A counter example to Slater's conjecture on basic gaps

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Abstract

Using the technique of cut-and-project quasicrystals with two-dimensional internal space and one-dimensional model space, we construct a counter example to the conjecture of Slater, which states that the distances in the set $\{n \in \mathbb{Z}_+ : \{n\alpha\} < A \text{ and } \{n\beta\} < B\}$, where $\alpha, \beta, \alpha\beta^{-1} \in \mathbb{R} \setminus \mathbb{Q}$, $A, B \in (0,1)$, can always be expressed as linear combinations with non-negative integer coefficients of at most three basic distances.

1 Introduction

Studying in [13, 14, 15] the distribution of integers N for which $\{N\theta\} < \varphi$, where $\{\cdot\}$ denotes the fractional part, Slater made in 1959 the following conjecture ([14], Section 9.6, Page 204, or [15], Section 3). Let α and β be two irrational numbers such that $\alpha\beta^{-1}$ is also irrational. For $A, B \in (0,1)$ denote by $N(\alpha, \beta, A, B)$ the set $\{n \in \mathbb{Z}_+ : \{n\alpha\} < A \text{ and } \{n\beta\} < B\}$. This is a uniformly discrete subset of $\mathbb{R}$. Slater conjectured that there always exist at most three minimal distances (gaps) between the elements of $N(\alpha, \beta, A, B)$ such that all other distances can be written as linear combinations with non-negative integer coefficients of these basic distances. Or, more generally, Slater conjectured that, starting from n irrational numbers that are linearly independent over $\mathbb{Q}$, we always get $n+1$ basic distances in the corresponding set with the above property. In [13] this was proved for the case $n = 1$ and is now known as the *Three Gap Theorem*. This theorem states that there are always at most three minimal distances in the corresponding set $N(\alpha, A)$ and, if there are exactly three distances, one of them is always the sum of the other two. The case $n = 1$ and its various generalizations were extensively studied by many authors, see for example [1, 2, 3, 7, 14, 16]. Applications, in particular to physics and chemistry, include the study of molecular vibrations [14], where the problem of finding the statistical distribution of the time lags of a given sum of vibrations translates into the mathematical problem of the Three Gap Theorem and its generalizations. Meanwhile, motivated by the experimental discovery of new materials called quasicrystals [12], there have also appeared a number of papers during the last decade studying minimal distances in quasicrystals associated with quadratic irrationalities and corresponding recursions, see for example [4, 5, 6] and references therein. These questions are quite similar to the original question studied by Slater and, in particular, one always is in the situation of the Three Gap Theorem in these cases.

In [15], Section 3, it is written that in [14], Chapter 9, there is a strong computational evidence that Slater's conjecture on the number of basic distances is true. The aim of the present paper is to show that already for $n = 2$ the conjecture of Slater is not true. The evidence for the existence of a counter example to Slater's conjecture comes from the recent paper [11], which studies the minimal distances in special cut-and-project quasicrystals associated with cubic irrationalities. This type of irrationalities leads to a cut-and-project scheme with a rectangular acceptance window in a two-dimensional internal space and a one-dimensional model space. The authors of [11] provide a complete description of the minimal distances in these quasicrystals (which depend on the size of the acceptance window) and from this description it follows that in some cases the set of basic distances (i.e. those from which all others are obtained as linear combinations with non-negative integer coefficients) contains indeed 4 and not 3 elements. However, one has to remark that the results of [11] imply that this situation is quite rare, namely, for the quasicrystals studied in [11] only approximately every tenth randomly chosen window leads to a quasicrystal with 4 basic gaps.

It is not very difficult to realize the set $N(\alpha, \beta, A, B)$ as a cut-and-project quasicrystal associated with a cut-and-project scheme with a quadrangular acceptance window in a two-dimensional internal space and a one-dimensional model space. Hence it seems natural to expect that the behavior of the minimal distances in $N(\alpha, \beta, A, B)$ should be analogous to the behavior of the minimal distances in the quasicrystals from [11]. It turns out that this is indeed the case and in this paper we show the existence of α, β, A, B such that the number of basic distances in $N(\alpha, \beta, A, B)$ is 4, which gives a counter example to the conjecture of Slater. In fact, our main result is the following.

Theorem 1. *For every $\alpha, \beta, \in (0,1) \setminus \mathbb{Q}$ with $\alpha\beta^{-1} \notin \mathbb{Q}$ there exist infinitely many $A, B \in (0,1)$ such that the (unique) set of basic distances in $N(\alpha, \beta, A, B)$ consists of 4 elements.*

From our realization of the set of Slater it is quite easy to derive that the set of minimal distances in such a set associated with n irrationalities has at most 2^n elements. The results of the present paper give quite a strong evidence that this bound is exact, that is that one can always choose α_i and A_i, $i = 1, \dots, n$, such that the corresponding Slater's set has exactly 2^n minimal distances. This is formulated as a conjecture at the end of the paper.

The paper is organized as follows. In Section 2 we recall the notion of cut-and-project quasicrystals and give a quasicrystal realization for Slater's set. In Section 3 we prove our main result, which provides a counter example to Slater's conjecture for $n = 2$. We finish the paper with a conjecture for the general case.

2 A cut-and-project quasicrystal associated with the set of Slater

Let M be a subset of $\mathbb{R}^k \oplus \mathbb{R}^l$ and $\Omega \subset \mathbb{R}^l$. The set $\Sigma(M, \Omega) = \{x \in \mathbb{R}^k :$ there exists $y \in \Omega$ such that $(x, y) \in M\}$ is the *model* of M in the *model space* $\mathbb{R}^k$ with respect to the *acceptance window* Ω in the *internal space* $\mathbb{R}^l$, see [8, 9, 10]. If M is a finitely generated lattice in $\mathbb{R}^k \oplus \mathbb{R}^l$ and the set $\Sigma(M, \Omega)$ is aperiodic, then it is called a *quasicrystal*. The preimage of $\Sigma(M, \Omega)$ in M will be called the *pre-model* of $\Sigma(M, \Omega)$ and will be denoted by $\Sigma'(M, \Omega)$.

Let α, β, A, B be elements in $(0, 1)$ such that $\alpha, \beta, \alpha\beta^{-1} \notin \mathbb{Q}$ (we remark that we can assume $\alpha, \beta \in (0, 1)$ without loss of generality). Our aim is to associate with $N(\alpha, \beta, A, B)$ a certain quasicrystal, which we denote by $Q(\alpha, \beta, A, B)$ and construct as follows. We consider $M = \mathbb{Z}^3 \subset \mathbb{R}^3$ and define the model space as the one-dimensional subspace of $\mathbb{R}^3$, which is parallel to the vector $(1, -\beta, \alpha - \beta)$. The internal space is defined as $\{(x_1, x_2, 0) \in \mathbb{R}^3\}$ and the acceptance window $\Omega = \Omega(\alpha, \beta, A, B)$ is the set of all $(x_1, x_2, 0)$ such that $0 \le \alpha x_1 + x_2 < A$ and $0 \le \beta x_1 + x_2 < B$. We remark that, by definition, the model space and the internal space depend neither on A nor on B. It is only the acceptance window Ω, which depends on these two parameters. It is obvious that the closure $\overline{\Omega}$ of Ω is the parallelogram in the internal space with vertices $(0, 0, 0)$, $(\alpha-\beta)^{-1}(A-B, \alpha B-\beta A, 0)$, $(\beta - \alpha)^{-1}(B, -\alpha B, 0)$ and $(\alpha - \beta)^{-1}(A, -A\beta, 0)$.

The relation between $N(\alpha, \beta, A, B)$ and $Q(\alpha, \beta, A, B)$ is clarified by the following statement:

Lemma 1. *Let $n \in \mathbb{N}$. Then $n \in N(\alpha, \beta, A, B)$ if and only if there exist $x_2, x_3 \in \mathbb{Z}$ such that $(n, x_2, x_3) \in \Sigma'(M, \Omega)$. Moreover, such x_2 and x_3 are uniquely defined.*

Proof. By definition, $(n, x_2, x_3) \in \Sigma'(M, \Omega)$ is equivalent to

$$(n, x_2, x_3) - \frac{x_3}{\alpha - \beta}(1, -\beta, \alpha - \beta) \in \Omega.$$

From the definition of Ω we get that the last condition is equivalent to the system

$$\left\{\begin{array}{lllll} 0 & \le & \alpha n - \dfrac{x_3\alpha}{\alpha - \beta} + x_2 + \dfrac{x_3\beta}{\alpha - \beta} & < & A \\ 0 & \le & \beta n - \dfrac{x_3\beta}{\alpha - \beta} + x_2 + \dfrac{x_3\beta}{\alpha - \beta} & < & B \end{array}\right. .$$

This system reduces to

$$\left\{\begin{array}{lllll} 0 & \le & \alpha n + x_2 - x_3 & < & A \\ 0 & \le & \beta n + x_2 & < & B \end{array}\right. ,$$

which, in turn, is equivalent to the condition $n \in N(\alpha, \beta, A, B)$. Moreover, the elements x_2 and x_3 are obviously unique as both A and B are smaller than 1. $\square$

Define the map $\varphi : N(\alpha, \beta, A, B) \to \Sigma'(M, \Omega)$ by $\varphi(n) = (n, x_2, x_3)$, where (n, x_2, x_3) is the unique element, given by Lemma 1. For $n, n' \in N(\alpha, \beta, A, B)$ set furthermore $\psi(n, n') = \varphi(n) - \varphi(n') \in \mathbb{Z}^3$. The next lemma relates the linear dependence of vectors in $N(\alpha, \beta, A, B)$ and $\Sigma'(M, \Omega)$.

Lemma 2. *There exists $\varepsilon > 0$ such that for every $A, B < \varepsilon$ the following condition is satisfied: If $n^{(j)}, n_1^{(j)}, \dots, n_k^{(j)} \in N(\alpha, \beta, A, B)$, $j = 1, 2$, and $a_i \in \mathbb{Z}_+$ are such that $n^{(2)} - n^{(1)} = \sum_{i=1}^k a_i(n_i^{(2)} - n_i^{(1)})$, $n^{(2)} - n^{(1)} > 0$, $n_i^{(2)} - n_i^{(1)} > 0$ for all i, and there exists $x \in \Sigma'(M, \Omega)$ such that $x + \sum_{i=1}^k a_i \psi(n_i^{(2)}, n_i^{(1)}) \in \Sigma'(M, \Omega)$, then $\psi(n^{(2)}, n^{(1)}) = \sum_{i=1}^k a_i \psi(n_i^{(2)}, n_i^{(1)})$.*

Proof. By our choice of α and β, the model space $\langle(1, -\beta, \alpha - \beta)\rangle$ is irrational in the sense that every translation of it contains at most one point from M. This, in particular, implies that the projection of $\Sigma'(M, \Omega)$ on Ω along this line is everywhere dense in Ω. Consider the elements $v = \psi(n^{(2)}, n^{(1)}) \in \mathbb{Z}^3$ and $w = \sum_{i=1}^k a_i \psi(n_i^{(2)}, n_i^{(1)}) \in \mathbb{Z}^3$. By our assumptions, there exist x and y in $\Sigma'(M, \Omega)$ such that $v + x$ and $w + y$ are again in $\Sigma'(M, \Omega)$. The elements $n^{(2)} - n^{(1)}$ and $\sum_{i=1}^k a_i(n_i^{(2)} - n_i^{(1)})$ are the first coordinates of the vectors v and w, respectively, and we thus get that $v - w = (0, x_2, x_3)$ for some $x_2, x_3 \in \mathbb{Z}$. Projecting $\{(0, x_2, x_3) : x_2, x_3 \in \mathbb{Z}\}$ onto the internal space along the model space we certainly get a uniformly discrete set in the internal space. Moreover, this operation is clearly injective. Let d be the minimal distance in this set. Now choose ε such that the diameter $\mathrm{diam}(\Omega)$ of Ω is smaller than $d/3$. Certainly, this can easily be achieved for all sufficiently small A and B. Consider the projections of x, y, v and w onto the internal space along the model space. Since projection is a linear operator, we obtain that all images x', y', $v' + x'$ and $w' + y'$ are in Ω. In particular, the lengths of both v' and w' are smaller than $\mathrm{diam}(\Omega)$ and the distance between x' and y' is also smaller than $\mathrm{diam}(\Omega)$. Hence the length of $v' - w'$ does not exceed the sum of these three numbers and thus is strictly smaller than $3\,\mathrm{diam}(\Omega) = d$. This implies that $v' = w'$ and hence $v - w = 0$ by injectivity of the projection. This means that $v = w$ and completes the proof. $\square$

3 The main result

A set $S \subset \mathbb{N}$ is called a *minimal set of basic distances* in $N(\alpha, \beta, A, B)$ provided that every element in S is a minimal distance in $N(\alpha, \beta, A, B)$ and every minimal distance in $N(\alpha, \beta, A, B)$ is a linear combination of distances from S with non-negative integer coefficients. We will denote by $DC(\alpha, \beta, A, B)$ the minimal possible cardinality for a minimal set of basic distances in $N(\alpha, \beta, A, B)$ and call this number the *distance complexity* of $N(\alpha, \beta, A, B)$.

Analogously, by a *basic vector* in $\Sigma'(M, \Omega)$ we mean an element $v \in \mathbb{Z}^3$, such that there exists $x \in \Sigma'(M, \Omega)$ with $x + v \in \Sigma'(M, \Omega)$ and such that the images of x and $x + v$ are neighbors in $\Sigma(M, \Omega)$. Moreover, we will also require that $(v, (1, -\beta, \alpha - \beta)) > 0$, where $(\cdot, \cdot)$ is the standard inner product in $\mathbb{R}^3$.

The main result of the present paper is the following statement.

Theorem 2. *For every $\alpha, \beta \in (0,1)$ such that $\alpha, \beta, \alpha\beta^{-1} \notin \mathbb{Q}$ there exist (infinitely many pairs) $(A, B) \in (0,1)^2$ such that $DC(\alpha, \beta, A, B) = 4$.*

The strategy of the proof of Theorem 2 is the following. We start by considering the pre-model set $\Sigma'(M, \Omega)$, constructed in Section 2, instead of $N(\alpha, \beta, A, B)$. For this set the minimal distances (considered as distances in $\Sigma(M, \Omega)$ and not in $\Sigma'(M, \Omega)$) are given by minimal vectors, which are elements in $\mathbb{Z}^3$ by definition. A set S of vectors will be called a *minimal set of basic vectors* in $\Sigma'(M, \Omega)$ provided that every element in S corresponds to a minimal distance in $\Sigma'(M, \Omega)$, and every other vector corresponding to a minimal distance in $\Sigma'(M, \Omega)$ is a linear combination of elements from S with non-negative integer coefficients. By an abuse of notation, we will use the same letter for the minimal set of basic distances in $\Sigma'(M, \Omega)$ and in $N(\alpha, \beta, A, B)$ (which are in a natural bijection for sufficiently small A and B by Lemma 2). We will show that any minimal set of basic vectors in $\Sigma'(M, \Omega)$ contains at least 3 elements. Then we show that, decreasing A and B, one always arrives at the situation where any minimal set of basic vectors contains at least 4 elements. Finally, choosing A and B such that they satisfy the conditions of Lemma 2 we will be able to transfer the linear independence of vectors in $\Sigma'(M, \Omega)$ to the linear independence for the corresponding distances in $N(\alpha, \beta, A, B)$ and thus complete the proof.

If V is a vector space and $S \subset V$, then by the *positive cone* of S we will mean the set of all finite linear combinations of elements from S with non-negative integer coefficients. The projection on the internal space along the model space will be called the *internal projection*, and the projection on the model spaces along the internal space will be called the *model projection*.

Lemma 3. *Let S be a minimal set of basic vectors for $\Sigma'(M, \Omega)$. Then $|S| \geq 3$.*

Proof. Assume that $S = \{v, w\}$ and consider the internal projection of $\Sigma'(M, \Omega)$ on Ω. Let v' and w' be the projections of v and w, respectively. The elements $e_1 = (-1, \alpha, 0)$ and $e_2 = (1, -\beta, 0)$ form a basis in the internal space and the vectors v' and w' have, by construction and the choice of α and β, nonzero coordinates throughout with respect to this basis, that is $v' = a_1 e_1 + a_2 e_2$ and $w' = b_1 e_1 + b_2 e_2$ with a_i, $b_i \neq 0$, $i = 1, 2$.

Denote by T the positive cone of $\{v', w'\}$. This set is certainly a discrete cone in the internal space. Denote by T' the convex hull of T, which is a convex cone in $\mathbb{R}^2$ and does not coincide with $\mathbb{R}^2$. In particular, since Ω is convex and bounded, there exists a translation of T' such that the intersection of T' with $\overline{\Omega}$ contains exactly one point, p say. This means that $p + t \notin \Omega$ for all $0 \neq t \in T$. Since T is discrete, we obtain that for all u' sufficiently close to p and for all $0 \neq t \in T$ one has $u' + t \notin \Omega$. Indeed, an $u \in \Sigma'(M, \Omega)$ such that u' is close enough to p can always be found, because the image of $\Sigma'(M, \Omega)$ is everywhere dense in Ω. This implies that the sum of u with every non-zero element from the positive cone of $\{v, w\}$ does not belong to $\Sigma'(M, \Omega)$. Let $u_1 \in \Sigma'(M, \Omega)$ be such that $u_1 - u$ is a basic vector. Then $u_1 - u$ must be a new element of S, a contradiction. $\square$

For $v \in M$ and for the image v' of v under the internal projection we will call *the* *signature* of v (and v') the pair $(\varepsilon_1, \varepsilon_2) \in \{\pm 1\}^2$, which corresponds to the signs of the coefficients a_1 and a_2 in the decomposition of v' with respect to the basis e_1 and e_2.

Lemma 4. *Let S be a minimal set of basic vectors in $\Sigma'(M, \Omega)$. Then the signatures of all elements in S are different. In particular, $|S| \leq 4$.*

Proof. Let $v, w \in S$ be such that the signatures of v and w coincide. Recall that the model space has irrational angles with respect to the internal space and thus the internal projections of different elements from M are different. In particular, since $(v, (1, -\beta, \alpha - \beta)) > 0$ and $(w, (1, -\beta, \alpha - \beta)) > 0$, these projections have different length. Let us assume that $(v - w, (1, -\beta, \alpha - \beta)) > 0$.

Since the elements v and w have the same signature, there exists a vertex x in $\overline{\Omega}$, such that $x + v' \in \Omega$ and $x + w' \in \Omega$. From this it follows that there exists an inner point, $y \in \Omega$, such that $y + (v' - w') \in \Omega$. This then implies the existence of $u \in \Sigma'(M, \Omega)$ with $(v - w) + u \in \Sigma'(M, \Omega)$.

Every minimal distance (given by a vector) in $\Sigma'(M, \Omega)$ lies in the positive cone of S. Hence the same holds also for all (non-minimal) distances, and in particular also for $v - w$. However, the vector v cannot contribute to this decomposition, because its length is bigger than that of $v - w$, and thus $v - w$ lies in the positive cone of $S \setminus \{v\}$. But this implies that v also lies in the positive cone of $S \setminus \{v\}$, which contradicts the minimality of S and completes the proof. $\square$

Let $(\varepsilon_1, \varepsilon_2)$ be a signature. Since $\Sigma'(M, \Omega)$ is uniformly discrete, there exists a unique element $v_{(\varepsilon_1, \varepsilon_2)}$ with this signature among the elements of $\Sigma'(M, \Omega)$, such that its model projection has the minimal possible length.

Lemma 5. *Let S be a minimal set of basic vectors for $\Sigma'(M, \Omega)$. Then*

$$S \subset \{v_{(1,1)}, v_{(1,-1)}, v_{(-1,1)}, v_{(-1,-1)}\}.$$

Proof. The arguments are analogous to those of Lemma 4. Assume that we have $v \in S$ and $v \notin \{v_{(1,1)}, v_{(1,-1)}, v_{(-1,1)}, v_{(-1,-1)}\}$. Let $(\varepsilon_1, \varepsilon_2)$ be the signature of v. Then $(v - v_{(\varepsilon_1, \varepsilon_2)}, (1, -\beta, \alpha - \beta)) > 0$ and hence $v - v_{(\varepsilon_1, \varepsilon_2)}$ lies in the positive cone of $S \setminus \{v\}$. The vector $v_{(\varepsilon_1, \varepsilon_2)}$ also lies in the positive cone of $S \setminus \{v\}$ by length arguments and hence also v lies in the positive cone of $S \setminus \{v\}$, which contradicts the minimality of S and thus proves the claim. $\square$

Lemma 6. *Let S be a minimal set of basic vectors for $\Sigma'(M, \Omega)$. Assume that $|S| = 3$ and $v_{(\varepsilon_1, \varepsilon_2)} \notin S$. Then $v_{(\varepsilon_1, \varepsilon_2)} = a v_{(-\varepsilon_1, \varepsilon_2)} + b v_{(\varepsilon_1, -\varepsilon_2)}$ for some $a, b \in \mathbb{N}$.*

Proof. Since $v_{(\varepsilon_1, \varepsilon_2)} \notin S$, we have that $S = \{v_{(-\varepsilon_1, \varepsilon_2)}, v_{(\varepsilon_1, -\varepsilon_2)}, v_{(-\varepsilon_1, -\varepsilon_2)}\}$ by the Lemmas 3 and 5. Moreover,

$$v_{(\varepsilon_1, \varepsilon_2)} = a v_{(-\varepsilon_1, \varepsilon_2)} + b v_{(\varepsilon_1, -\varepsilon_2)} + c v_{(-\varepsilon_1, -\varepsilon_2)}$$

for some $a, b, c \in \mathbb{Z}_+$. As the signature of $v_{(-\varepsilon_1, -\varepsilon_2)}$ is opposite to that of $v_{(\varepsilon_1, \varepsilon_2)}$, the vector $a v_{(-\varepsilon_1, \varepsilon_2)} + b v_{(\varepsilon_1, -\varepsilon_2)}$ has the same signature as $v_{(\varepsilon_1, \varepsilon_2)}$, but the length of the model projection of $av_{(-\varepsilon_1,\varepsilon_2)} + bv_{(\varepsilon_1,-\varepsilon_2)}$ is smaller than that of $v_{(\varepsilon_1,\varepsilon_2)}$ if $c > 0$. Since $v_{(\varepsilon_1,\varepsilon_2)}$ is that element of signature $(\varepsilon_1, \varepsilon_2)$ with the smallest length, we obtain $c = 0$. Then clearly $a, b \neq 0$ and the lemma is proved. $\square$

Corollary 1. *The minimal set of basic vectors for $\Sigma'(M, \Omega)$ is unique.*

Proof. If $DC(\alpha, \beta, A, B) = 4$ then the set of basic vectors for $\Sigma'(M, \Omega)$ coincides with $D = \{v_{(1,1)}, v_{(1,-1)}, v_{(-1,1)}, v_{(-1,-1)}\}$ by Lemma 5. If $DC(\alpha, \beta, A, B) = 3$, then, again by Lemma 5, S is a proper subset of D. Let $v_{(\varepsilon_1,\varepsilon_2)}$ be an element in D, which belongs to the positive cone of $D \setminus \{v_{(\varepsilon_1,\varepsilon_2)}\}$. We aim to show that in this case any other $v_{(\varepsilon'_1,\varepsilon'_2)}$ does not belong to the positive cone of $D \setminus \{v_{(\varepsilon'_1,\varepsilon'_2)}\}$. Indeed, by Lemma 6 it is enough to show that $v_{(\varepsilon'_1,\varepsilon'_2)}$ does not belong to the positive cone of $D \setminus \{v_{(-\varepsilon'_1,\varepsilon'_2)}, v_{(\varepsilon'_1,-\varepsilon'_2)}\}$. The positive cone of $\{v_{(-\varepsilon_1,\varepsilon_2)}, v_{(\varepsilon_1,-\varepsilon_2)}\}$ contains $v_{(\varepsilon_1,\varepsilon_2)}$ and hence cannot contain $v_{(-\varepsilon_1,-\varepsilon_2)}$, which proves the statement for $v_{(-\varepsilon_1,-\varepsilon_2)}$. Furthermore, neither of the elements $v_{(-\varepsilon_1,\varepsilon_2)}$ and $v_{(\varepsilon_1,-\varepsilon_2)}$ can belong to the positive cone of $\{v_{(\varepsilon_1,\varepsilon_2)}, v_{(-\varepsilon_1,-\varepsilon_2)}\}$, because the lengths of $v_{(-\varepsilon_1,\varepsilon_2)}$ and $v_{(\varepsilon_1,-\varepsilon_2)}$ are strictly smaller than that of $v_{(\varepsilon_1,\varepsilon_2)}$. Hence $S = D \setminus \{v_{(\varepsilon_1,\varepsilon_2)}\}$, which completes the proof. $\square$

After Corollary 1 it makes sense to denote the minimal set of basic vectors for $\Sigma'(M, \Omega)$ by $S(\alpha, \beta, A, B)$.

Corollary 2. *Let $A' \leq A$, $B' \leq B$ and suppose that the element $v \in S(\alpha, \beta, A, B)$ occurs as a distance between two points from $\Sigma'(M, \Omega(\alpha, \beta, A', B'))$. Then $v \in S(\alpha, \beta, A', B')$.*

Proof. Suppose otherwise. Then v is in the positive cone of $S(\alpha, \beta, A', B')$, and moreover, the sum of all coefficients in the corresponding linear combination is bigger than 1. But every element from $S(\alpha, \beta, A', B')$ is in the positive cone of $S(\alpha, \beta, A, B)$, and the length arguments imply that v must belong to the positive cone of $S(\alpha, \beta, A, B) \setminus \{v\}$, which contradicts the minimality of $S(\alpha, \beta, A, B)$ and thus proves the claim. $\square$

Proof of Theorem 2. We will prove Theorem 2 by starting with A, B sufficiently small, that is $A, B \leq \varepsilon$, where ε is given by Lemma 2, and then decreasing A and B until we arrive at the situation where $DC(\alpha, \beta, A, B) = 4$. Our aim is to show that this is always possible and then the theorem follows.

Assume in the following that A and B are small enough and that $DC(\alpha, \beta, A, B) = 3$. Moreover, assume that $(\varepsilon_1, \varepsilon_2)$ is such that $v_{(\varepsilon_1,\varepsilon_2)} \notin S(\alpha, \beta, A, B)$.

Let δ_1 denote the minimal difference between the length of the side of $\overline{\Omega}$, connecting the vertices $(0, 0, 0)$ and $(\beta - \alpha)^{-1}(B, -\alpha B, 0)$ (i.e. the side parallel to e_1), and the absolute values of the e_1-coefficients of the elements from $S(\alpha, \beta, A, B) \cup \{v_{(\varepsilon_1,\varepsilon_2)}\}$. Similarly, let δ_2 denote the minimal difference between the length of the side of $\overline{\Omega}$, which is adjacent to the previous one, and the absolute values of the e_2-coefficients of the elements from $S(\alpha, \beta, A, B) \cup \{v_{(\varepsilon_1,\varepsilon_2)}\}$. It follows immediately from Corollary 2 that for all $0 < a < \delta_1$ and $0 < b < \delta_2$ we have $S(\alpha, \beta, A, B) = S(\alpha, \beta, A - a, B - b)$.

Now consider the sets $S(\alpha, \beta, A - \delta_1, B - b)$ and $S(\alpha, \beta, A - a, B - \delta_2)$, where $0 < a < \delta_1$ and $0 < b < \delta_2$. Recall that the sides of $\overline{\Omega}$ are not parallel to any element in $\mathbb{Z}^3$ by construction. This implies that in both cases (that is both for δ_1 and for δ_2) exactly one of the elements from the set $S(\alpha,\beta,A,B)\cup\{v_{(\varepsilon_1,\varepsilon_2)}\}$ does not occur any more as a distance between two elements in the set $\Sigma'(M,\Omega(\alpha,\beta,A',B'))$ for A' and B' given by the choice of a or b above. Now a case study is needed.

Case 1. Assume that the vanishing vector is $v_{(\varepsilon_1,\varepsilon_2)}$ and let $v^{(1)}_{(\varepsilon_1,\varepsilon_2)}$ denote the new minimal vector with the signature $(\varepsilon_1,\varepsilon_2)$. This vector represents some distance in the original set $\Sigma'(M,\Omega(\alpha,\beta,A,B))$ and hence belongs to the positive cone of $S(\alpha,\beta,A,B)=S(\alpha,\beta,A',B')$. From Lemma 6 we get that $v^{(1)}_{(\varepsilon_1,\varepsilon_2)}$ already belongs to the positive cone of $\{v_{(-\varepsilon_1,\varepsilon_2)},v_{(\varepsilon_1,-\varepsilon_2)}\}\subset S(\alpha,\beta,A',B')$. Since the intersection of the positive cone of $\{v'_{(-\varepsilon_1,\varepsilon_2)},v'_{(\varepsilon_1,-\varepsilon_2)}\}$ with Ω is finite, Case 1 cannot occur infinitely many times and thus eventually one of the following cases becomes relevant.

Case 2. Assume that the vanishing vector is $v_{(-\varepsilon_1,-\varepsilon_2)}$. Then $v_{(\varepsilon_1,\varepsilon_2)}$ is still in the positive cone of $v_{(-\varepsilon_1,\varepsilon_2)}$ and $v_{(\varepsilon_1,-\varepsilon_2)}$, and thus $S(\alpha,\beta,A',B')=(S(\alpha,\beta,A,B)\setminus v_{(-\varepsilon_1,-\varepsilon_2)})\cup\{v^{(1)}_{(-\varepsilon_1,-\varepsilon_2)}\}$, where $v^{(1)}_{(-\varepsilon_1,-\varepsilon_2)}$ is the new minimal vector of signature $(-\varepsilon_1,-\varepsilon_2)$.

Since the set of all distances is discrete and bounded from below, Case 1 and Case 2 cannot occur infinitely many times after each other. Hence, in a finite number of steps we will always arrive at Case 3 below.

Case 3. Assume that the vanishing vector is either the vector $v_{(\varepsilon_1,-\varepsilon_2)}$ or the vector $v_{(-\varepsilon_1,\varepsilon_2)}$.

Start by considering the case where $v_{(\varepsilon_1,\varepsilon_2)}=v_{(-\varepsilon_1,\varepsilon_2)}+v_{(\varepsilon_1,-\varepsilon_2)}$. Assume without loss of generality that $v_{(\varepsilon_1,-\varepsilon_2)}$ is the vanishing vector (the arguments for $v_{(-\varepsilon_1,\varepsilon_2)}$ are similar), which is replaced by the new element $v^{(1)}_{(\varepsilon_1,-\varepsilon_2)}$. Then we have by Lemma 4 that

$$S(\alpha,\beta,A',B')\subset D=\{v^{(1)}_{(\varepsilon_1,-\varepsilon_2)},v_{(\varepsilon_1,\varepsilon_2)},v_{(-\varepsilon_1,\varepsilon_2)},v_{(-\varepsilon_1,-\varepsilon_2)},\}$$

and now the length arguments imply that $v_{(\varepsilon_1,\varepsilon_2)}$ does no longer belong to the positive cone of $\{v^{(1)}_{(\varepsilon_1,-\varepsilon_2)},v_{(-\varepsilon_1,\varepsilon_2)}\}$. If the element $v^{(1)}_{(\varepsilon_1,-\varepsilon_2)}$ does not belong to the positive cone of $\{v_{(\varepsilon_1,\varepsilon_2)},v_{(-\varepsilon_1,-\varepsilon_2)}\}$, we get that $S(\alpha,\beta,A',B')=D$ and hence $DC(\alpha,\beta,A',B')=4$. If the element $v^{(1)}_{(\varepsilon_1,-\varepsilon_2)}$ belongs to the positive cone of $\{v_{(\varepsilon_1,\varepsilon_2)},v_{(-\varepsilon_1,-\varepsilon_2)}\}$, then no element of signature $(-\varepsilon_1,\varepsilon_2)$ can belong to this cone. Since the element $v_{(\varepsilon_1,\varepsilon_2)}=v_{(-\varepsilon_1,\varepsilon_2)}+v_{(\varepsilon_1,-\varepsilon_2)}$ has signature $(\varepsilon_1,\varepsilon_2)$ it follows that, going back to $\Omega(\alpha,\beta,A,B)$ and decreasing the other parameter (that is A instead of B or vice versa), we will come to the situation, where the vanishing vector is uniquely given by $v_{(-\varepsilon_1,\varepsilon_2)}$. The new vector $v^{(1)}_{(-\varepsilon_1,\varepsilon_2)}$ will then have signature $(-\varepsilon_1,\varepsilon_2)$, and therefore cannot belong to the positive cone of $\{v_{(\varepsilon_1,\varepsilon_2)},v_{(-\varepsilon_1,-\varepsilon_2)}\}$. By the arguments above this implies that we have arrived at the situation where $DC(\alpha,\beta,A',B')=4$.

Now consider the case that $v_{(\varepsilon_1,\varepsilon_2)}=av_{(-\varepsilon_1,\varepsilon_2)}+bv_{(\varepsilon_1,-\varepsilon_2)}$ for positive integers a,b which are such that at least one of them is different from 1. Consider furthermore the vector $w=v_{(-\varepsilon_1,\varepsilon_2)}+v_{(\varepsilon_1,-\varepsilon_2)}$. Since the signatures $(-\varepsilon_1,\varepsilon_2)$ and $(\varepsilon_1,-\varepsilon_2)$ are opposite, it follows that w occurs as a distance between some elements in $\Sigma'(M,\Omega(\alpha,\beta,A,B))$ and also as a distance between some elements in $\Sigma'(M,\Omega(\alpha,\beta,A',B'))$. However, if one of $v_{(-\varepsilon_1,\varepsilon_2)}$ or $v_{(\varepsilon_1,-\varepsilon_2)}$ disappears, Lemma 2 implies that w immediately becomes the vector of minimal length among vectors of its signature. Applying Lemma 2 once more we see that w, which belongs to the positive cone of $\{v_{(-\varepsilon_1,\varepsilon_2)}, v_{(\varepsilon_1,-\varepsilon_2)}\}$, cannot belong also to the positive cone of $\{v_{(-\varepsilon_1,-\varepsilon_2)}, v_{(\varepsilon_1,\varepsilon_2)}\}$. Hence, if one of the elements $v_{(-\varepsilon_1,\varepsilon_2)}$ or $v_{(\varepsilon_1,-\varepsilon_2)}$ disappears, the new element $v^{(1)}_{(-\varepsilon_1,\varepsilon_2)}$ or $v^{(1)}_{(\varepsilon_1,-\varepsilon_2)}$ must coincide with w, and, moreover, w is contained in $S(\alpha,\beta,A',B')$. In particular, we see that the set $S(\alpha,\beta,A',B')$ is obtained from $S(\alpha,\beta,A,B)$ by adding w and taking $v_{(-\varepsilon_1,\varepsilon_2)}$ or $v_{(\varepsilon_1,-\varepsilon_2)}$ away. The element $v_{(\varepsilon_1,\varepsilon_2)}$ still belongs to the positive cone of $\{v^{(1)}_{(-\varepsilon_1,\varepsilon_2)}, v_{(\varepsilon_1,-\varepsilon_2)}\}$ (or of $\{v^{(1)}_{(\varepsilon_1,-\varepsilon_2)}, v_{(-\varepsilon_1,\varepsilon_2)}\}$), and the coefficients in the corresponding linear combination have not increased, but at least one has decreased.

This implies that after a finite number of steps corresponding to the Cases 1, 2, and 3 above, we always arrive at the situation where $v_{(\varepsilon_1,\varepsilon_2)} = v_{(-\varepsilon_1,\varepsilon_2)} + v_{(\varepsilon_1,-\varepsilon_2)}$. As explained earlier, this situation corresponds to some A' and B' such that $DC(\alpha,\beta,A',B') = 4$, which completes the proof. $\square$

It is very easy to see that a statement analogous to the statement of Theorem 2 holds for every quasicrystal associated with a two-dimensional internal space, a one-dimensional model space and an acceptance window in the form of a parallelogram. That is for such quasicrystals one always obtains a quasicrystal of distance complexity 4 by a suitable shrinking of the acceptance window.

We would also like to remark that the results of [11] give a strong evidence that the description of all minimal distances in $N(\alpha,\beta,A,B)$ in the general case is a very difficult problem.

Theorem 2 suggests the following conjecture:

Conjecture 1. *Let a_i, $i = 1,\dots,N$, be irrational numbers linearly independent over $\mathbb{Q}$, and let $A_i \in (0,1)$ for $i = 1,\dots N$. Then the minimal number of basic distances in the set $\{n \in \mathbb{N} : na_i < A_i, i = 1,\dots,N\}$ is at least $N+1$ and at most 2^N. Moreover for every $\{a_i\}$ one can choose $\{A_i\}$ such that this number is an arbitrary fixed integer between $N+1$ and 2^N.*

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