Rigidity of generalized Verma modules

Oleksandr Khomenko and Volodymyr Mazorchuk *

Abstract

We prove that generalized Verma modules, induced from generic Gelfand-Zetlin modules, and generalized Verma modules, associated with Enright-complete modules, are rigid. Their Loewy length and quotients of the unique Loewy filtrations are calculated for the regular block of the corresponding category $\mathcal{O}(\mathfrak{p}, \Lambda)$.

1 Introduction

Let $\mathfrak{g}$ be a semi-simple complex finite-dimensional Lie algebra with a fixed triangular decomposition, $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and $\mathfrak{p} \supset \mathfrak{h} \oplus \mathfrak{n}_+$ be its parabolic subalgebra. In what follows there will be a distinguished special case, namely when the semi-simple part $\mathfrak{a}$ of the Levi factor $\mathfrak{a}'$ is isomorphic to a direct product of some $\mathfrak{sl}(n_i, \mathbb{C})$. This will be assume all the time when we will discuss generic Gelfand-Zetlin modules. Let $\mathfrak{n}$ be the nilpotent radical of $\mathfrak{p}$.

This paper continues the study of certain parabolic generalizations, $\mathcal{O}(\mathfrak{p}, \Lambda)$, of the celebrated BGG category $\mathcal{O}$ of $\mathfrak{g}$-modules ([BGG2]). These categories, associated with $\mathfrak{p}$ and certain admissible category Λ of $\mathfrak{a}$-modules, were introduced in [FKM1], where it was shown that their blocks correspond to the so-called standardly stratified algebras in the sense of [CPS] (and even to the smaller class of properly stratified algebras, which was recently introduced in [Dl]). In particular, there is an analogue of the BGG-reciprocity formula, which involves besides the indecomposable projective modules and simple modules a class of intermediate modules, which are called generalized Verma modules. Later in [FKM2, FKM3] the algebra of the principal block of $\mathcal{O}(\mathfrak{p}, \Lambda)$ was given a combinatorial description, analogous to Soergel's description of $\mathcal{O}$, [S].

The basic example of $\mathcal{O}(\mathfrak{p}, \Lambda)$ was constructed in terms of the so-called generic Gelfand-Zetlin modules (see [DOF1]). In [KM1, KM2] this example was related to a certain subcategory of $\mathcal{O}$, which can be described in terms of Enright's completion functors ([E]). The last categories carry a "strange" abelian structure, which is not inherited from that on $\mathcal{O}$. Where this structure appears from was explained in [KM2] after realizing (Mathieu's version, [M], of) Enright's completion functor via the approximation functor with respect to a suitably chosen injective module in $\mathcal{O}$.

*Junior Associate of the ICTP

From [I2, I3] (combined with the reduction to integral case in [S]) it is known that all Verma modules and big projective modules in the principal blocks of $\mathcal{O}$ (this can be viewed as the first extremal case of $\mathcal{O}(\mathfrak{p},\Lambda)$, which corresponds to $\mathfrak{a}=0$) are rigid, i.e. their socle and radical filtrations coincide and form the unique Loewy filtration (the shortest filtration with semi-simple quotients). In [BGS] this rigidity result was reobtained and extended to all Verma modules using the machinery of Koszul rings and some deep geometrical results. Let $\mathcal{O}(\mathfrak{p},\Lambda)$ be the basic example mentioned above. The blocks of $\mathcal{O}(\mathfrak{p},\Lambda)$ contain generalized Verma modules induced from generic Gelfand-Zetlin modules and big projective modules. So, there arise a natural question: are these modules also rigid? By [KM1] there is a full functor, say $\mathcal{F}$, from $\mathcal{O}(\mathfrak{p},\Lambda)$ to $\mathcal{O}$, which sends each generalized Verma module to a Verma module and each big projective (in $\mathcal{O}(\mathfrak{p},\Lambda)$) to a big projective in $\mathcal{O}$. The main problem is that the abelian structure on the image of $\mathcal{F}$ does not coincide with the abelian structure on $\mathcal{O}$. Hence the results of [I2, I3, BGS] can not be applied directly.

The aim of the paper is to give a positive answer to the first part of the question above. Our main result is the following.

Theorem 1. *Let Λ be the admissible category, generated by a simple generic Gelfand-Zetlin module, or the admissible category of Enright-complete modules, and $\mathcal{O}(\mathfrak{p},\Lambda)$ be the corresponding parabolic analogue of $\mathcal{O}$, [FKM1]. Then generalized Verma modules in $\mathcal{O}(\mathfrak{p},\Lambda)$ are rigid.*

In contrast with the classical case, the big projective modules in $\mathcal{O}(\mathfrak{p},\Lambda)$ fail to be rigid in the general case. We present an $sl(3)$-counterexample in Section 4. We also remark that, in the second extremal case, namely $\mathfrak{g}=\mathfrak{a}$, all GVMs are simple, hence rigid of Loewy length 1 and the rigidity of the big projective module (=the unique indecomposable projective in the principal block, =the standard module) was proved already in [KM1].

The paper is organized as follows. The next section contains all necessary preliminary information, in particular, we define $\mathcal{O}(\mathfrak{p},\Lambda)$ and recall how the situation is related to the category $\mathcal{O}$. Section 3 is devoted to the study of Loewy series on generalized Verma modules. Here we prove rigidity, calculate Loewy length and quotients of the unique Loewy filtration. In Section 4 we compute an $\mathfrak{sl}(3,\mathbb{C})$-example, and, in particular, give an example of a non-rigid big projective module in $\mathcal{O}(\mathfrak{p},\Lambda)$. We finish the paper with a short discussion of properties of standard modules in Section 5.

2 Notation and preliminary results

2.1 Generalized Verma modules and generic Gelfand-Zetlin modules

By [KM1], the category constructed here is a special case of the categories considered in the next subsection (we recall that for Gelfand-Zetlin modules we assume that $\mathfrak{a}$ is a product of $\mathfrak{sl}(n_i,\mathbb{C})$). The advantage of this construction is that for these categories the notions of simple objects and simple $\mathfrak{g}$-modules coincide.

Let V be an $\mathfrak{a}'$-module. Extending it trivially to a $\mathfrak{p}$-module we can form the module $M_{\mathfrak{p}}(V) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$, which is called a *generalized Verma module* (GVM in the sequel), provided V is simple.

Let $[l] = (l_{i,j})_{i=1,\dots,n}^{j=1,\dots,i}$ be a *tableau* with complex entries such that $l_{i,j} - l_{i,k} \notin \mathbb{Z}$ for all $i < n$ and all possible j, k, and $V([l])$ be the corresponding *generic Gelfand-Zetlin* $\mathfrak{gl}(n,\mathbb{C})$-*module* (generic GZ-module in the sequel), as defined in [DOF1]. $V([l])$ can be restricted in a natural way to the canonical copy of $\mathfrak{sl}(n,\mathbb{C}) \subset \mathfrak{gl}(n,\mathbb{C})$ and thus we can talk about generic GZ-modules over $\mathfrak{sl}(n,\mathbb{C})$. Now over a direct product of some $\mathfrak{sl}(n_i,\mathbb{C})$ (e.g. over $\mathfrak{a}$) we define generic GZ-modules as tensor product of the above generic GZ-modules over the components.

Fix a simple generic GZ-module, V, over $\mathfrak{a}$ and consider the category Λ of all subquotients in $V \otimes F$, where F runs through all finite-dimensional $\mathfrak{gl}(n,\mathbb{C})$-module. This category extends in a natural diagonalizable way to $\mathfrak{a}'$. The blocks of this category will be module categories over local finite-dimensional associative algebras, [FKM1]. For a simple object, $V \in \Lambda$, we denote by $\tilde{V}$ the projective cover of V.

Following [FKM1] we define $\mathcal{O}(\mathfrak{p},\Lambda)$ as the full subcategory of the category of all $\mathfrak{g}$-modules, which consists of finitely-generated $\mathfrak{n}$-locally finite $\mathfrak{g}$-modules, which decompose into a direct sum of modules from Λ, when viewed as $\mathfrak{a}'$-modules. The blocks of $\mathcal{O}(\mathfrak{p},\Lambda)$ are module categories over properly stratified finite-dimensional associative algebras, [FKM1]. GVMs $M_{\mathfrak{p}}(V)$, V is a simple generic GZ-module over $\mathfrak{a}'$, are objects of $\mathcal{O}(\mathfrak{p},\Lambda)$ and they are also *proper standard modules* for the corresponding properly stratified algebras. The modules $\Delta(V) = M_{\mathfrak{p}}(\tilde{V})$, V is simple in Λ, are *standard modules* for $\mathcal{O}(\mathfrak{p},\Lambda)$.

2.2 Enright completions and $\mathcal{S}$-subcategories in $\mathcal{O}$

Let $\mathfrak{R}$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and π be the basis of $\mathfrak{R}$, which corresponds to our fixed triangular decomposition. Fix some Weyl-Chevalley basis, $\{X_\alpha | \alpha \in \mathfrak{R}\}$, $\{H_\alpha | \alpha \in \pi\}$, in $\mathfrak{g}$. Denote by W the Weyl group of $\mathfrak{R}$ and by s_α the reflection, corresponding to $\alpha \in \mathfrak{R}$. We denote by $l : W \to \mathbb{Z}_+$ the length function and by $\hat{w}$ the unique longest element of W.

For a simple root, α, denote by U_α the localization of $U(\mathfrak{g})$ with respect to $\{X_{-\alpha}^k | k \in \mathbb{N}\}$ (see [M]) and define the completion functor $E_\alpha : \mathcal{O} \to \mathcal{O}$ as the composition of $U_\alpha \otimes_{U(\mathfrak{g})} -$, $-|_{U(\mathfrak{g})}$ and the functor of taking the locally X_α-finite part of a module (see [FKM2, FKM3, KM1, KM2]). Clearly $E_\alpha \circ E_\alpha = E_\alpha$.

For $S \subset \pi$ let $\mathcal{O}(S)$ denote the full subcategory of $\mathcal{O}$, which consists of all modules on which all $X_{-\alpha}$, $\alpha \in S$, act injectively. Then for $\alpha, \beta \in S$ the restrictions of E_α and E_β on $\mathcal{O}(S)$ satisfy braid relations (see [De] or [KM2] for a short proof). Hence, if $s_{\alpha_1} \dots s_{\alpha_k} = w \in W_S = \langle s_\alpha | \alpha \in S \rangle$ the functor $E_w = E_{\alpha_1} \circ \dots \circ E_{\alpha_k} : \mathcal{O}(S) \to \mathcal{O}(S)$ is well-defined. Let $w_S \in W_S$ be the longest element. The functor E_{w_S} is called the *S-completion* functor and a module, $M \in \mathcal{O}$, is called *S-complete* if $M \in E_{w_S}(\mathcal{O}(S))$. The category $\mathcal{O}(S)_{st}$ of all S-complete modules is an abelian category with usual kernels and cokernels defined as follows: if $f : M \to N$, $M, N \in \mathcal{O}(S)_{st}$, then the cokernel of f is $E_{w_S}(N/E_{w_S}(f(M)))$ ([FKM3], see [KM2] for another description in terms of approximation with respect to an injective module).

Now let S be the set of simple roots of $\mathfrak{a}$. Then, by [KM1], for appropriate $\mathfrak{a}$, there exists a blockwise equivalence of categories, $\mathcal{F}:\mathcal{O}(\mathfrak{p},\Lambda)\to\mathcal{O}(S)_{st}$, which sends GVMs to Verma modules. Again by [KM1], the functor $\mathcal{F}^{-1}:\mathcal{O}(S)_{st}\to\mathcal{O}(\mathfrak{p},\Lambda)$ is also well-defined. In particular, we can set $\Lambda=\mathcal{O}(S)_{st}$ and consider $\mathcal{O}(\mathfrak{p},\Lambda)$ in the case of arbitrary $\mathfrak{a}$. We emphasize once more that the advantage of Gelfand-Zetlin modules is that in this case simple objects of $\mathcal{O}(\mathfrak{p},\Lambda)$ are simple $\mathfrak{g}$-modules. Finally, we recall the following properties of E_{w_S}, [FKM3]:

1. for a simple $L\in\mathcal{O}$ the module E_{w_S} is either 0 or simple in $\mathcal{O}(S)_{st}$;
2. $E_{w_S}(M)\subset E_{w_S}(N)$ for $M\subset N$;
3. $E_{w_S}(N/M)\supset E_{w_S}(N)/E_{w_S}(M)$ for $M\subset N$.

2.3 Loewy length and Loewy filtrations

Let M be a module of finite length. A filtration, $0\subset M_1\subset\cdots\subset M_k=M$ is called a *Loewy filtration* if all M_i/M_{i-1} are semi-simple and k is the minimal possible. This k is called the *Loewy length* of M and is denoted by $ll(M)$ (if M is an object of two different categories and we want to underline that it's Loewy length is considered inside, say the category A, we will write $ll_A(M)$). Associated with M there are two Loewy filtrations: the socle filtration $0\subset\operatorname{soc}^1 M\subset\operatorname{soc}^2 M\subset\cdots\subset\operatorname{soc}^{ll(M)}=M$ (here $\operatorname{soc}^{i+1}M/soc^i M=\operatorname{soc}(M/\operatorname{soc}^i M)$) and the radical filtration $0=\operatorname{rad}^{ll(M)}M\subset\cdots\subset\operatorname{rad}^1 M\subset\operatorname{rad}^0 M=M$ (here $\operatorname{rad}^i(M)=\operatorname{rad}(\operatorname{rad}^{i-1}M)$). If $0\subset M_1\subset\cdots\subset M_k=M$ is a Loewy filtration of M, there holds $\operatorname{rad}^{ll(M)-i}M\subset M_i\subset\operatorname{soc}^i M$. The module M is called *rigid* if the socle filtration and the radical filtration coincide and thus there is only one Loewy filtration. We will denote the layers of the socle and the radical filtrations by $\operatorname{soc}_i M=\operatorname{soc}^i M/\operatorname{soc}^{i-1}M$ and $\operatorname{rad}_i M=\operatorname{rad}^i M/\operatorname{rad}^{i+1}M$ correspondingly.

2.4 Soergel's combinatorics of $\mathcal{O}$ and its generalization to $\mathcal{O}(\mathfrak{p},\Lambda)$

For $\lambda\in\mathfrak{h}^*$ we will denote by $M(\lambda)$ the Verma module with highest weight λ, by $L(\lambda)$ the unique simple quotient of $M(\lambda)$, by $P(\lambda)$ the indecomposable projective cover of $L(\lambda)$ and by $I(\lambda)$ the indecomposable injective envelope of $L(\lambda)$. If ρ is the half-sum of all positive root, we recall the standard $\cdot$-action of W on $\mathfrak{h}^*$, defined as follows: $w\cdot(\lambda)=w(\lambda+\rho)-\rho$.

Let χ be an integral central character of $\mathfrak{g}$ and $\mathcal{O}_\chi$ be the block of $\mathcal{O}$, corresponding to χ. By the main result of [S], any indecomposable block of $\mathcal{O}$ is equivalent to $\mathcal{O}_\chi$ for some χ, however, this equivalence may force to change $\mathfrak{g}$. The character χ thus corresponds to an integral W-orbit, $W\cdot\lambda$, where λ is antidominant. Let W_λ denote the stabilizer of λ. Then the simples in $\mathcal{O}_\chi$ are naturally parameterized by cosets $\xi\in W/W_\lambda$. If $w\in\xi$ is the longest element, then we set $L(\xi)=L(w\cdot\lambda)$. Analogously we define $P(\xi)=P(w\cdot\lambda)$, $\Delta(\xi)=\Delta(w\cdot\lambda)$ and $M(\xi)=M(w\cdot\lambda)$. Set $P_\lambda=\oplus_{\xi\in W/W_\lambda}P(\xi)$ and $A_\lambda=\operatorname{End}_{\mathcal{O}}(P_\lambda)$. A_λ is a basic, finite-dimensional, associative and quasi-hereditary algebra. The Verma modules $M(\xi)$ are standard modules with respect to the quasi-hereditary structure. Set $C_\lambda = \mathrm{End}_{\mathcal{O}}(P(\lambda))$. By [S, Endomorphismensatz 7], the algebra C_λ is the subalgebra of W_λ-invariants in the coinvariant algebra C, which is the quotient of $\mathbb{C}[\mathfrak{h}^*]$ modulo the ideal generated by W-invariant, with respect to the usual W-action, polynomials without constant term. In particular, $C_\lambda = C$ if λ is regular. Let e be the primitive idempotent of A_λ, corresponding to $\xi = W_\lambda$ (which means $C_\lambda = \mathrm{End}_{A_\lambda Ae}$). Then, by [S, Struktursatz 9], $A_\lambda \simeq \mathrm{End}(A_\lambda e_{C_\lambda})$. This is usually called the *double centralizer property.* For a primitive idempotent, f, of A_λ we will denote the C_λ-module fAe by D_f.

Now we recall the analogous description of $\mathcal{O}(\mathfrak{p},\Lambda)$, the latter as defined in Subsection 2.2 ([FKM2, FKM3]). For this we consider the equivalent category $\mathcal{O}(S)_{st}$ and set $\mathcal{O}_\lambda^S = \mathcal{O}(S)_{st} \cap \mathcal{O}_\lambda$. We emphasize that the abelian structure on $\mathcal{O}_\lambda^S$ is not inherited from that on $\mathcal{O}_\lambda$. The simples of $\mathcal{O}_\lambda^S$ are indexed by those $\xi \in W/W_\lambda$ such that the longest representative w of ξ is at the same time the shortest element in a coset from W/W_S. Let $T = \{\xi_1, \dots, \xi_k\}$ be the complete set of parameters of simples in $\mathcal{O}_\lambda^S$. Set $P_\lambda^S = \oplus_{\xi\in T} P(\xi)$ and $A_\lambda^S = \mathrm{End}_{\mathfrak{g}}(P_\lambda^S)$. Denote by e^S the primitive idempotent of A_λ^S corresponding to $P(\lambda)$. Then, by [FKM3, Section 5], we have $A_\lambda^S = \mathrm{End}((A_\lambda^S e^S)_{C_\lambda})$.

3 Rigidity of generalized Verma modules

In this section we prove the main result of this paper, the rigidity of GVMs. Our proof follows the ideas of [BGS]. In fact, we are going to realize GVMs as graded modules over graded algebras and then apply the following lemma ([BGS, Proposition 2.4.1]).

Lemma 1. *Let R be a $\mathbb{Z}$-graded ring, generated by R_0 and R_1, such that R_0 is semi simple. Let M be a graded R-module of finite length. If* $\mathrm{soc}(M)$ *(resp.* $\mathrm{rad}(M)$*) is simple, then the socle (resp. radical) filtration on M coincides with the grading filtration, up to shift.*

The coinvariant algebra C is graded in a natural way, and we fix a grading, in which the generators $(\mathfrak{h}^*)$ have degree 2. Hence the algebra C_λ is graded as well. Both modules $A_\lambda e$ and $A_\lambda^S e^S$ are graded C_λ-modules (see Remark after [S, Lemma 7]). Moreover, as $A_\lambda^S = tA_\lambda t$ for some, in general case non-primitive, idempotent t of A_λ, we can naturally consider $A_\lambda^S e^S$ as a graded submodule of $A_\lambda e$. The decomposition $A_\lambda e = \oplus_f D_f$, where f runs through the set of all primitive idempotents of A_λ, is a decomposition of $A_\lambda e$ into a direct sum of indecomposable modules and all summands are graded. If f corresponds to $L(\xi)$, $\xi \in W/W_\lambda$, then the minimal non-zero degree of D_f is exactly $ll(M(\xi))$. Via the double centralizer property we get a positive grading on A_λ, which corresponds to the "mixed" structure on $\mathcal{O}$. By [BGS, Section 4], this grading is Koszul, in particular, $(A_\lambda)_0$ is semi simple and A_λ is generated by $(A_\lambda)_1$ over $(A_\lambda)_0$.

Now we want to move this picture to A_λ^S, which is a positively graded subalgebra of A_λ via the double centralizer property. This trivially implies that $(A_\lambda^S)_0$ is semi simple as well. Unfortunately, in the general case A_λ^S **is not** generated by $(A_\lambda^S)_1$ over $(A_\lambda^S)_0$. We refer the reader to Section 4 for the corresponding $\mathfrak{sl}(3,\mathbb{C})$-example. However, we can find some convenient set of generators for A_λ^S.

Since the algebra C_λ is commutative the double centralizer property implies that it is a subalgebra of A_λ^S and, moreover, is a subalgebra of the center of A_λ^S. We also recall that C_λ appears in [S] as the image of the action of $Z(\mathfrak{g})$ on $P(\lambda)$.

Proposition 1. *A_λ^S is generated by $(A_\lambda^S)_0$, $(A_\lambda^S)_1$ and C_λ.*

Proof. Let f_1 and f_2 be two primitive idempotents of A_λ^S. Our aim is to decompose any graded element from $f_1 A_\lambda^S f_2$ into a product of elements from $(A_\lambda^S)_0$, $(A_\lambda^S)_1$ and C_λ. But any element from $f_1 A_\lambda^S f_2$ corresponds to a map from an indecomposable projective $P(\xi_1)$ to indecomposable projective $P(\xi_2)$ in $\mathcal{O}(\mathfrak{p},\Lambda)$. from the definition of projective modules it follows that the map $\varphi : P(\xi_1) \to P(\xi_2)$ can not be decomposed non-trivially into a product of other maps between projectives only in the case, when $\varphi(P(\xi_1)) \notin \mathrm{rad}^2(P(\xi_2))$. Hence, it is enough to prove that, under this assumption, we have $\varphi \in (A_\lambda^S)_0$ or $\varphi \in (A_\lambda^S)_1$ or $\varphi \in C_\lambda$. We have to consider several cases.

First, let $\xi_1 = \xi_2$ and φ be an isomorphism. Then the assumption that φ is a graded element implies $\varphi \in (A_\lambda^S)_0$.

Now, let $\xi_1 \neq \xi_2$. Because of the duality on $\mathcal{O}(\mathfrak{p},\Lambda)$ we may assume $\xi_1 \not\leq \xi_2$ with respect to the order of properly stratified structure. Then, necessarily $L(\xi_1)$ is a top of $\mathrm{rad}(P(\xi_2))$. Now we recall that, according to [FKM1], all projectives in $\mathcal{O}(\mathfrak{p},\Lambda)$ are filtered by standard modules. Because of $\xi_1 \not\leq \xi_2$ and the BGG-reciprocity, [FKM1], the standard module $\Delta(\xi_1)$ occurs in this filtration. Hence there should exist a quotient of $P(\xi_2)$, which is an extension of $\Delta(\xi_1)$ by $\Delta(\xi_2)$, such that the unique copy of $L(\xi_1)$ in the top of $\Delta(\xi_1)$ is exactly the one, covered by φ. But this means that φ covers the whole $\Delta(\xi_1)$ as it has a simple top. So, the image of φ does not belong to $\mathrm{rad}(P(\xi_2))$ in the category $\mathcal{O}$ as well. This implies that φ is a degree 1 map in A_λ, as it is graded. Hence $\varphi \in (A_\lambda^S)_1$.

Finally, we consider the case $\xi_1 = \xi_2 = \xi$ and $\varphi(P(\xi)) \subset \mathrm{rad}(P(\xi))$. Again we recall that $P(\xi)$ is filtered by standard modules and there is only one of them (the last one), isomorphic to $\Delta(\xi)$. As all other standard modules have different tops, φ maps the top of $P(\xi)$, which is also the top of $\Delta(\xi)$, into some other copy of $L(\xi)$ in $\Delta(\xi)$. Our claim will easily follow if we prove that the natural action of C_λ on $\Delta(\xi)$ surjects onto $\mathrm{End}_{A_\lambda}(\Delta(\xi))$. Realize $\Delta(\xi)$ as $M_\mathfrak{p}(\tilde{V})$, where $\tilde{V}$ is projective in Λ. Then the exactness of induction guarantees $\mathrm{End}_{A_\lambda}(\Delta(\xi)) = \mathrm{End}_\Lambda(\tilde{V})$. The S-Harish-Chandra homomorphism, [DOF2], restricts the action of $Z(\mathfrak{g})$ on $\Delta(\xi)$ to the action of $Z(\mathfrak{a})$ on $\tilde{V}$, and the statement follows from [S, Endomorphismensatz]. This completes the proof. □

Lemma 2. *$(C_\lambda)_{>0}$ annihilates $M(\xi)$.*

Proof. The action of C_λ comes from the action of $Z(\mathfrak{g})$, which acts by scalars on $M(\xi)$, [FC, DOF2]. □

Lemma 3. *$M(\xi)$ is a graded A_λ^S-module.*

Proof. It follows directly from [BGS], as $M(\xi)$ is a graded A_λ-module and A_λ^S is a graded subalgebra of A_λ. But one can also get this from our graded picture and the double-centralizer. Obviously, $A_\lambda^S e^S$ is a positively graded finite-dimensional A_λ^S-module. Moreover, it is filtered by GVMs and, by BGG-reciprocity, [FKM1], each GVM from $\mathcal{O}_\lambda^S$ occurs as a subquotient of $A^S_\lambda e^S$. The biggest GVM is naturally identified with $\mathrm{soc}(A^S_\lambda e^S)$, the latter viewed as C_λ-module, hence graded. The statement follows by induction. $\square$

We are now ready to state our main result.

Theorem 2. *Any GVM $M(\xi)$ in $\mathcal{O}^S_\lambda$ is rigid.*

Proof. By Lemma 3, $M(\xi)$ is a graded module over graded algebra A^S_λ. By Lemma 2, it is even a graded module over a graded algebra $A^S_\lambda/((C_\lambda)_{>0})$. By Proposition 1, the last is generated in degrees 0 and 1. It is well-known that GVMs have simple socles and tops, [FC, MO]. Hence the statement follows from Lemma 1. $\square$

Now we can calculate the Loewy length of GVMs for regular λ. We recall that in this case they are parameterized by $\xi \in W/W_S$. We start with the following easy observation.

Lemma 4. *The inequality $ll_{\mathcal{O}(\mathfrak{p},\Lambda)}(\mathcal{F}^{-1} \circ E_{w_S}(M)) \leq ll_{\mathcal{O}}(M)$ is true for any $M \in \mathcal{O}$.*

Proof. Follows from properties of E_{w_S}, described in Subsection 2.2. $\square$

Now we can calculate the Loewy length of $M(\xi)$.

Lemma 5. *Let λ be regular and $\xi \in W/W_S$. Then $ll(M(\xi)) = l(w^\xi) + 1$.*

Proof. The inequality $ll(M(\xi)) \geq l(w^\xi) + 1$ follows from the analogue of BGG-criterion for inclusion of GVMs, [MO]. By this criterion, each $M(\xi)$ has a filtration, $0 \subset M(\xi_1) \subset \cdots \subset M(\xi_r) = M(\xi)$, where $r = l(w^\xi) + 1$ and all quotients of the filtration are non-zero. As each GVM has a simple top, $M(\xi_i)$ lies in the radical of $M(\xi_{i+1})$ and hence $ll(M(\xi)) \geq r$.

Let us now prove that $ll(M(\xi)) \leq l(w^\xi) + 1$. Consider two Verma modules $M(w_\xi\lambda)$ and $M(w^\xi\lambda)$. By [De] $E_{w_S}(M(w^\xi\lambda)) = M(w_\xi\lambda)$ and, as we have already mentioned, $\mathcal{F}^{-1}(M(w_\xi\lambda)) = M(\xi)$. Hence, by Lemma 4, we have $ll_{\mathcal{O}(\mathfrak{p},\Lambda)}(M(\xi)) \leq ll_{\mathcal{O}}(M(w_\xi\lambda)) = l(w^\xi) + 1$. This completes the proof for GVMs. $\square$

We remark that in the case of singular λ one can also calculate the Loewy length in terms of height of an ideal in the poset W/W_λ. However, this does not give any closed formula as in Lemma 5.

Combining the above results we can describe the layers of the unique Loewy filtration of $M(\xi)$.

Corollary 1. *Let λ be regular, $\xi \in W/W_S$, w the shortest representative of ξ, and $i \in \mathbb{Z}_+$. Then $\mathrm{soc}_i(M(\xi)) = \mathrm{rad}_{ll(M(\xi))-i+1}(M(\xi)) = \mathcal{F}^{-1} \circ E_{w_S}(\mathrm{soc}_i(M(w \cdot \lambda)))$.*

Proof. As it was already mentioned, $\mathcal{F}^{-1} \circ E_{w_S}$, sends any Loewy filtration of $M(w \cdot \lambda)$ to a filtration of $M(\xi)$ with semi-simple subquotients. But by Lemma 5, we have $ll_{\mathcal{O}}(M(w \cdot \lambda)) = ll_{\mathcal{O}(\mathfrak{p},\Lambda)}(M(\xi))$ and both modules are rigid. The statement follows. $\square$

4 $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$, $\mathfrak{a} = \mathfrak{sl}(2,\mathbb{C})$ – example

In this section we assume $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{C})$ with the standard Cartan subalgebra and $\{\alpha, \beta\}$ is a basis of Δ. We assume that $\mathfrak{a} = \mathfrak{sl}(2,\mathbb{C})$ is the subalgebra corresponding to the root α. We fix λ regular antidominant. Then $\mathcal{O}_\lambda$ contains 6 simple modules indexed by $1 = \lambda$, $2 = s_\alpha \cdot \lambda$, $3 = s_\beta \cdot \lambda$, $4 = s_\alpha s_\beta \cdot \lambda$, $5 = s_\beta s_\alpha \cdot \lambda$ and $6 = s_\alpha s_\beta s_\alpha \cdot \lambda$. The Verma modules have the following radical filtrations (the number on the left is the index i of rad_i):

	$M(1)$	$M(2)$	$M(3)$	$M(4)$	$M(5)$	$M(6)$
0	1	2	3	4	5	6
1		1	1	2 3	2 3	4 5
2				1	1	2 3
3						1

The big projective module $A_\lambda e$ is rigid and filtered by all Vermas with all multiplicities equal to 1. The graded picture of $A_\lambda e$ as C_λ module is the following one, where by i we denote the simple subquotients, which come from $\mathfrak{g}$-simple subquotients of A_λ, isomorphic to $L(i)$. This is what we called D_f before. Here we will call these C_λ-direct summands of $A_\lambda e$ by D_i respectively. The number of the left is the degree of the corresponding graded component.

0	1					
1		2	3			
2	1 1			4	5	
3		2 2	3 3			6
4	1 1			4	5	
5		2	3			
6	1					

In particular, the algebra C_λ, as a graded left module over itself, is isomorphic to D_1. Going to $\mathcal{O}(\mathfrak{p}, \Lambda)$ we take (see [FKM3, KM2]) the category of injectively copresented modules with respect to the direct sum of indecomposable injectives indexed by the shortest representatives in W/W_S. This means that the only indices, corresponding to these shortest representatives, namely 1, 3 and 5, will survive. Hence, the graded structure of $A^S_\lambda e^S$ will be:

0	1		
1		3	
2	1 1		5
3		3 3	
4	1 1		5
5		3	
6	1		

Now let us look at GVMs and standard modules, which we will naturally index by 1, 3 and 5. As all multiplicities are one, all GVMs are uniserial and have the following radical filtrations:

	$M_{\mathfrak{p}}(1)$	$M_{\mathfrak{p}}(3)$	$M_{\mathfrak{p}}(5)$
0	1	3	5
1		1	3
2			1

To obtain $\Delta(5)$ one has to find the trace of D_5 in $A_\lambda^S e^S$. It is easy to see that this will be the following part of $A_\lambda^S e^S$:

2			5
3		3	
4	1		5
5		3	
6	1		

Analogously, $\Delta(3)$ is presented by the following part of $A_\lambda^S e^S$:

1		3
2	1	
3		3
4	1	

By direct calculation one easily gets that these graded filtrations are in fact Loewy filtrations of the corresponding standard modules. The module $\Delta(1)$ is, obviously, a self-extension of $M_{\mathfrak{p}}(1)$. It is presented by the following part of $A_\lambda^S e^S$:

0	1
1	
2	1

And here we see the difference: to obtain the radical filtration of $\Delta(1)$ one has to re-scale the grading. The last causes the fact that the big projective module $A_\lambda^S e^S$ in $\mathcal{O}(\mathfrak{p}, \Lambda)$ is not rigid. Indeed, let us look at the first three graded components of it:

0	1	
1		3
2	1 1	

Clearly, the dimension of C_λ-homomorphisms of degree 1 from D_1 to D_3 is 1. But the dimension of C_λ-homomorphisms of degree 1 from D_3 to D_1 is also 1 (e.g since D_1 is injective). Hence their composition has dimension at most 1, which means that one composition factor of $\mathrm{rad}_1(D_1)$ belongs to $\mathrm{rad}_1(A_\lambda^S e^S)$. The above description of Loewy filtrations of standard modules (it is enough to have it for $\Delta(5)$) implies that the Loewy length of $A_\lambda^S e^S$ is 7. hence the graded filtration is a Loewy one and does not coincide with the radical filtration. Therefore $A_\lambda^S e^S$ is not rigid.

5 What can be said about standard modules?

Based on the example above we formulate the following conjecture:
Conjecture. Standard modules in $\mathcal{O}(\mathfrak{p},\Lambda)$ are rigid.

We also have to remark that even the calculation of Loewy length of standard modules seems to be a very non-trivial problem. As the above example shows, the answer will be much more difficult than that for GVMs. However, it is not difficult to get the following inequalities:

Lemma 6. *Let $M = M_{\mathfrak{p}}(V)$ be a generalized Verma module and $\Delta = \Delta(\tilde{V})$ be the corresponding standard module. Then*

$$ll(M) + ll(\tilde{V}) - 1 \leq ll(\Delta) \leq ll(M) + 2ll(\tilde{V}) - 1.$$

Proof. To prove the left inequality we recall that $\tilde{V}$ has simple socle, V, and M is a submodule of Δ with simple top $L_{\mathfrak{p}}(V)$ and all other composition factors different from $L_{\mathfrak{p}}(V)$. Hence $M \subset rad_{ll(\tilde{V})-1}(\Delta)$, which implies necessary inequality.

The right inequality follows from the double centralizer property. Indeed, it implies that the big projective is graded as A_λ-module. Moreover, by BGG reciprocity, it is filtered by standard modules, and by induction one derives that all standard modules are in fact graded modules over A_λ. It is straightforward to see that the length of the grading filtration inherited from the big projective module is precisely $ll(\Delta) \leq ll(M) + 2ll(\tilde{V}) - 1$. Indeed, $ll(\Delta) \leq ll(M)$ corresponds to the Verma submodule M of Δ and its grading filtration and $2ll(\tilde{V})$ corresponds to the grading filtration of $\tilde{V}$ as C_λ-module (we recall that C_λ is even-graded). This implies the necessary inequality. $\square$

We note that both extremal cases of equalities are possible. This can be read off from the example in Section 4. Indeed, the left equality holds for $\Delta(1)$ and the right equality holds for both $\Delta(3)$ and $\Delta(5)$. It is easy to construct other examples where $ll(\Delta)$ satisfies two strict inequalities from the above formula. However, for this one has to take $\mathfrak{a}$ of rank greater than 1.

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O.K.: Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Eckerstrasse 1, D-79104, Freiburg im Breisgau, Germany,
e-mail: `khomenko@mathematik.uni-freiburg.de`

V.M.: Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-5311, Bonn, Germany,
e-mail: `mazor@mpim-bonn.mpg.de`;
on leave from: Chalmers Tekniska Högskola och Göteborgs Universitet, Göterorg, Sweden, e-mail: `volody@math.chalmers.se`