SUBMODULE STRUCTURE OF GENERALIZED VERMA MODULES INDUCED FROM GENERIC GELFAND-ZETLIN MODULES

VLADIMIR MAZORCHUK AND SERGEI OVSIENKO

Abstract. For complex Lie algebra $sl(n,\mathbb{C})$ we study the submodule structure of generalized Verma modules induced from generic Gelfand-Zetlin modules over some subalgebra of type $sl(k,\mathbb{C})$. We obtain necessary and sufficient conditions for the existence of a submodule generalizing the BGG theorem for Verma modules.

1. Introduction

One of the basic facts in the theory of highest weight representations of finite - dimensional complex Lie algebras, established by Bernstein - Gelfand - Gelfand ([1], [4]) is the description of the submodule structure of Verma modules in terms of the Weyl group action on the weight space, considered as a parameter space of the isomorphism classes of Verma modules. The original theorem asserts that

if $M(\lambda)$ and $M(\mu)$ are Verma modules with highest weights $\lambda-\rho$ and $\mu-\rho$, then $M(\mu)\subseteq M(\lambda)$ if and only if there exist reflections $w_{\beta_1}, w_{\beta_2},\dots,w_{\beta_r}$ in the Weyl group such that

$$\mu = w_{\beta_r}w_{\beta_{r-1}}\dots w_{\beta_1}(\lambda)\leq w_{\beta_{r-1}}\dots w_{\beta_1}(\lambda)\leq\dots\leq w_{\beta_1}(\lambda)\leq\lambda.$$

A generalization of these results in the case of generalized Verma modules (GVM) induced from the infinite-dimensional simple weight $sl(2,\mathbb{C})$-modules with no highest or lowest weight was obtained in the case of $sl(n,\mathbb{C})$ in [11] and in the case of arbitrary simple algebra except G_2 in [8]. Those articles deal with the so-called α-stratified GVM, i.e. those induced from simple weight $sl(2,\mathbb{C})$-module with no highest or lowest weight. A criterion of the existence of a non-trivial homomorphism between α-stratified GVM's is formulated (as in the classical case) in terms of the Weyl group action on some algebraic variety which parametrizes a set of corresponding GVM's.

We note some peculiar features of that description:

- the space of parameters $\mathcal{A}$ is a 2-sheets covering of the variety of the isomorphism classes of GVM's;
- the submodules are parametrized by the cosets W/H, where H is the subgroup of W, corresponding to the root system $\{\alpha,\,-\alpha\}$;
- an action of the Weyl group on $\mathcal{A}$ is not originated from the action on the weight space $\mathfrak{H}^*$ and does not induce an action on the isomorphism classes of GVM.

In the present paper we apply the technique from [11], [8] for a new class of modules. We consider the standard inclusion $sl(k,\mathbb{C}) \hookrightarrow sl(n,\mathbb{C})$ on the first k rows and columns and investigate the submodule structure of GVM's associated with this inclusion. We induce generalized Verma $sl(n,\mathbb{C})$-module $M(\lambda,\, V)$ from the so-called simple generic Gelfand-Zetlin $sl(k,\mathbb{C})$-modules V ([5]). This seems to be a natural generalization of the results discussed above. The answer is obtained in the same terms as in [1],[11],[8]. Moreover, our results coincide with classical BGG-theorem in the case $k=1$ and with results of [11],[8] in the case $k=2$.

Note, that modules discussed in our paper are not "weight modules" in a standard sense. More precisely, in the case $k>2$ their weight spaces (with respect to a Cartan subalgebra) are infinite-dimensional.

We use a technique of Gelfand - Zetlin modules developed in [5]. These modules are defined by some finite-dimensionality conditions with respect to the so called Gelfand - Zetlin subalgebra Γ instead of $S(\mathfrak{H})$ in the classical case.

We define a special partial order on the set of parameters of GVM and obtain our main result in terms of the Weyl group action on this set. We also reformulate our results in terms of Bruhat order on the quotients S_n/S_k where S_k is the Weyl group corresponding to subalgebra $sl(k,\mathbb{C})$ and give a criterion for GVM to be simple.

Let us briefly describe a structure of the paper. In section 2 we give some previous notations and results about generalized Verma modules. In section 3 we collect all the preliminaries about Gelfand-Zetlin modules, following closely [5] (see also [13]). The only difference is a way of parametrizing of such modules. In section 4 we discuss modules induced from subalgebra of corank one. In section 5 we introduce a set which parametrizes GVM and define an action of the Weyl group on this set. Theorem 3 of this section is an analogue of Harish-Chandra isomorphism theorem ([4, Theorem 7.4.5]). In section 6 we define an analogue of the Kostant function and show that it describes the growth of GVM. This allows us to prove some analogue of [4, Theorem 7.6.6]. Finally, in section 7 we obtain the BGG-like criterion for the existence of a submodule in GVM induced from a simple generic GZ-module. As a corollary we give a criterion for GVM to be simple and obtain some results related to the Bruhat order on the quotients of the Weyl group.

2. Generalized Verma Modules

Let $\mathbb{C}$ denotes the complex numbers, $\mathbb{Z}$ denotes the ring of integers, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{Z}_+$ denotes the set of all non-negative integers.

Let $\mathfrak{G}$ be a simple complex finite-dimensional Lie algebra. We fix a Cartan subalgebra $\mathfrak{H}$ in $\mathfrak{G}$. Let Δ be the corresponding root system and W be the Weyl group of Δ. For some base π of Δ let $\Delta=\Delta_+\cup\Delta_-$ be the partition of Δ into positive and negative roots with respect to π. For $\alpha\in\Delta$ we denote by $\mathfrak{G}_\alpha$ the corresponding root space in $\mathfrak{G}$. Set $\rho=\frac{1}{2}\sum\limits_{\alpha\in\Delta_+}\alpha$ and let $<\cdot,\cdot>$ be the standard form on $\mathfrak{H}^*$. For $\alpha\in\Delta$ we denote by $X_\alpha\in\mathfrak{G}_\alpha$ an element from the fixed Weyl-Chevalle basis.

For a Lie algebra $\mathfrak{A}$ we will denote by $U = U(\mathfrak{A})$ the universal enveloping algebra of $\mathfrak{A}$, by $Z(\mathfrak{A})$ the center of U and by $\mathfrak{A}$-mod its category of finitely generated left modules.

Let $\mathfrak{n}_{\pm}$ be the Lie subalgebras in $\mathfrak{G}$, generated by $X_{\pm\alpha}$, where α runs through the set of all positive roots. Then we have the following triangular decomposition of $\mathfrak{G}$:

$$\mathfrak{G} = \mathfrak{n}_- \oplus \mathfrak{H} \oplus \mathfrak{n}_+.$$

Consider a subset $S \subset \pi$ and let $\Delta_S = (\Delta_S)_+ \cup (\Delta_S)_-$ be the root system generated by vectors of S together with the standard decomposition with respect to the base S. Denote by W_S the Weyl group of Δ_S (W_S is a subgroup of W). Let $\mathfrak{G}_S$ be a semisimple Lie subalgebra of $\mathfrak{G}$, generated by X_α for all $\alpha \in \Delta_S$. We will denote by $\mathfrak{H}_S$ its Cartan subalgebra and by U_S its enveloping algebra. We denote by $\mathfrak{n}_S$ $(= \mathfrak{n}_S^+)$ the subalgebra of $\mathfrak{n}_+$ generated by vectors $\mathfrak{G}_\alpha$ for $\alpha \notin \Delta_S$ and by $\mathfrak{n}_S^-$ the corresponding subalgebra of $\mathfrak{n}_-$. Set $\mathfrak{B}_S = \mathfrak{G}_S + \mathfrak{H} + \mathfrak{n}_S$.

For a $\mathfrak{G}$-module V and $\lambda \in \mathfrak{H}^*$ set

$$V_\lambda = \{v \in V \,:\, hv = \lambda(h)v, \ \forall h \in \mathfrak{H}\}.$$

If $V_\lambda \neq 0$, then we will call λ *a weight* of V and V_λ the corresponding *weight space*. We will call a $\mathfrak{G}$ – module V *weight module* if it is a direct sum of V_λ where λ runs through $\mathfrak{H}^*$ (we remark, that we do not assume the finite-dimensionality of the spaces V_λ.) For a weight module V and $\lambda \in \mathfrak{H}^*$ a non-zero element $v \in V_\lambda$ is called *S-primitive (with respect to π)* provided $\mathfrak{n}_S v = 0$.

Consider a weight $\mathfrak{G}_S$-module V and $\lambda \in (\mathfrak{H}^S)^*$, where $\mathfrak{H}^S = (\mathfrak{H}_S)^\perp$ is the orthogonal subspace in $\mathfrak{H}$ with respect to the standard form. Let $\rho_S = \frac{1}{2} \sum_{\beta \in (\Delta_S)_+} \beta$ and $\rho^S = \rho - \rho_S$. Putting $\mathfrak{n}_S v = 0$ for all $v \in V$ and $hv = (\lambda - \rho^S)(h)v$ for all $h \in \mathfrak{H}^S$ we turn V into $\mathfrak{B}_S$-module. Then we can define a $\mathfrak{G}$-module

$$M(\lambda, V) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{B}_S)} V.$$

This module is called *Generalized Verma Module (GVM)* corresponding to the module V and $\mathfrak{H}^S$-weight λ provided V is simple.

From now on all the modules of the form $M(\lambda, V)$ are assumed to be Generalized Verma Modules.

First of all we recall some basic properties of GVM:

Proposition 1. 1. *$M(\lambda, V)$ is a weight module.*
2. *$M(\lambda, V)$ has the unique maximal submodule.*
3. *Every S-primitive vector of $M(\lambda, V)$ generates a submodule isomorphic to some module $M(\mu, V_1)$, where $\mu \in (\mathfrak{H}^S)^*$ and V_1 is a weight $\mathfrak{G}_S$-module.*
4. *Let T be a weight $\mathfrak{G}$-module generated by S–primitive vector v of weight $\lambda - \rho$ and $T_S = U_S v$ be a simple U_S-module. Then there exists a canonical epimorphism $\varphi : M(\lambda^S, T_S) \to T$, where $\lambda^S = \lambda|_{\mathfrak{H}^S}$, such that $\varphi(1 \otimes t_s) = t_s$, $t_s \in T_S$.*

5. *If $X \subset M(\lambda, V)$ is a nontrivial submodule then X contains a submodule Y generated by S-primitive element.*
6. *The module $M(\lambda, V)$ is $U(\mathfrak{n}_S^-)$-free with V as a space of free generators.*

Proof. 1 follows easily from the fact that the module, generated by a weight vector over parabolic subalgebra is a weight module. The proof of 2 is quite analogues to that of [4, Proposition 7.1.11]. 3 follows from the PBW theorem by standard arguments. 4 follows from the universal property of tensor product. 5, 6 are obvious. □

We denote by $L(\lambda, V)$ the unique simple quotient of $M(\lambda, V)$.

In this paper we will consider the case of the Lie algebra $sl(n, \mathbb{C})$ with a Cartan subalgebra $\mathfrak{H}$ consisting of all diagonal matrices with zero trace. We denote by $\{\alpha_1, \dots, \alpha_{n-1}\}$ the standard set of simple roots. The root space corresponding to the root $\gamma = \pm(\alpha_i + \dots + \alpha_j)$ is generated by X_γ, that is a matrix unit $e_{i\,j+1}$ or $e_{j+1\,i}$ in the case $+$ or $-$ respectively.

3. Gelfand-Zetlin algebra and Gelfand-Zetlin modules

Consider the Lie algebras $\mathfrak{G}_m = gl(m, \mathbb{C})$ and fix the notations $U_m = U(\mathfrak{G}_m)$ and $Z_m = Z(\mathfrak{G}_m)$, $m \geq 1$. Let us fix some $n \geq 1$, set $\mathfrak{G} = \mathfrak{G}_n$, $U = U_n$ and identify $\mathfrak{G}_m$ for $m \leq n$ with the Lie subalgebra of $\mathfrak{G}$ generated by the matrix units $\{e_{ij} \mid i, j = 1, \dots, m\}$. Then we obtain the inclusions

$$\mathfrak{G}_1 \subset \mathfrak{G}_2 \subset \dots \mathfrak{G}_n = \mathfrak{G}$$

and the induced inclusions

$$U_1 \subset U_2 \subset \dots U_n = U.$$

Let Γ $(= \Gamma_n)$ be the subalgebra of U generated by $\{Z_m \mid m = 1, \dots, n\}$. We will call it *Gelfand-Zetlin subalgebra* (GZ-subalgebra) of U.

Proposition 2 ([13])**.** *The algebra Z_m is a polynomial algebra in m variables $\{c_{mk} \mid k = 1, \dots, m\}$, where*

$$c_{mk} = \sum_{i_1, \dots, i_k \in \{1, \dots, m\}} e_{i_1 i_2} e_{i_2 i_3} \dots e_{i_k i_1}$$

The algebra Γ is a polynomial algebra in $\dfrac{n(n+1)}{2}$ variables c_{ij}, $1 \leq j \leq i \leq n$.

We need more convenient set of generators in GZ-algebra Γ. Following [13] to present new generators for Γ we will use the notion of *tableaux*.

Let $\mathcal{L} = \mathbb{C}^{\frac{n(n+1)}{2}}$. The elements of $\mathcal{L}$ will be called *tableaux* and considered as double indexed families

$$[\,l\,] = \{l_{k\,m} \mid k = 1, \dots, n; m = 1, \dots, k\}.$$

We denote by δ^{kl} the Kronecker tableaux, i.e. $\delta^{kl}_{kl} = 1$ and other coordinates are equal 0.

For arbitrary tableaux $[\,l\,]$ we will denote

- $[\,l\,]_i = \{l_{ij} \mid j = 1, 2, \dots, i\}$ — the i-th row of $[\,l\,]$;

- $[l]^i = \{l_{kj} \mid k = 1, 2, \ldots, i; j = 1, 2, \ldots, k\}$ — the tableaux formed by the first i rows of $[l]$.

For a vector $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ we denote

$$\mathcal{L} < a >= \{l \in \mathcal{L} \mid l_{ni} = a_i,\ i = 1, \ldots, n\}.$$

We consider a polynomial algebra Λ in $\frac{n(n+1)}{2}$ variables λ_{mi}, where $m = 1, \ldots, n$ and $i = 1, \ldots, m$. Putting $\lambda_{mi}([l]) = l_{mi}$ we identify Λ with the algebra of polynomial functions on $\mathcal{L}$. The product of the symmetrical groups $G = S_1 \times S_2 \times \ldots \times S_n$ acts on $\mathcal{L}$ as follows: S_k permutes elements of $[l]_k$. Clearly this induces an action of G on the algebra Λ.

Proposition 3 ([13],[5])**.** *Let* $\mathbf{i} : \Gamma \to \Lambda$ *be a homomorphism, which maps*

$$c_{mk} \mapsto \sum_{i=1}^{m} (\lambda_{mi} + m)^k \prod_{j \neq i} \left(1 - \frac{1}{\lambda_{mi} - \lambda_{mj}}\right). \tag{1}$$

The image of $\mathbf{i}$ *coincides with the set of* G*-invariant elements in* Λ*.*

We will identify Γ with it's image in Λ.

We choose a new set $\{\gamma_{ij}\}$, $1 \leq j \leq i \leq n$ of generators of the polynomial ring Γ, where γ_{ij} denotes $j-$th elementary symmetrical function on $\lambda_{i1}, \ldots, \lambda_{ii}$.

By $[\mathbf{l}]$ we denote the *universal tableaux* which contains element λ_{ij} on the place ij.

For any $n \geq 1$ denote

$$\begin{array}{rcll}
\mathcal{L}_0(n) & = & & \{[l] \mid l_{ij} \in \mathbb{Z}, 1 \leq j \leq i \leq n-1; l_{nj} = 0, 1 \leq j \leq n\}; \\
\mathcal{L}_1(n) & = & & \{[l] \mid l_{mi} - l_{mj} \notin \mathbb{Z}, 1 \leq i < j \leq m \leq n-1\}; \\
\mathcal{L}_2(n) & = & \mathcal{L}_1(n) \cap & \{[l] \mid l_{mi} - l_{m+1\,j} \notin \mathbb{Z}, \\
 & & & 1 \leq i \leq m; 1 \leq j \leq m+1; 1 \leq m \leq n-1\}; \\
\mathcal{L}_3(n) & = & \mathcal{L}_2(n) \cap & \{[l] \mid l_{ni} - l_{nj} \notin \mathbb{Z}, 1 \leq i < j \leq n\}.
\end{array}$$

Remark that if $[l] \in \mathcal{L}_2(n)$ then $[l]^{n-1} \in \mathcal{L}_3(n-1)$.

The set $\mathcal{L}_0(n)$ is obviously a lattice in $\mathbb{C}^{\frac{n(n+1)}{2}}$. To simplify our notations we will write $\mathcal{L}_i$ instead of $\mathcal{L}_i(n)$, $i = 0, 1, 2, 3$ since n will be fixed.

Denote by $\mathcal{S}_i$, $i = 1, 2, 3$ the multiplicative sets in Λ, where

- $\mathcal{S}_1$ is generated by $(\lambda_{mi} - \lambda_{mj} - t)$, $1 \leq m \leq n-1$, $i \neq j$, $t \in \mathbb{Z}$;
- $\mathcal{S}_2$ is generated by $\mathcal{S}_1$ and $(\lambda_{mi} - \lambda_{m+1\,j} - t)$, $1 \leq i \leq m$, $1 \leq j \leq m+1$, $1 \leq m \leq n-1$, $t \in \mathbb{Z}$;
- $\mathcal{S}_3$ is generated by $\mathcal{S}_2$ and $(\lambda_{ni} - \lambda_{nj} - t)$, $1 \leq i < j \leq n$, $t \in \mathbb{Z}$.

Let Λ_i be the localization of Λ by $\mathcal{S}_i$, $i = 1, 2, 3$. Obviously, $\mathcal{L}_i$ can be identified with the set of maximal ideals of Λ_i.

An U-module V is called *Gelfand-Zetlin module (GZ-module)* provided it is a direct sum of finite - dimensional Γ-modules. For $\chi : \Gamma \to \mathbb{C}$ set

$$V^{\chi} = \left\{v \in V \mid (c - \chi(c))^t v = 0 \text{ for all } c \in \Gamma \text{ and some } t \in \mathbb{N}\right\}.$$

We will call V^{χ} the *GZ-root subspace* corresponding to the root $\chi \in \Gamma^*$. Then the direct sum above can be rewritten as $\bigoplus_{\chi\in\Gamma^*} V^{\chi}$. We will write $\operatorname{Supp} V$ for the set of all non-zero *GZ-roots* of V, i.e. for the set of all $\chi \in \Gamma^*$ such that $V^{\chi} \neq 0$.

The subspace V^{χ} is called *GZ - weight subspace* provided an action of Γ is diagonalizable on it. We will denoted such a subspace by V_{χ}.

For a commutative complex algebra A we define a *family $\mathcal{V}$ of U-modules over A* as an U-A-bimodule (equivalently, as $\mathcal{U}_A = U \otimes A$-module), free as A-module. Denote by $\operatorname{Specm} A$ the space of maximal ideals of A endowed with Zariski topology. For a point $\chi \in \operatorname{Specm} A$, which we identify with the corresponding homomorphism $\chi : A \to \mathbb{C}$, we define *the specialization $\mathcal{V}_{\chi}$ of $\mathcal{V}$ in the point* χ as $\mathcal{V} \otimes_A \mathbb{C}_{\chi}$, where $\mathbb{C}_{\chi}$ is the field $\mathbb{C}$ endowed by χ with an A-module structure. As an example of such family over $S[\mathfrak{H}^S]$ one can consider, following PBW theorem, the family of all modules $M(\lambda, V)$ with fixed V.

We will identify $\mathcal{L}_i$ with $\operatorname{Specm}\Lambda_i$ and will consider the families of U-modules over Λ_i. We will also say that such a family is parametrized by $\mathcal{L}_i$.

The GZ-formulae allows us to define *a universal GZ-module* as a family over Λ_1 and *a universal generic GZ-module* as a family over Λ_2.

Proposition 4. *Consider a free Λ_1-module $\mathcal{V}$ with free generators $\{v_{[l]} | \, [l] - [\mathbf{1}] \in \mathcal{L}_0\}$.*

1. *The following formulae defines an $U \otimes \Lambda_1$ – module structure on $\mathcal{V}$:*

$$c_{mk} v_{[l]} = c_{mk}([l]) v_{[l]}, \; E^{\pm}_m v_{[l]} = \sum_{i=1}^{m} a^{\pm}_{mi}([l]) v_{[l\pm\delta^{mi}]}$$

where $E^+_m = e_{m,m+1}$, $E^-_m = e_{m+1,m}$, $m = 1, \dots, n-1$ and

$$c_{mk}([l]) = \sum_{i=1}^{m} (l_{mi} + m)^k \prod_{j\neq i} \left(1 - \frac{1}{l_{mi} - l_{mj}}\right)$$

$$a^{\pm}_{mi}([l]) = \mp \frac{\prod_j (l_{m\pm1 j} - l_{mi})}{\prod_{j\neq i}(l_{mj} - l_{mi})}.$$

2. *If $[l], [l'] \in \mathcal{L}_1$, then $\mathcal{V}_{[l]} \simeq \mathcal{V}_{[l']}$ if and only if there exists $g \in G$, such that $[l'] - g[l] \in \mathcal{L}_0$.*
3. *A specialization $\mathcal{V}_{[l]}$ of $\mathcal{V}$, where $[l] \in \mathcal{L}_2$ is a simple U-module.*

Proof. Follows from [5, Proposition 21] and Harish-Chandra theorem [5, Proposition 22]. □

A simple module $\mathcal{V}_{[l]}$ will be called *generic* GZ-module provided $[l] \in \mathcal{L}_2$ and *strongly generic* provided $[l] \in \mathcal{L}_3$.

We can naturally view $\mathcal{V}_{[l]}$ as $sl(n, \mathbb{C})$-module. Fix the standard inclusion map $sl(n, \mathbb{C}) \subset gl(n, \mathbb{C})$. This inclusion allows us to consider every $gl(n, \mathbb{C})$-module as $sl(n, \mathbb{C})$-module. Then the sets $\mathcal{L}_i$ will also parametrize the corresponding sets of $sl(n, \mathbb{C})$-modules with the only difference that the layer of an isomorphism class is not countable (as for $gl(n, \mathbb{C})$), but 1-dimensional. Analogously, tableaux will parametrize the weights of *GZ–algebra* $\Gamma \cap U(sl(n,\mathbb{C}))$ for $sl(n,\mathbb{C})$.

From now on we will consider all modules as $sl(n,\mathbb{C})$-modules but keep all notation from this chapter. It will not lead us to the ambiguity. *The main goal of our paper is to describe the submodule structure of $M(\lambda, V)$ for a generic GZ-module V.*

Let S be the subset of π corresponding to the subalgebra $sl(k,\mathbb{C})$ (we will say that such subset is well-defined). Let V be a simple generic GZ-module over $sl(k,\mathbb{C})$ of the form $\mathcal{V}_{[l]}$ and $M(\lambda, V)$ be the corresponding GVM. The modules $M(\lambda, V)$ in a natural way form a family over $\operatorname{Specm}\Lambda_1^S$, where $\Lambda_1^S = \Lambda_1 \otimes S(\mathfrak{H}^S)$. Let $U_i = U_i(\mathfrak{G})$ ($U_i^S(\mathfrak{B}_S)$) denotes the algebra $\mathcal{U}(\mathfrak{G}) \otimes \Lambda_i$ $(U(\mathfrak{B}_S) \otimes \Lambda_i^S)$, $i = 1, 2, 3$. Then $\mathcal{V}^S = \mathcal{V} \otimes S(\mathfrak{H}^S)$ is endowed with a natural structure of $U_i^S(\mathfrak{B}_S)$-module by setting $\mathfrak{n}_S \mathcal{V} = 0$. Denote

$$\mathcal{M}(\mathcal{V}) = U_i \bigotimes_{U_i^S(\mathfrak{B})} \mathcal{V}^S.$$

We will call the Λ_2^S-family $\mathcal{M}(\mathcal{V})$ *universal generic GVM.* At the same way we will call *universal strongly generic GVM* the restriction of this family on the parameter set $\mathcal{L}_3^S$. Clearly, we can obtain GVM $M(\lambda, V)$ induced from generic GZ-module $V = \mathcal{V}_{[l]}$ by specialization of $\mathcal{M}(\mathcal{V})$ with respect to $[\,l\,]$ and λ.

For a GZ-module V we will also denote by V_l the root space corresponding to a tableaux $[\,l\,]$ and say that any non-trivial $v \in V_l$ has *GZ-tableaux* $[\,l\,]$.

Corollary 1 (of Theorem 30 in [5]). *For $[\,l\,] \in \mathcal{L}_1$ there exists a unique simple GZ-module V with $V_l \neq 0$.*

We will also need the following lemma [5, Proposition 21].

Lemma 1. *If V is a GZ-module, V_l is it's root space with the tableaux $[\,l\,] \in \mathcal{L}$, then*

$$e_{i\,i+1}(V_l) \subset \bigoplus_{j=1}^{i} V_{l+\delta^{ij}},\, i = 1, \dots, n-1;\; e_{i+1\,i}(V_l) \subset \bigoplus_{j=1}^{i} V_{l-\delta^{ij}},\, i = 2, \dots, n.$$

For a vector $x = (x_1, x_2, \dots, x_n)$ we define $\pi_i x = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. We will write $x \simeq y$ if there exists $\sigma \in S_n$ such that $x = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$.

Theorem 1. *Let V be a simple GZ-module over $sl(k,\mathbb{C})$ and $\lambda \in (\mathfrak{H}^S)^*$. A module $M(\lambda, V)$ is GZ-module over $sl(n,\mathbb{C})$.*

Proof. Since the module $M(\lambda, V)$ is generated over $sl(n,\mathbb{C})$ by some eigenvector for Γ it should be GZ-module ([5]). □

4. Modules induced from subalgebra of corank one

The aim of this section is to establish some sufficient conditions for the existence of a submodule in $M(\lambda, V)$ in some special case (see below). Theorem 2 is a substitution in the proof of the main theorem 8 of the theory of finite-dimensional representations of the algebra $sl(2,\mathbb{C})$ (see [4, proposition 7.1.15]*), as it used in* [1],[4].

Let $\mathfrak{G} = gl(n+1, \mathbb{C})$, $\pi = \{\alpha_1, \dots, \alpha_n\}$ be the standard set of simple roots, $S = \{\alpha_1, \dots, \alpha_{n-1}\}$, $\mathfrak{G}_S = gl(n, \mathbb{C}) \subset gl(n+1, \mathbb{C})$. Consider a strongly generic GZ-module V over $\mathfrak{G}_S$.

In this case we are able to give an alternative construction of GVM induced from strongly generic GZ-module over $\mathfrak{G}_S$, as a family over the polynomial ring $\Lambda_3[\mathbf{p}]$, where $\mathbf{p}$ is a free variable. Let $v \in V$ be a GZ-vector which has a tableaux $[\,l\,]$. Since V is strongly generic it follows that so is $[\,l\,]$.

For $p \in \mathbb{C}$ consider an algebra homomorphism $\Phi_{n,p} : \Lambda_1(n+1) \to \Lambda_3(n)[\mathbf{p}]$ induced from the following map of tableaux: $[\,l\,] \mapsto [\,l|p\,]$, where $([\,l|p\,])^n = [\,l\,]$ and $[\,l|p\,]_{n+1} = ([\,l\,]_n, p)$.

Lemma 2. *Let* $\mathcal{T} \subset \mathcal{L}_0$

$$\mathcal{T} = \left\{ [\,t\,] \mid t_{n+1\,j} = 0\,\forall\, j;\ t_{nj} \in \mathbb{Z}^- \cup \{0\}\,\forall j;\ t_{ij} \in \mathbb{Z}\,\forall j, i < n \right\}.$$

and

$$\mathcal{I} = \{[\,l|p\,] + [\,t\,] \mid [\,t\,] \in \mathcal{T}\} = \mathcal{I}([\,l|p\,]).$$

Then for $\Lambda_3[\mathbf{p}](n)$*–family* $\mathcal{V}_{\mathbf{p}} = \mathcal{V} \bigotimes_{\Lambda_1} \Lambda_3[\mathbf{p}]$ *a* $\Lambda_3[\mathbf{p}]$*–submodule* $\mathcal{M}_{\mathbf{p}} = \mathcal{V}_p|_I$ *is also* U*–submodule.*

Proof. Follows from proposition 4. □

A specialization of $\mathcal{M}_{\mathbf{p}}$ in the point $[\,l|p\,]$ will be denoted by $M_{[\,l|p\,]}$. The following statement is an easy corollary of GZ-formulae.

Lemma 3. *A restriction of the module* $M_{[\,l|p\,]}$ *on the GZ-weights*

$$[\,l|p\,] + \{[\,t\,] \in \mathcal{T} \mid t_{nj} = 0, j = 1, 2, \dots, n\},$$

is a simple $\mathfrak{G}_S$*-module isomorphic to* $V = V_{[\,l\,]}$.

Remark, that for any GZ-module M and any S–primitive element $v \in M$ with weight decomposition $v = \sum a_{[\,t\,]} v_{[\,t\,]}$ $(v_{[\,t\,]} \in M_t)$ all the components $v_{[\,t\,]}$ are also S–primitive.

We will call an element $[\,s\,] \in \mathcal{I}$ *S-primitive tableaux* if there exists $j \in \{1, \dots, n+1\}$ such that $[\,s\,]_n \simeq \pi_j [\,s\,]_{n+1}$.

Lemma 4. *An element* $v_{[\,s\,]} \in M_{[\,l|p\,]}$ *which has a tableaux* $[\,s\,]$ *is S-primitive if and only if* $[\,s\,]$ *is S-primitive tableaux.*

Proof. Let $v_{[\,s\,]}$ be an S-primitive vector. Then by GZ-formulae (proposition 4, lemma 1)

$$e_{n\,n+1} v_{[\,s\,]} = \sum_{i=1}^{n} a^+_{ni}(s) v_{[\,l+\delta^{ni}\,]} = 0$$

so $a^+_{ni}(s) = 0$ for all i. Thus $[\,s\,]$ is S-primitive.

Conversely, suppose that $[\,s\,]$ is an S-primitive tableaux. It follows easily from GZ–formulae that $e_{n\,n+1} v_{[\,s\,]} = 0$. Assume that for some $1 < i < n+1$ holds $e_{i\,n+1} v_{[\,s\,]} = 0$ and show that $e_{i-1\,n+1} v_{[\,s\,]} = 0$. One have

$$e_{i-1\,n+1} v_{[\,s\,]} = [e_{i-1\,i}, e_{i\,n+1}] v_{[\,s\,]} = e_{i-1\,i} e_{i\,n+1} v_{[\,s\,]} - e_{i\,n+1} e_{i-1\,i} v_{[\,s\,]} =$$

$$0 - e_{i\,n+1}(e_{i-1\,i}v_{[\,s\,]}) = 0.$$

Hence $v_{[\,s\,]}$ is S-primitive. □

Lemma 5. 1. *If $l_{ni} - p \notin \mathbb{N}$ holds for all i then $M_{[\,l|p\,]}$ is simple.*

2. *If there exists k, $1 \le k \le n$ such that $l_{nk} - p \in \mathbb{N}$ then $M_{[\,l|p\,]}$ contains the unique submodule isomorphic to $M_{[\,s|l_{nk}\,]}$, where $[\,s\,]$ is defined as follows:*

$$[\,s\,]_i = [\,l\,]_i,\ i \ne n\ ;\ [\,s\,]_{nj} = [\,l|p\,]_{nj},\ j \ne k\ ;\ [\,s\,]_{nk} = p.$$

Moreover, such k is uniquely defined.

Proof. The reducibility of $M_{[\,l|p\,]}$ causes the existence of a non-trivial S-primitive tableaux in $\mathcal{I}$ by lemma 4. But in the case 1 there were no S-primitive tableaux in I except of the form $[\,l|p\,] + [\,t\,]$, where

$$[\,t\,] \in \mathcal{T}' = \{[\,t_{ij}\,] \in \mathcal{T}\ :\ t_{nj} = 0\}\,.$$

This proves 1.

The same arguments imply that in the second case there exists a unique (up to shift on $\mathcal{T}'$) tableaux $[\,s\,]$ which is S-primitive. Clearly, it defines the unique submodule $M_{[\,s\,]}$. Lemma is proved. □

Lemma 6. *Let V be an simple generic GZ-module over $\mathfrak{G}_S$ and $\lambda \in (\mathfrak{H}^S)^*$. Then $M(\lambda, V)$ has a composition series.*

Proof. We denote by v some GZ-weight generator of V. For $\mu \in \operatorname{Supp} M(\lambda, V)$ consider the $\mathfrak{G}_S$ – module $N = U_S M(\lambda, V)_\mu$. Obviously, N is GZ-module over $\mathfrak{G}_S$. It follows from lemma 1 that there exist only a finite number of GZ-weights in N up to a shift on $\mathcal{T}'$ (see proof of lemma 5). But then one have

$$N = \sum_{(\alpha_1,\ldots,\alpha_s)} X_{\alpha_1} \ldots X_{\alpha_s} U_S v,$$

where each $\alpha_i \in \Delta_- \setminus (\Delta_S)_-$.

Following lemma 1 for a fixed GZ-tableaux $[\,t'\,]$ of N there is finitely many GZ-tableaux $[\,t\,]$ of V such that for any GZ-element v which has GZ-tableaux $[\,t\,]$ holds

$$\Gamma X_{\alpha_1} \ldots X_{\alpha_s} v \cap N_{[\,t'\,]} \ne 0.$$

This means that all GZ-root spaces of N are finite - dimensional and belong to the finite number of the cosets $[\,u\,]+\ \mathcal{L}_0(n)$. Therefore N has a composition series as U_S -module. By [6, Proposition 3] there exist only finitely many cosets $\mu+\ (\mathfrak{H}_S)^*$ with the property that there can exist a subquotient of $M(\lambda, V)$ with S-primitive vector of weight in $\mu + (\mathfrak{H}_S)^*$. This implies that U-module $M(\lambda, V)$ has a composition series.

□

Lemma 7. *Let $M_{[\,l|p\,]}$ be a module satisfying the conditions of lemma 5.1 and generated by GZ-weight S-primitive vector v. Denote $V_{[\,l\,]} = U_S v$ and let $\lambda -\ \rho^S$ be a $\mathfrak{H}^S$-weight of v. Then the canonical epimorphism*

$$p : M(\lambda, V_{[\,l\,]}) \longrightarrow M_{[\,l|p\,]}$$

is an isomorphism (Note, that λ does not depend on the choice of v.)

Proof. Clearly, $M_{[l|p]}$ appears exactly ones as a subquotient in a composition series for $M(\lambda, V_{[l]})$ and the lemma follows. □

Lemma 8. *Let $M_{[l|p]}$ contains the unique submodule $M_{[s]}$ (equivalently, $M_{[l|p]}$ satisfy conditions of lemma 5.2)) and λ and $V_{[l]}$ are defined as in lemma 7. Then there exists an S-primitive element in $M(\lambda, V_{[l]})$ that has GZ-tableaux $[s]$.*

Proof. By universal property of $M(\lambda, V_{[l]})$ we can consider an epimorphism

$$\varphi : M(\lambda, V_{[l]}) \to M_{[l|p]}.$$

The same arguments as in lemma 7 imply that a composition series of $\ker\varphi$ contains only $M_{[s]}$ as subquotients. Let v be an element of $M(\lambda, V_{[l]})$ such that $\varphi(v) = v_{[s]}$, where $v_{[s]}$ denotes a canonical generator of $M_{[s]}$. Then $e_{i\,n+1}v \in \ker\varphi$, since $\varphi(e_{i\,n+1}v) = e_{i\,n+1}v_{[s]} = 0$. But by lemma 1 any GZ-weight of $e_{i\,n+1}v$ does not belong to $\mathrm{Supp}\ker\varphi$ and thus $e_{i\,n+1}v = 0$. Finally, we obtain that v is S-primitive vector. □

Let $M(\lambda, V)$ be GVM generated by S-primitive vector v which has the GZ-tableaux $[l|p]$ (note that now we consider the case of generic, but not necessary strongly generic module V). We assume that there exists $j \in \{1, \dots, n\}$ such that $[l]_{nj} - p \in \mathbb{N}$ and denote by $[s]$ the following tableaux:

$$[s]_{ik} = [l]_{ik}, i \neq n;\ [s]_{nk} = [l]_{nk}, k \neq j;\ [s]_{nj} = p.$$

Theorem 2. *Under the conditions above there exists an S-primitive vector v in the module $M(\lambda, V)$ such that v has an S-primitive GZ-tableaux $[s|l_{nj}]$.*

Proof. It follows from lemmas 7,8 that the statement is true in the case of strongly generic tableaux $[l]$.

We consider the modules $M(\lambda, V)$ as a Λ_2^S-family. Put $[t] = [l|p] - [s|l_{nj}]$. First of all we will show that the set

$$\big\{[u] : \text{there exists } S\text{-primitive vector in } M(\lambda, V_{[u]^n}) \text{ which has the tableaux } [u] - [t]\big\}$$

is Zariski closed in $(\mathfrak{H}^S)^* \times \mathcal{L}_2(n)$.

For $\pi = \{\alpha_1, \dots, \alpha_n\}$ and $1 \leq k \leq n$ we denote

$$\gamma_k = \alpha_k + \cdots + \alpha_n \in \Delta.$$

For $N \in \mathbb{N}$ we consider a finite dimensional space $F_N \subset U$ generated by all elements of the form

$$f(k_1, \dots, k_n) = X_{-\gamma_n}^{k_1} X_{-\gamma_{n-1}}^{k_1} \dots X_{-\gamma_1}^{k_n}$$

such that $\sum_{i=1}^{n} k_i = N$. One can choose N such that an U_S-module $U_S F_N\, v$ contains a non-zero element which has GZ-tableaux $[s]$.

By lemma 1 there exists only tableau $[\,q\,]$ such that for any GZ-element $v_{[\,q\,]}$ of the module V with GZ-tableaux $[\,q\,]$ holds:

$$\Gamma F_N\, v_{[\,q\,]} \cap M(\lambda, V)_{[\,s\,]} \neq 0.$$

Clearly, this intersection depends only on N and $[\,t\,]$. By PBW-theorem there exists a Λ_2^S-free submodule $\mathcal{N}$ of finite rank in $\mathcal{M}(\mathcal{V})$ containing all such tableaux in any specialization of $\mathcal{M}(\mathcal{V})$. Thus we can rewrite the condition for the existence of a non-trivial S-primitive element which has the GZ-tableaux $[\,p\,]-[\,t\,]$ as a closed condition in $(\mathfrak{H}^S)^* \times \mathcal{L}_2(n)$. Clearly, for some $k \in \mathbb{N}$ this condition is equivalent to the condition that Λ_2^S-linear operators x_{γ_s}, $s = 1, \ldots, n$

$$x_{\gamma_s} : F_N\mathcal{N} \longrightarrow (U(\mathfrak{n}_-))^k,\ m \mapsto X_{\gamma_s} m$$

have (after a specialization) common non-trivial kernel. And the last is equivalent to the condition that some determinant with coefficients from Λ_2^S equals zero.

To complete the proof we only need to note that the set of strongly generic modules is dense in topology considered. □

5. Weyl group action and Generalized Harish-Chandra isomorphism

Here we introduce a set of parameters of GVM's and define an action of the Weyl group on it. We prove an analogue of the classical Harish-Chandra isomorphism theorem that gives necessary conditions for the existence of a submodule in GVM.

The following statement follows easily from [5] and previous section, so we omit it's proof:

Lemma 9. *Let V be a $\mathfrak{G}_n$-module, generated by S-primitive vector v for some well-defined S. Suppose that v is GZ-weight vector which has a tableaux $[\,l\,]$. Consider a $\mathfrak{G}_{n+1}$-module $V_1 = M(\lambda, V)$ for some $\lambda \in (\mathfrak{H}^S)^*$. Then the module V_1 is generated by S-primitive vector $v_1 = 1 \otimes v$, moreover v_1 is a GZ-vector and there exists $p \in \mathbb{C}$ (depending on λ) such that v_1 has the tableaux $[\,s\,] = [\,l|p\,]$.*

This statement allows us to introduce the main construction. We return to the situation where $\mathfrak{G}_S = gl(k, \mathbb{C}) \subset gl(n, \mathbb{C}) = \mathfrak{G}$. Consider a map $\Phi : \mathbb{C}^n \oplus \mathcal{L}(k-1) \to \mathcal{L}(n)$ defined as follows: for $(x, [\,y\,]) \in \mathbb{C}^n \oplus \mathcal{L}(k-1)$, $x = (x_1, \ldots, x_n)$ we set $\Phi(x, [\,y\,]) = [\,l\,]$, where

$$[\,l\,]^{k-1} = [\,y\,]; [\,l\,]_{ij} = x_j, i \geq k.$$

We will also denote $\Phi(x, [\,y\,])$ by $[\,y{\uparrow}x\,]$ and we will denote by $\mathcal{A}$ the image of Φ. Define an action of the symmetrical group $S_n = W$ (recall that it is the Weyl group of Δ) on the set $\mathcal{A}$ as follows: for $\sigma \in S_n$ set

$$[\,y{\uparrow}x\,]^\sigma = [\,y{\uparrow}\sigma x\,],$$

where $\sigma x = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. W acts also on the quotient space $\mathcal{A}_S = \mathcal{A}/\Phi(\vec{0} \oplus \mathcal{L}(k-1))$. We will call this action *the action of the Weyl group on the space of parameters of GVM* and denote it by $W_{\Gamma,S}$.

The following lemma is obvious:

Lemma 10. 1. *Let* $[\,y\,] \in \mathcal{L}_3(k-1)$, $x \in \mathbb{C}^n$. *Let* V *be a simple GZ-module over* $\mathfrak{G}_S$, *generated by vector which has a tableaux* $[\,y{\uparrow}x\,]^k$. *Then there exists a unique* $\lambda \in (\mathcal{H}^S)^*$ *such that* $M(\lambda, V)$ *is generated by GZ-weight vector which has tableaux* $[\,y{\uparrow}x\,]$.

2. *Let* V *be a simple GZ-module over* $\mathfrak{G}_S$ *generated by an* S*-primitive element which has tableaux* $[\,y\,]$. *Assume that GVM* $M(\lambda, V)$ *is generated by GZ-weight vector* $v_{[\,l\,]}$ *which has tableaux* $[\,l\,]$. *Then* $[\,l\,]$ *is of the form* $[\,[\,y\,]^{k-1}{\uparrow}x\,]$ *for suitable* $x \in \mathbb{C}^n$.

We will also work with the subset $\mathcal{B} \subset \mathcal{A}$ of parameters corresponding to the modules that are induced from generic GZ-module. Clearly, $\mathcal{B}$ is invariant under $W_{\Gamma,S}$. Lemma 10 allows us to define GVM $M([\,l\,])$ for $[\,l\,] \in \mathcal{B}$. In this way the action of W on the set of parameters of GVM is correctly defined (note, that this action does not induce an action on the isomorphism classes).

Lemma 11. *Consider a natural inclusion* $W_S \simeq S_k \hookrightarrow W \simeq S_n$ *as a subgroup, permuting first* k *components. Then for* $\sigma \in W_S$ *holds* $M_{[\,l\,]} \simeq M_{[\,l\,]^\sigma}$.

Proof. Follows from the second part of proposition 4. □

Let φ_S denotes an S-homomorphism of Harish-Chandra (see [6] for more details). Consider an automorphism γ of $S(\mathfrak{H})$ defined by $\gamma(p)(\lambda) = p(\lambda - \rho)$ for $\lambda \in \mathfrak{H}^*$ and polynomial function p on $\mathfrak{H}^*$. Similarly define an automorphism γ_S of $S(\mathfrak{H}_S)$ by $\gamma_S(p)(\lambda) = p(\lambda - \rho_S)$. Set $\gamma^S = \gamma|_{S(\mathfrak{H}^S)}$.

Define a homomorphism $i_S : Z(\mathfrak{G}_S) \otimes S(\mathfrak{H}^S) \to S(\mathcal{A}^*)$ as a composition

$$Z(\mathfrak{G}_S) \otimes S(\mathfrak{H}^S) \xrightarrow{\;p\;} \Gamma \xrightarrow{\;i\;} S(\mathcal{A}^*),$$

where i is defined in (1) and p is the canonical inclusion. We consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & S(\mathcal{A}^*) & \xrightarrow{\;j\;} & S(\mathfrak{H}) \\
 & & {\scriptstyle i_S}\uparrow & & \uparrow \\
Z(\mathfrak{G}) & \xrightarrow{(1\otimes\gamma^S)\circ\varphi_S} & Z(\mathfrak{G}_S) \otimes S(\mathfrak{H}^S) & \xrightarrow{(\gamma_S\circ\varphi^S)\otimes 1} & S(\mathfrak{H}_S)^{W_S} \otimes S(\mathfrak{H}^S)
\end{array}$$

Here j is an isomorphism which commutes with the actions of W on $S(\mathcal{A}^*)$ and on $S(\mathfrak{H})$ and which makes the diagram above commutative. One can construct j applying this diagram to the primitive generators of classical Verma module.

We can extend our commutative diagram by the row of the invariant algebras:

$$\begin{array}{ccccc}
 & & S(\mathcal{A}^*) & \xrightarrow{\;j\;} & S(\mathfrak{H}) \\
 & & {\scriptstyle i_S}\uparrow & & \uparrow \\
Z(\mathfrak{G}) & \xrightarrow{(1\otimes\gamma^S)\circ\varphi_S} & Z(\mathfrak{G}_S) \otimes S(\mathfrak{H}^S) & \xrightarrow{(\gamma_S\circ\varphi^S)\otimes 1} & S(\mathfrak{H}_S)^{W_S} \otimes S(\mathfrak{H}^S) \\
 & & {\scriptstyle q}\uparrow & & \uparrow \\
 & & S(\mathcal{A}^*)^{W_{\Gamma,S}} & \xrightarrow{\;j_S\;} & S(\mathfrak{H})^W
\end{array}$$

Here q denotes the canonical inclusion and j_S is isomorphism. Consider the composition $\psi_S = q^{-1} \circ (1 \otimes \gamma^S) \circ \varphi_S$. The commutativity of the diagram above gives us the following generalization of Harish-Chandra theorem:

Theorem 3.

$$Z(\mathfrak{G}) \overset{\psi_S}{\simeq} S(\mathcal{A}^*)^{W_{\Gamma,S}}$$

Denote by $M([\,l\,])$ the GVM generated by S-primitive element which has GZ-tableaux $[\,l\,]$. Assume that $[\,l\,]^k \in \mathcal{L}_2$. Denote by $\theta_{[\,l\,]}$ the central character of GVM module $M([\,l\,])$.

Corollary 2. *The following statements are equivalent*

1. $\theta_{[\,l\,]} = \theta_{[\,l'\,]}$;
2. *there exists* $g \in W_{\Gamma,S}$ *such that* $\Phi^{-1}([\,l\,] - g[\,l'\,]) \in \vec{0} \times \mathcal{L}_0(k-1)$.

Proof. Is analogues to [4, Proposition 7.4.7] and [8, Theorem 6.3]. □

Corollary 3. *Let* $M(\lambda, V)$ *be a GVM, isomorphic to* $M([\,l\,])$ *for some* $[\,l\,]$ *with* $[\,l\,]^k \in \mathcal{L}_2(k)$. *Then*

1. *For arbitrary tableaux* $[\,t\,] \in \operatorname{Supp} M(\lambda, V)$ *holds* $[\,t\,]^k - [\,l\,]^k \in (\mathcal{L}_0(k+1))^k$;
2. *for any* S*-primitive element* v *and for any* $c \in Z(\mathfrak{G})$ *holds*

$$cv = \psi_S(c)v.$$

6. Kostant S-function and Gelfand-Kirillov dimension

In this section we establish two basic facts (corollary 4, 5) about GVM which are the analogues of [4, proposition 7.6.3] *and* [4, proposition 7.6.6]. *To prove these facts we need some* S*-analogue for the notion of the dimension of the weight space. As a substitute we propose the length of certain composition series. Moreover, we calculate this value using the notion of Kostant* S *-function.*

Let Δ' be some root system with a fixed base π'. Consider some $S' \subset \pi'$. For

$$\gamma = \sum_{\beta \in \pi'} a_\beta \beta$$

set

$$\psi_{S'}(\gamma) = \sum_{\beta \in (\pi' \setminus S')} a_\beta \beta.$$

We recall that $\pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$, $S = \{\alpha_1, \ldots, \alpha_{k-1}\}$, $\pi \setminus S = \{\alpha_k, \ldots, \alpha_{n-1}\}$. Define the *Kostant* S*-function* $c_S : (\mathbb{Z}_+)^{n-k} \to \mathbb{N}$ as follows: for $x = \{x_k, \ldots, x_{n-1}\} \in (\mathbb{Z}_+)^{n-k}$ set $c_S(x)$ to be a number of the following decompositions

$$x_k\alpha_k + x_{k+1}\alpha_{k+1} + \ldots x_{n-1}\alpha_{n-1} = \sum_{\alpha \in \Delta_+ \setminus (\Delta_S)_+} n_\alpha \psi_S(\alpha)$$

with integer $n_\alpha \geq 0$.

Let $M(\lambda, V)$ be GVM induced from a generic GZ-module V with S-primitive generator $v_{[\,l\uparrow x\,]}$ which has tableaux $[\,l\uparrow x\,] \in \mathcal{A}$ (see lemma 10). For the non-negative integers s_k,

$\dots,\ s_{n-1}$ we define the growth function $H_{[\,l\!\uparrow\!x\,]}(s_k,\dots,\ s_{n-1}) \in \mathbb{Z}$ as the length of the $\mathfrak{G}_S$ – module $U_S M(\lambda,V)_\mu$, $\mu = \lambda - \rho^S - s_k\alpha_k - \dots - s_{n-1}\alpha_{n-1}$.

Theorem 4. 1. *$H_{[\,l\uparrow x\,]}(s_k,\dots,s_{n-1})$ is a polynomial growth function $H(s_k,\dots,s_{n-1})$ that does not depend on $[\,l\!\uparrow\!x\,]$.*
2. *$H(s_k,\dots,s_{n-1}) = c_S(s_k,\dots,s_{n-1})$*

Proof. Let $M(\lambda,V)$ be a GVM generated by S–primitive vector $v_{[\,l\uparrow x\,]}$ which has tableaux $[\,l\!\uparrow\!x\,]$ such that $[\,l\!\uparrow\!x\,]^k \in \mathcal{L}_3$. Suppose that $M(\lambda,V)$ is simple. Then, following lemma 7 and general construction of tableaux modules from section 4, one can consider $M(\lambda,V)$ as a tableaux module $M_{[\,l\uparrow x\,]}$, which is a submodule of corresponding specialization $\mathcal{V}_{[\,l\uparrow x\,]}$. Thus it follows from GZ-formulae that $U_S M(\lambda,V)_\mu$ is a direct sum of simple generic modules over U_S. Hence both statements of the theorem can be easily obtained by direct calculations.

Following corollary 1 one can check that all the subquotients of $\mathfrak{G}_S$-module $V^{S,\mu} = U_S M(\lambda,V)_\mu$ are generic GZ-modules. If a simple generic $\mathfrak{G}_S$ – module T generated by GZ-vector $v_{[\,t\,]}$ has multiplicity $\mu_T(V^{S,\mu})$ in a composition series of $V^{S,\mu}$ then $\dim_{\mathbb{C}} V^{S,\mu}_{[\,t\,]} = \mu_T(V^{S,\mu})$.

Consider $M(\lambda,V)$ as a specializations of Λ_2^S-family $\mathcal{M}(\mathcal{V})$. At the same way as in theorem 2 one can show that for any $[\,s\,] \in \mathcal{L}_0(n)$ there exists a free Λ_2^S-direct summand $\mathcal{N}^{[\,s\,]}$ of $\mathcal{M}(\mathcal{V})$ such that after any specialization $\mathcal{M}_{[\,l\uparrow x\,]}$ all GZ-root subspaces of the weight $[\,t\,] = [\,l\!\uparrow\!x\,] + [\,s\,]$ belong to the subspace $\mathcal{N}^{[\,s\,]}_{[\,l\uparrow x\,]} \subset \mathcal{M}_{[\,l\uparrow x\,]}$. Consider a Λ_2^S-linear map $\mathcal{X}: \mathcal{N}^{[\,s\,]} \to \mathcal{M}(\mathcal{V})^{\frac{k(k+1)}{2}}$ defined by

$$x \mapsto ((\gamma_{ij} - \mathbf{l}_{ij})^N x)_{1\le i<j\le k}$$

for N large enough. Then every specialization $\mathcal{M}_{[\,l\uparrow x\,]}(\mathcal{V})$ induces a linear map $\mathcal{X}_{[\,l\uparrow x\,]}$. Obviously, the kernel of $\mathcal{X}_{[\,l\uparrow x\,]}$ is the root space of $\mathcal{M}_{[\,l\uparrow x\,]}(\mathcal{V})$. Thus the property of $\mathcal{X}_{[\,l\uparrow x\,]}$ to have the smallest possible dimension of the kernel is open in $\mathcal{L}_2(k)^S = \mathcal{L}_2(k) \oplus (\mathfrak{H}^S)^*$.

We will prove that the value of $H_{[\,l\uparrow x\,]}(s_k,\ \dots,\ s_{n-1})$ with fixed $s_k,\ \dots,\ s_{n-1}$ can only increase in some neighborhood of any $[\,l\!\uparrow\!x\,]$. Then the last two facts together with the observation that there exists simple $M([\,l'\!\uparrow\!x'\,])$ in every neighborhood of $[\,l\!\uparrow\!x\,]$ complete the proof.

Fix $M(\lambda',V)$ generated by an S-primitive vector of weight $\lambda' - \rho^S$ and some $s_k,\dots,s_{n-1}$. Denote $\mu = \lambda' - \rho^S - s_k\alpha_k - \dots - s_{n-1}\alpha_{n-1}$. Let $\{f_1,\dots,f_c\} \subset U$ be the set of all elements of the form $X^{t_1}_{-\beta_1}\dots X^{t_l}_{-\beta_l}$ with $\beta_i \in \Delta_S \cup \Delta_+$ such that

$$t_1\beta_1 + \dots + t_l\beta_l = s_k\alpha_k + \dots + s_{n-1}\alpha_{n-1}.$$

It is obvious that $U_S M(\lambda,V)_\mu = \sum_{i=1}^{b} U_S f_i v_{[\,l\uparrow x\,]}$, where $v_{[\,l\uparrow x\,]}$ is an S-primitive generator of $M(\lambda,V)$.

Consider the standard filtration of U_S:

$$U_{S,0} \subset U_{S,1} \subset \dots,$$

where $U_{S,i}$ is the set of all linear combinations of monomials whose length less or equal i. By PBW theorem we have $U_S = \bigcup_{i=0}^{\infty} U_{S,i}$.

Denote by $p(i)$ the dimension of the space $P_i = < U_{S,i} f_1 v_{[l\uparrow x]}, \ldots, U_{S,i} f_t v_{[l\uparrow x]} >$. One can easily check that if for two modules $M(\lambda_1, V_1)$ and $M(\lambda_2, V_2)$ holds $H_1(s_{k+1}, \ldots, s_n) \geq H_2(s_{k+1}, \ldots, s_n)$, then there exists j such that for all $i > j$ holds $p_1(i) > p_2(i)$. Conversely, if for all i large enough $p_1(i) \geq p_2(i)$, then $H_1(s_{k+1}, \ldots, s_n) \geq H_2(s_{k+1}, \ldots, s_n)$. On the other hand consider the Λ_2^S-module $\mathcal{P}_i \subset \mathcal{M}(\mathcal{V})$ generated by $U_{S,i} f_1 v_{[l\uparrow x]}, \ldots, U_{S,i} f_t v_{[l\uparrow x]}$. It's specialization coincides with P_i. Thus the property of $\mathcal{P}_i$ to have the specialization of maximal possible dimension is an open property in Λ_2^S. Hence $p(i)$ can not decrease in a small neighborhood and neither can $H(s_{k+1}, \ldots, s_n)$. □

Let A and B be associative algebras, $B \subset A$, A is finitely generated over B, say $A = B\{x_1, x_2, \ldots, x_m\}$. One can associate with this pair a filtration of $B - B$ bimodules

$$B = B_0 \subset B_1 \subset B_2 \subset \ldots,$$

where

$$B_i = \sum_{1 \leq j_1, \ldots, j_s \leq m; s \leq i} B x_{j_1} B x_{j_2} B \ldots B x_{j_s} B.$$

Let M be an A-module, $N \subset M$ be a B-submodule such that $M = AN$ and for every i the module $B_i N$ is of finite length $l(B_i N)$ as B-module. We define B-Gelfand-Kirillov dimension d_B of an A-module M via

$$d_B(M) = \overline{\lim_{i \to \infty}} \frac{\log l(B_i N)}{\log i}. \tag{2}$$

One can check that $d_B(M)$ does not depend on the choice of N and on the choice of $\{x_1, \ldots, x_m\}$.

For a module M we will write

$$ST(M) = \lim_{i \to \infty} \frac{l(B_i N)}{i^{d_B(M)}},$$

in the case when this limit exists.

In our case one can calculate the $U(\mathfrak{G}_S)$-Gelfand-Kirillov dimension of the module $M(\lambda, V)$.

For $s = (s_{k+1}, \ldots, s_n)$ set $|s| = s_{k+1} + \cdots + s_n$. For $l \geq 0$ let

$$G(l) = \sum_{s : |s| \leq l} H(s).$$

Obviously, G is a polynomial growth function. Let $D_S(M(\lambda, V))$ be a growth grade of G.

Lemma 12. *For generic GZ-modules V, W and $\lambda, \mu \in (\mathfrak{H}^S)^*$ holds*

1. $D_S(M(\lambda, V)) = d_{U(\mathfrak{G}_S)}(M(\lambda, V))$;
2. $d_{U(\mathfrak{G}_S)}(M(\lambda, V)) = d_{U(\mathfrak{G}_S)}(M(\mu, W))$;
3. $ST(M(\lambda, V)) = ST(M(\mu, W))$.

4. $d_{U(\mathfrak{G}_S)}(M(\lambda,V)/N) < d_{U(\mathfrak{G}_S)}(M(\lambda,V))$ *for every non-trivial* $N\subset M(\lambda,V)$.

Proof. The first statement can be obtained by direct calculation using (2), where $A = U(\mathfrak{G})$, $B = U(\mathfrak{G}_S)$, $x_1 = e_{k\,k+1},\dots, x_{n-k} = e_{n-1\,n}$, $x_{n-k+1} = e_{k+1\,k},\dots, x_{2(n-k)} = e_{n\,n-1}$, $N = V$. In this case it is easy to see that $l(B_iV) = G(i)$.

The second and the third statements follows from the fact that G is a polynomial function and theorem 4.

Every non-trivial submodule N of $M(\lambda,V)$ contains some S-primitive vector. Thus we have $d_{U(\mathfrak{G}_S)}(N) \geq D_S(M(\lambda,V))$ and from $N\subset M(\lambda,V)$ we obtain, that $d_{U(\mathfrak{G}_S)}(N) = D_S(M(\lambda,V))$. Moreover, $ST(N)\geq ST(M(\lambda,V))$ and from $N\subset M(\lambda,V)$ we obtain, that $ST(N) = ST(M(\lambda,V))$. Thus the last statement follows. □

Corollary 4. *Let $M(\lambda,V)$ be GVM induced from a generic simple GZ-module V. Every two non-trivial submodules of $M(\lambda,V)$ have a non-trivial intersection. In particular, there exists the minimal submodule in $M(\lambda,V)$.*

Proof. Using lemma 12,4 one can prove this following the proof of analogues statement [9, Lemma 2.2.2]. □

The last corollary immediately implies the following.

Corollary 5. *Let $M(\lambda_i,V_i)$ be GVMs induced from generic simple generalized Verma modules V_i, $i=1,2$. Then*

$$\dim \operatorname{Hom}_U(M(\lambda_1,V_1),M(\lambda_2,V_2))\leq 1$$

and every non-zero element of this space is an injective map.

Proof. First we note that for every $v\in M(\lambda_i,V_i)$ $(i=1,2)$ there exists an S-primitive generator w and an element $u\in U(\mathfrak{n}_-)$ such that $v = uw$. Now the proof is analogues to the proof of [4, Theorem 7.6.6]. □

7. The submodule structure of $M(\lambda,V)$

We are in position now to obtain our main result: the criterion for $M([\,l{\uparrow}x\,])$ to be a submodule in $M([\,s{\uparrow}y\,])$.

We have to note that the methods we will use to prove our main results are similar to the original methods from [1]. For today there exists a simple modern technique to obtain the BGG-theorem for Verma modules which use the notion of Shapovalov form and Jantzen filtration (see for example [12]). It seems to be rather non-trivial to develop an analogue of the Shapovalov form on GVMs induced from generic GZ-modules since Γ has no bimodule complement in U. This easily follows from the existence of simple GZ-module which has a non-trivial GZ-root (not weight!) subspace ([7]).

Recall that for $[\,l\,]\in\mathcal{A}$ we will denote by $M([\,l\,])$ GVM generated by S-primitive GZ-vector which has tableaux $[\,l\,]$.

Let $\Pi:\mathcal{A}\to\mathcal{L}(k-1)$ be a canonical projection $\Pi([\,l\,]) = [\,l\,]^{k-1}$. For $r\in\mathcal{L}_2(k-1)$ denote $\mathcal{A}_r = \Pi^{-1}(r)$. It is obvious that $W_{\Gamma,S}\mathcal{A}_r\subset\mathcal{A}_r$. One can prove the following by standard arguments:

Lemma 13. *Let $r - r' \notin \mathcal{L}_0(k-1)$ and $[\,l\,] \in \mathcal{A}_r$, $[\,s\,] \in \mathcal{A}_{r'}$. Then*

$$\dim \operatorname{Ext}^{\,i}_U(M([\,l\,]), M([\,s\,])) = 0,\ i \geq 0.$$

Lemma 14. *A module M generated by S-primitive GZ-weight vector v such that the module $U_S v$ is a generic GZ-module will be a GVM if and only if M is $U(\mathfrak{n}_S^-)$-free.*

Proof. The necessity of this condition is obvious. To prove it's sufficiency consider a GVM $M([\,l\,])$, generating by S-primitive GZ-weight element w which has the same tableaux as v. Clearly $[\,l\,] \in \mathcal{B}$. Then under the conditions of lemma, a map $\psi : M([\,l\,]) \to M$, $(w \mapsto v)$ is monomorphism and thus $M \simeq M([\,l\,])$. □

Lemma 15. *Let $r \in \mathcal{L}_2(k-1)$ and $[\,l\,] \in \mathcal{A}$. Then the set*

$$\{[\,s\,] \in \mathcal{A}_r \ : \ M([\,s\,] - [\,l\,]) \subset M([\,s\,])\}$$

is Zariski closed in $\mathcal{A}_r$.

Proof. Based on lemma 14 the proof is quite analogues to the proof of theorem 2 from section 4 and to the proof of [8, Proposition 7.1] and [4, Proposition 7.6.12]. □

For $x, y \in \mathbb{C}^m$ we will write $x \leq y$ if there exists $z \simeq x$ such that $y - z \in \mathbb{Z}_+^m$. We define a partial order on $\mathcal{A}_r$ by setting $[\,l\,] \leq [\,s\,]$ provided $[\,l\,]_i \leq [\,s\,]_i$ holds for all i, $k \leq i \leq n$.

Lemma 16. 1. *Consider a subset $\mathcal{T}_k \subset \mathcal{L}_0$*

$$\mathcal{T}_k = \{[\,t\,] \,|\, t_{nj} = 0, \forall\, j;\ t_{ij} \in -\mathbb{Z}_+, \forall j \leq i,\ k \leq i < n;\ t_{ij} \in \mathbb{Z}, \forall\, j \leq i < k\}\,.$$

For $x \in \mathbb{C}^n$ such that $x_i - x_j \notin \mathbb{Z}$ for $i \neq j$; $i, j < n$ and for $y \in \mathcal{L}_3(k-1)$ denote

$$\mathcal{I}_y^x = \{[\,y {\uparrow} x\,] + [\,t\,] \,|\, [\,t\,] \in \mathcal{T}_k\}\,.$$

Then for $\Lambda_3[\mathbf{x}_1, \ldots, \mathbf{x}_n](k)$–family $\mathcal{V}_{\mathbf{x}} = \mathcal{V} \otimes_{\Lambda_1} \Lambda_3[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ a $\Lambda_3[\mathbf{x}_1, \ldots, \mathbf{x}_n]$–submodule $\mathcal{M}_{\mathbf{x}} = \mathcal{V}_x|_I$ is also U–submodule.

2. *If $x \in \mathbb{C}^n$ such that there exists the unique $i < n$ with property $x_i - x_n \in \mathbb{N}$ then $\mathcal{M}_{\mathbf{x}}$ has the unique submodule for every $y \in \mathcal{L}_3(k-1)$.*

Proof. Follows immediately from GZ-formulae. □

Proposition 5. *If w is a reflection in $W_{\Gamma,S}$ and $w[\,l\,] \leq [\,l\,]$ for some $[\,l\,] \in \mathcal{A}$ then $M(w[\,l\,]) \subset M([\,l\,])$.*

Proof. We fix $t = [\,l\,]_n - w[\,l\,]_n$. It follows from lemma 15 that the set

$$\mathcal{C}(t) = \{[\,s\,] \in A_r \,|\, M([\,s'\,]) \subset M([\,s\,]);\ [\,s\,]_n - [\,s'\,]_n = t,\ s' \in \mathcal{A}_r\}$$

is Zariski closed in $\mathcal{A}_r$. Consider a set

$$C_w^t = \{[\,s\,] \in \mathcal{A}_r \,|\, ([\,s\,] - w[\,s\,])^n = t\}\,.$$

Using lemma 16 and following the proof of [11, Lemma 4.3] one can show that there exists a set $\mathcal{D} \in \mathcal{A}_r$ such that the closure of $\mathcal{D}$ in Zariski topology contains C_w^t and for every $[\,s\,] \in \mathcal{D}$ holds $M([\,s'\,]) \subset M([\,s\,])$ for $[\,s'\,] \in \mathcal{A}_r$ with $[\,s\,]^n - [\,s'\,]^n = t$. □

For a Verma module $M(\lambda)$, $\lambda \in \mathfrak{H}^*$ we will denote by χ_λ its central character. For the next step we need the following fact from [10]:

Theorem 5 ([10]). *Let*

- *V be a $\mathfrak{G}$-module with a central character χ_λ for some $\lambda \in \mathfrak{H}^*$;*
- *V_μ be a finite-dimensional module with $\{\mu_1, \dots, \mu_s\}$ as the set of all distinct weights.*

Then for any $c \in Z(\mathfrak{G})$ an element

$$\prod_{i=1}^{s}(c - \chi_{\lambda+\mu_i}(c))$$

annihilates $V \otimes V_\mu$.

This theorem immediately implies the following:

Corollary 6. *Let V be a simple GZ-module and F be a finite-dimensional module over U. Then* $\operatorname{Supp} V \otimes F \subset \operatorname{Supp} V + \operatorname{Supp} F$. *In particular, if* $\operatorname{Supp} V \subset [\,s\,] + \mathcal{L}_0$, *then* $\operatorname{Supp} V \otimes F \subset [\,s\,] + \mathcal{L}_0$. *Moreover $V \otimes F$ is Γ-diagonalizable as soon as V is strongly generic.*

Proof. Can be easily obtained by the restriction of $V \otimes F$ to subalgebras $sl(m, \mathbb{C})$ for $m \leq n$. □

Lemma 17. *Let F be a finite-dimensional $U(\mathfrak{G})$-module and $[\,l\,] \in \mathcal{A}$. Then there exists a filtration*

$$0 \subset M_0 \subset M_1 \subset \cdots \subset M_m = M([\,l\,]) \otimes F$$

such that for all $i = 1, \dots, m$ holds $M_i/M_{i-1} \simeq M([\,l_i\,])$ for some $[\,l_i\,] \in \mathcal{A}$.

Proof. Using corollary 6 and lemma 14 one can obtain the statement following the proof of [11, Lemma 4.7]. □

The following statement follows easily from comparison of dimensions of GZ-root spaces:

Corollary 7. *For all $p_i \in \operatorname{Supp} F$ a module $M([\,l\,]) \otimes F$ has a subquotient isomorphic to $M([\,l\,] + [\,p_i\,])$.*

Consider a complex vector space $\mathcal{A}_0$ of dimension n. One can choose the base $\{x_1, x_2, \dots, x_n\}$ in $\mathcal{A}_0$ in such a way that $W_{\Gamma,S}$ acts as the Weyl group of the root system of type A_{n-1} with the base $\{x_1, \dots, x_{n-1}\}$ and $W_{\Gamma,S} x_n = x_n$. Define the Weyl chambers in $\langle x_1, \dots, x_{n-1}\rangle$ as connected components in $\langle x_1, \dots, x_{n-1}\rangle \setminus \mathcal{E}$, where $\mathcal{E}$ denotes the set of all points x satisfying $< \Re x, x_i >= 0$ for some i, $1 \leq i \leq n-1$ (here $< \cdot, \cdot >$ denotes the standard $W_{\Gamma,S}$-invariant form). For a Weyl chamber C in $\langle x_1, \dots, x_{n-1}\rangle$ we will say that $C + \langle x_n\rangle$ is a Weyl chamber in $\mathcal{A}_0$. For $r \in \mathcal{L}(k-1)$ and a Weyl chamber C in $\mathcal{A}_0$ we define $C + [\,r{\uparrow}0\,]$ to be a Weyl chamber in $\mathcal{A}_r$.

The proof of the subsequent theorem 7 is based on the following reformulation of [1, Theorem 2] (see also [4, Theorem 7.6.23]):

Theorem 6. *Let A be an algebraicaly open subspace of a $\mathbb{C}$-space parametrizing (not necessary one-to-one) the set of isomorphism classes of $\mathfrak{G}$-modules $\{M(x) \in \mathfrak{G}\text{-mod},\ x \in A\}$. Suppose that there exists an action of W on A satisfying the following conditions:*

1. *W is a Weyl group of some root system $\Sigma \subset A$ with a fixed base B.*
2. *The stabilizer of isomorphism class is a subgroup W_1 which is a Weyl group of the root system generated by some $C \subset B$.*
3. *If $<$ denotes the standard order on A corresponding to B then for any simple root $\beta \in B$ such that $w_\beta x < x$ holds $M(w_\beta x) \subset M(x)$.*
4. *For any $x \in A$ and any finite-dimensional module F the module $M(x) \otimes F$ has a filtration with quotients of type $M(x_i)$, $i = 1, \dots, k$. Moreover the set $\{x - x_i; i = 1, \dots, k\}$ depends only on module F and is a subset of $\mathbb{Z}B$.*
5. *For any x, $y \in A$ central characters of modules $M(x)$ and $M(y)$ coincide if and only if $x \in Wy$.*
6. *If a module N has a filtration with quotients $\{M(x_i) : i = 1, \dots, k\}$ and x_j is maximal in the set $Wx_j \cap \{x_i : i = 1, \dots, k\}$ then $M(x_j) \subset N$.*
7. $\dim \operatorname{Hom}(M(x), M(y)) \leq 1$ *for any x, $y \in A$ and every nonzero homomorphism is a monomorphism.*

Then $\operatorname{Hom}(M(x), M(y)) \neq 0$ *implies that there exists a sequence $\beta_1, \dots, \beta_s \in \Sigma_+$, such that*

$$x < w_{\beta_1} x < \dots < w_{\beta_s} \dots w_{\beta_1} x = y.$$

Proof. One can prove this theorem following step by step the original proof of [1, Theorem 2]. □

Theorem 7. *Let $[\,l\,], [\,s\,] \in \mathcal{A}_r \cap \mathcal{B}$ for some $r \in \mathcal{L}_2(k-1)$. If $M([\,l\,]) \subset M([\,s\,])$ then there exist $w_1, \dots, w_t$ — reflections in $W_{\Gamma,S}$ such that*

$$[\,l\,] \leq w_1[\,l\,] \leq \dots \leq w_t \dots w_1[\,l\,] = [\,s\,].$$

Proof. We set $A = \mathcal{B} \cap \mathcal{A}_r$. $W_{\Gamma,S}$ acts on A as a Weyl group with the chosen fixed root systems and the Weyl chambers defined above. Then conditions 1,5 of theorem 6 follows from Generalized Harish-Chandra theorem (section 5); 3 follows from theorem 2; 4 follows from lemma 17 and corollary 6; 2 follows from lemma 11; 6 is obvious and 7 follows from corollary 5. Now the statement follows immediately from theorem 6 and previous lemmas. □

Combining the results obtained in this section one can prove the following main theorem:

Theorem 8. *Let $[\,l\,], [\,s\,] \in \mathcal{B} \cap \mathcal{A}_r$ for some $r \in \mathcal{L}_2(k-1)$. Then the following statements are equivalent:*

1. *$M([\,l\,]) \subset M([\,s\,])$;*
2. *$M([\,s\,])$ has $L([\,l\,])$ as a subquotient in a composition series;*
3. *There exists reflections $w_1, \dots, w_t$ in $W_{\Gamma,S}$ such that*

$$[\,l\,] \leq w_1[\,l\,] \leq \dots \leq w_t \dots w_1[\,l\,] = [\,s\,].$$

As a corollary we obtain a criterion for $M([\,l\,])$ to be simple:

Corollary 8. *$M([\,l{\uparrow}x\,])$ is simple if and only if for all $i > k$ and all $j < i$ holds $x_j - x_i \notin \mathbb{N}$.*

8. Some relations with Bruhat order

Consider the subset $P_S^{++} \subset \mathcal{A}$ containing all $[\,l\,]$ such that for all $w \in W_S$ holds $w[\,l\,] \leq [\,l\,]$. We will call P_S^{++} the set of S-dominant weights. Recall that W_S denotes the Weyl group of the root system with the base S.

Theorem 9. *For $w_1, w_2 \in W_{\Gamma,S}$ and $[\,l\,] \in P_S^{++}$, $M(w_1[\,l\,]) \subset M(w_2[\,l\,])$ if and only if*

$$w_1 W_S \geq_B w_2 W_S$$

(here $\geq_B$ denotes the Bruhat order on the quotients of $W_{\Gamma,S}/W_S$, see [2] fore more details).

Proof. It follows immediately from theorem 8 and definition of the Bruhat order on the quotients of Coxeter group [2]. □

9. Acknowledgments

The authors would like to thank Yu. A. Drozd and V. M. Futorny for helpful discussions. For the first author the work is partially supported by ISSEP, Grant Number PSU051083. Paper was completed during the visit of the first author to SFB 343, Bielefeld University, whose financial support is gratefully acknowledged. We also thank referee for helpful suggestions that led to the improvements in the paper.

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Mechanics and Mathematics Department
Kyiv Taras Shevchenko University
64, Volodymyrska st.
252033 Kyiv
Ukraine