcal{A})p$ it is enough to show that there exists a $*$-homomorphism $\psi : pM_N(\mathcal{A})p \to M_n(C^*(\mathcal{F}_2))$ such that $\psi(pM_N(\mathcal{A})p)$ is dense in $M_n(C^*(\mathcal{F}_2))$. In fact, since $(\pi_n(M_n(C^*(\mathcal{F}_2))^t)' = M_n(\mathbb{C})' \bar{\otimes} (\pi(C^*(\mathcal{F}_2)))')' = CI_n \bar{\otimes} (\pi(C^*(\mathcal{F}_2)))'$ for any $\pi \in \text{Rep} \, (C^*(\mathcal{F}_2))$, we would have in this case $C \in F_0(\pi)(pM_N(\mathcal{A})p)' = (\pi_n(M_n(C^*(\mathcal{F}_2))^t)' = CI_n \bar{\otimes} (\pi(C^*(\mathcal{F}_2)))'$, giving the statement. We know the existence of a $*$-homomorphism $\varphi : \mathcal{A} \to M_K(C^*(\mathcal{F}_2))$ satisfying this density condition (see the corresponding arguments in the proof of Lemma 5) and, moreover, the condition $\varphi(\mathcal{A}) \subset M_K(\mathbb{C}[\mathcal{F}_2])$. Let $\varphi_N$ be the $*$-homomorphism $M_N(\mathcal{A}) \to M_{NK}(\mathbb{C}[\mathcal{F}_2])$ induced by $\varphi$. Then

$$
\varphi_N(pM_N(\mathcal{A})p) = \varphi_N(p)\varphi_N(M_N(\mathcal{A}))\varphi_N(p)
$$

is dense in $\varphi_N(p)M_{NK}(\mathcal{F}_2))\varphi_N(p)$. $\varphi_N(p)$ is a projection in $M_{NK}(\mathbb{C}[\mathcal{F}_2])$ and therefore, by Lemma 7, is equivalent to a projection of type $q \otimes e$, where $q$ is a projection.
(say, of rank $n$) in $M_{NK}(\mathbb{C})$. Let $T \in M_{NK}(\mathbb{C}([\mathcal{F}_2]))$ be an invertible element giving the equivalence. Then

$$
\varphi_N(p)M_{NK}(C^*(\mathcal{F}_2))\varphi_N(p) = T^{-1}(q \otimes e)TM_{NK}(C^*(\mathcal{F}_2))T^{-1}(q \otimes e)T = T^{-1}(q \otimes e)M_{NK}(C^*(\mathcal{F}_2))(q \otimes e)T.
$$

Since $(q \otimes e)M_{NK}(C^*(\mathcal{F}_2))(q \otimes e) \simeq M_n(C^*(\mathcal{F}_2))$, we have $\varphi_N(p)M_{NK}(C^*(\mathcal{F}_2))\varphi_N(p) \simeq M_n(C^*(\mathcal{F}_2))$. If $\delta$ is the corresponding isomorphism, $\delta \circ \varphi_N : pM_N(A)p \to M_n(C^*(\mathcal{F}_2))$ is the required $*$-homomorphism $\psi : pM_N(A)p \to M_n(C^*(\mathcal{F}_2))$.

\begin{remark}
Using similar arguments one can prove that if $B$ is strongly $*$-wild, then $A$ is $*$-wild. We do not know if strong $*$-wildness can be replaced by $*$-wildness. This would be true if we could prove that any projection in $M_n(C^*(\mathcal{F}_2))$ is unitarily equivalent to an elementary one, that is $q \otimes e$, where $q$ is a projection in $M_n(\mathbb{C})$. However, at the moment we do not know how to prove the last statement (and we do not know if it is correct).
\end{remark}

\section{Proof of the main result}

The second statement of the theorem follows from Lemma 5. For the first statement we can assume that the algebra $A$ is indecomposable and basic using Lemma 8.

If $A$ is basic and is isomorphic to $A_1$, then it is obvious that $A(\text{Rad} - *) \simeq A$ is $*$-finite. In fact it has (up to a $*$-isomorphism) only one irreducible $*$-representation, which is the trivial one.

By Lemma 2, the radical $*$-double of both $A_n$ and $\tilde{A}_n$ is $*$-tame. In Remark 3 we have seen that the algebras $A_n$ and $\tilde{A}_n$ are characterized as those finite-dimensional basic indecomposable algebras whose dimensions of indecomposable representations do not exceed 2. To complete the proof it is now left to show that the radical $*$-double of a basic indecomposable algebra $A$, admitting a 3-dimensional indecomposable representation, $\pi$ say, is strongly $*$-wild. Let us denote by $B$ the quotient of $A$ modulo the annihilator of $\pi$. We note that $B$ is basic and indecomposable and that $\pi$ induces a 3-dimensional indecomposable representation of $B$. It is of course enough to show that the radical $*$-double of $B$ is strongly $*$-wild. We will now show that this essentially reduces to Lemma 4.

We have the following three possibilities for $\pi$:

1. $\pi$ has exactly one 1-dimensional subrepresentation which we denote by $\pi_1$, and exactly one 1-dimensional quotient representation which we denote by $\pi_2$.
2. $\pi$ has exactly one 1-dimensional subrepresentation which we denote by $\pi_1$, but more than one 1-dimensional quotient representations.
3. $\pi$ has exactly one 1-dimensional quotient representation which we denote by $\pi_2$, but more than one 1-dimensional subrepresentations.

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In what follows we are going to study all possibilities for \( B \) case by case.

Assume first that \( 1 \) is a primitive idempotent. Then the radical \( * \)-double of \( B \) and the usual \( * \)-double of \( B \) in the sense of [MT] coincide and the statement follows from [MT, Corollary 1] (remark that all \( * \)-wild \( * \)-doubles in [MT] are in fact strongly \( * \)-wild by the constructions used in [MT] and Remark 1).

Let us now assume that \( B \) has two non-equivalent orthogonal primitive idempotents \( f \) and \( 1 - f \). Since both these elements do not annihilate \( \pi \), we can assume that the image of \( f \) is 2-dimensional and the image of \( 1 - f \) is thus 1-dimensional. In the case (I) we have three possibilities:

1. \( 1 - f \) is not annihilated by \( \pi_1 \). In this case the algebra \( B \) is (via \( \pi \)) the algebra of the following matrices:

\[
B = \left\{ \begin{pmatrix} b & c & d \\ 0 & a & l \\ 0 & 0 & a \end{pmatrix} : a, b, c, d, l \in \mathbb{C} \right\},
\]

and one easily constructs an isomorphism to the algebra of Lemma 4(f). Using the latter lemma we conclude that \( B(\text{Rad} - *) \) is strongly \( * \)-wild.

2. \( 1 - f \) is not annihilated by \( \pi_2 \). In the same way as above, it is easy to see that in this case \( B \) is isomorphic to the algebra of Lemma 4(g) and hence \( B(\text{Rad} - *) \) is strongly \( * \)-wild.

3. \( 1 - f \) is annihilated by both \( \pi_1 \) and \( \pi_2 \). It is easy to see that in this case \( B \) is isomorphic to the algebra of Lemma 4(d) and hence \( B(\text{Rad} - *) \) is strongly \( * \)-wild.

In the case (II) we have two possibilities:

1. \( 1 - f \) is not annihilated by \( \pi_1 \). It is easy to see that in this case \( B \) is isomorphic to the algebra of Lemma 4(e) and hence \( B(\text{Rad} - *) \) is strongly \( * \)-wild.

2. \( 1 - f \) is annihilated by \( \pi_1 \). It is easy to see that in this case \( B \) is isomorphic to the algebra of Lemma 4(g) and hence \( B(\text{Rad} - *) \) is strongly \( * \)-wild.

In the case (III) we have two possibilities:

1. \( 1 - f \) is not annihilated by \( \pi_2 \). It is easy to see that in this case \( B \) is isomorphic to the algebra of Lemma 4(e) and hence \( B(\text{Rad} - *) \) is strongly \( * \)-wild.

2. \( 1 - f \) is annihilated by \( \pi_2 \). It is easy to see that in this case \( B \) is isomorphic to the algebra of Lemma 4(f) and hence \( B(\text{Rad} - *) \) is strongly \( * \)-wild.
Finally, let us assume that $B$ has three non-equivalent pairwise orthogonal primitive idempotents $e$, $f$ and $1 - f - e$. Then the rank of each of them under $\pi$ is 1-dimensional. Hence in the case (I) we get that $B$ is isomorphic to the algebra of Lemma 4(a) and hence $B(\text{Rad} - \ast)$ is strongly $\ast$-wild. In the case (II) we get that $B$ is isomorphic to the algebra of Lemma 4(c) and hence $B(\text{Rad} - \ast)$ is strongly $\ast$-wild. In the case (III) we get that $B$ is isomorphic to the algebra of Lemma 4(b) and hence $B(\text{Rad} - \ast)$ is strongly $\ast$-wild. This completes the proof.

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