On the semigroup of square matrices

Ganna Kudryavtseva and Volodymyr Mazorchuk

Abstract

We study the structure of nilpotent subsemigroups in the semigroup $M(n,\mathbb{F})$ of all $n \times n$ matrices over a field, $\mathbb{F}$, with respect to the operation of the usual matrix multiplication. We describe the maximal subsemigroups among the nilpotent subsemigroups of a fixed nilpotency degree and classify them up to isomorphism. We also describe isolated and completely isolated subsemigroups and conjugated elements in $M(n,\mathbb{F})$.

1 Introduction

The structure and combinatorics of the classical finite transformation semigroups is now relatively well understood. A lot of information about such semigroups as the full finite inverse symmetric semigroup $\mathcal{IS}_n$, the full transformation semigroup $\mathcal{T}_n$ and the semigroup $\mathcal{PT}_n$ of all partial transformations on $\{1, 2, \dots, n\}$ can be found for example in the monographs [CP, Hi, Ho, La, Li] or other numerous papers studying transformation semigroups. Two directions of study for transformation semigroups, which developed over the last 15 years, is the study of conjugated elements and nilpotent subsemigroups in these semigroups, which resulted into several nice structural and combinatorial results, see [GK1, GK2, GK3, GK4, GTS, GM, KM].

Certainly, the passage to infinite transformation semigroups completely changes the picture. However, it is still possible to obtain some information under reasonable "finiteness" conditions. One of the most classical examples of an infinite object possessing several properties, inherent in finite objects, is the algebra of all linear operators on a finite-dimensional vector space over a field. Forgetting the addition of linear operators one gets a semigroup, isomorphic to the semigroup $M(n,\mathbb{F})$ of all $n \times n$ square matrices with coefficients from a field $\mathbb{F}$ with respect to the operation of usual matrix multiplication. This semigroup has also been studied by many authors, but so far not as intensively as the classical transformation semigroups. Many interesting results about $M(n,\mathbb{F})$ can be found in the recent monograph [Ok2] and in its references.

The aim of the present paper is to contribute to the study of $M(n,\mathbb{F})$ with results in several directions. The main emphasis is made on the study of conjugated elements and nilpotent subsemigroups of $M(n,\mathbb{F})$. In the appendix we also address the problem of the study of isolated and completely isolated subsemigroups. These different directions are not immediately related with each other, however, when combined together, they give a very nice illustration for the fact that in the passage from the classical transformation semigroups to matrix semigroups one should expect that some results would be transfered to very similar results, while some other results would sound quite differently. And even in the case of similar results, the passage to matrix semigroups substantially increases the level of technical difficulties.

The notion of conjugated elements in group theory is very important and has a lot of applications (for example for the study of automorphisms, characters or representations). There are several ways to extend this notion for semigroups. Two most straightforward generalizations are: conjugation with respect to an invertible element, and the transitive closure of the $ab \sim ba$ relation. Both these notions provide some invariant on semigroups and hence can be applied for the study of automorphisms, endomorphisms and representation. The already existing literature, where these notions were studied (see for example [GK1, Li, KM]) suggests that the second generalization is more interesting than the first one. For $M(n, \mathbb{F})$ the description of equivalence classes with respect to the conjugation by invertible matrices is a classical problem, the answer to which is given by the Jordan normal form of a matrix (in the case of an algebraically closed field). In Section 2 of the present paper we describe the equivalence classes with respect to the transitive closure of the $ab \sim ba$ relation. It turns out that these are given by the "invertible part" of the Jordan normal form. This is very similar to the results obtained in [GK1, KM] for transformation semigroups. The results of Section 2 of the present paper and the results of [GK1, KM] were the main motivation for the abstract approach to the study of conjugation for semigroups, developed in [Ku].

The majority of the paper is devoted to the study of nilpotent subsemigroups of $M(n, \mathbb{F})$. To start with, we would like to remark that the notion of a nilpotent semigroup has been used in the literature in at least three different senses. The most classical one is the notion of *nilpotent semigroup in the sense of Maltsev*, [Ma], which is defined by means of the identities for nilpotent groups, rewritten without g^{-1} terms. The matrix semigroups, nilpotent in the sense of Maltsev, were recently studied in [Ok1]. However, in this paper we are going to use another notion of a nilpotent semigroup, which comes from the ring theory. A semigroup, S, with the zero element 0 is called *nilpotent* of *nilpotency degree* $\mathrm{nd}(S) = k$ provided that $a_1 a_2 \dots a_k = 0$ for any $a_1, \dots, a_k \in S$ while there exist $b_1, \dots, b_{k-1} \in S$ such that $b_1 \dots b_{k-1} \neq 0$. This notion is almost as old as the first one and goes back at least till [Shev]. From now on we will use only the last notion of nilpotent semigroups.

The study of nilpotent subsemigroups of certain semigroups of partial transformations, in particular, of the semigroup $\mathcal{IS}_n$, was originated in [GK4]. It happened that the combinatorial data, describing the maximal nilpotent subsemigroups of finite transformation semigroups, is usually a certain partial order on the underlined set, on which the semigroup acts. This philosophy was successfully used later in [GK2, GK3] and generalized on the infinite case in [Sh2, Sh1]. In [GTS] the combinatorial description of maximal nilpotent subsemigroups of $\mathcal{IS}(M)$ was used to determine the group of automorphisms for these subsemigroups.

In the present paper we generalize this technique to study nilpotent subsemigroups of the semigroup $M(n,\mathbb{F})$. A part of this study is very easy. It is well-known that the maximal nilpotent subalgebras of $M(n,\mathbb{F})$ correspond bijectively to complete flags in $\mathbb{F}^n$. An elementary exercise in linear algebra shows that a maximal nilpotent subsemigroup of $M(n,\mathbb{F})$ is in fact a subalgebra. Hence maximal nilpotent subsemigroup of $M(n,\mathbb{F})$ are also described by complete flags in $\mathbb{F}^n$. This is a perfect analogy with the combinatorial data describing maximal nilpotent subsemigroups in transformation semigroups. So, there is no problem to classify such subsemigroups, what we do in Section 3, where we also collect some other statement necessary for or main goal: to classify maximal nilpotent subsemigroups of $M(n,\mathbb{F})$ up to isomorphism. At this point we face our main difficulty in comparison to finite transformation subgroups. The fact that the cardinality of $M(n,\mathbb{F})$ is infinite for infinite $\mathbb{F}$ makes it impossible to distinguish non-isomorphic maximal nilpotent subsemigroups of $M(n,\mathbb{F})$ by their cardinality, as it was done in case of $\mathcal{IS}_n$ in [GK3]. Therefore we classify all maximal nilpotent subsemigroups of $M(n,\mathbb{F})$ up to isomorphism in Section 4 using completely different arguments.

In the Appendix we classify all isolated and completely isolated subsemigroups of $M(n,\mathbb{F})$ for finite $\mathbb{F}$. This is not directly related to the main content of the paper, however, from our point of view, these results, especially for isolated subsemigroups, are not immedeately expected, are non-trivial and again give a very nice (and not yet well-understood) illustration of similarity with the classical transformation semigroups. The results of the Appendix were also a part of the motivation for the study of isolated subsemigroups in the variant of $\mathcal{T}_n$, see [MT].

2 Conjugated elements

Let S be a monoid and G be its group of units. The elements $x, y \in S$ are said to be G-conjugated provided there exists $g \in G$ such that $x = g^{-1}yg$. This is denoted by $x \sim_G y$. The binary relation $\sim_G$ is an equivalence relation on S. The elements a and b of a semigroup, S, are called *primarily S-conjugated* if there exist such $x, y \in S$ that $a = xy$ and $b = yx$. The binary relation $\sim_{pS}$ of primary S-conjugation is reflexive and symmetric, but not transitive in general. We denote by $\sim_S$ the transitive closure of this relation. If $x \sim_S y$, the elements x and y will be called *S-conjugated.* Both $\sim_G$ and $\sim_S$ generalize the notion of conjugated elements in a group, and in the general case the relations $\sim_G$ and $\sim_S$ do not coincide. However, $\sim_G$ is always a subset of $\sim_S$.

The group of units in $M(n,\mathbb{F})$ is $\mathrm{GL}(n,\mathbb{F})$. If $\mathbb{F}$ is algebraically closed, then the description of $\mathrm{GL}(n,\mathbb{F})$-conjugated elements in $M(n,\mathbb{F})$ is the classical Jordan theorem of the basic linear algebra: $A \sim_{\mathrm{GL}(n,\mathbb{F})} B$ if and only if the Jordan normal forms of A and B coincide (up to a permutation of Jordan cells).

Let $A \in M(n,\mathbb{F})$. For each $k \geq 0$ we have the inclusion $\mathbb{F}^n \supset A^k(\mathbb{F}^n) \supset A^{k+1}(\mathbb{F}^n) \supset 0$, which implies $n \geq \dim(A(\mathbb{F}^n)) \geq \dots \geq \dim(A^k(\mathbb{F}^n)) \geq \dots \geq 0$. Since only at most n of these inequalities can be strict, we can assert that, starting from some power t, we have $A^t(\mathbb{F}^n) = A^{t+i}(\mathbb{F}^n)$ for each $i \geq 0$. Denote by A_s the linear operator defined as follows:

$A_s(v) = A(v)$, $v \in A^t(\mathbb{F}^n)$; $\ker(A_s) = \ker(A^t)$. Remark that for an algebraically closed field, $\mathbb{F}$, if we fix a Jordan basis for A, then A_s is obtained from A by replacing all Jordan cells with eigenvalue 0 by zero blocks. The main result of this section is the following.

Theorem 1. *Two matrices A and B from $M(n,\mathbb{F})$ are $M(n,\mathbb{F})$-conjugated if and only if A_s and B_s are $\mathrm{GL}(n,\mathbb{F})$-conjugated.*

Proof. Sufficiency. We start with the following lemma:

Lemma 2. *$A \sim_{M(n,\mathbb{F})} A_s$ for every $A \in M(n,\mathbb{F})$. In particular, $A \sim_{M(n,\mathbb{F})} 0$ for every nilpotent $A \in M(n,\mathbb{F})$.*

Proof. Denote by $V_i = \mathrm{Im}(A^i)$, $i \in \mathbb{N}$, $V_0 = \mathbb{F}^n$. Clearly, $V_i \subset V_{i-1}$ for all i. Let V_i' be a complement to V_i, $i \in \mathbb{Z}_+$, such that $V_i' = \ker(A^i)$ for all $i \geq t$. Set $B_i = e(V_i, V_i')Ae(V_{i-1}, V_{i-1}')$. Then $B_1 = A$ and $B_n = A_s$. Hence, to complete the proof it is enough to show that $B_i \sim_{pM(n,\mathbb{F})} B_{i+1}$ for all i. For a fixed i denote $u_i = e(V_i, V_i')$ and $v_i = e(V_i, V_i')Ae(V_{i-1}, V_{i-1}') = B_i$. Then $u_i v_i = v_i = B_i$. Moreover,

$$v_i u_i = e(V_i, V_i')Ae(V_{i-1}, V_{i-1}')e(V_i, V_i') = \\ = e(V_i, V_i')Ae(V_i, V_i') = e(V_{i+1}, V_{i+1}')Ae(V_i, V_i') = B_{i+1}.$$

□

Let A_s and B_s be $\mathrm{GL}(n,\mathbb{F})$-conjugated. By Lemma 2, we have $A \sim_{M(n,\mathbb{F})} A_s$ and $B \sim_{M(n,\mathbb{F})} B_s$. Hence the transitivity of the relation $\sim_{M(n,\mathbb{F})}$ implies $A \sim_{M(n,\mathbb{F})} B$.

Necessity. Clearly, it is enough to consider only the case when A and B are primarily $M(n,\mathbb{F})$-conjugated. So, let $A = XY$ and $B = YX$ for some $X, Y \in M(n,\mathbb{F})$. Let $V_1 = \mathrm{Im}(A^n)$, $W_1 = \mathrm{Im}(B^n)$, $V_2 = \ker(A^n)$ and $W_2 = \ker(B^n)$. Then we have $\mathbb{F}^n = V_1 \oplus V_2 = W_1 \oplus W_2$ and

$$rank(A^n) = rank(A^i) \quad \text{and} \quad rank(B^n) = rank(B^i) \quad \text{for all} \quad i > n. \tag{1}$$

Since $B(\ker(X)) = 0$ and $B(W_1) = W_1$ it follows that $\ker(X) \cap W_1 = 0$, analogously $\ker(Y) \cap V_1 = 0$. Let $x \in V_2$. Then $A^{n+1}x = XB^nYx = 0$, and $\ker(X) \cap W_1 = 0$ forces $Yx \in W_2$, that is $Y : V_2 \to W_2$. Analogously one shows that $X : W_2 \to V_2$. If $x \in V_1$ then $x = A^n x'$ for some $x' \in V_1$ since $A^n : V_1 \to V_1$ is bijective by (1). Hence $Yx = YA^n x' = B^n Yx' \in W_1$. This implies $Y : V_1 \to W_1$ and, analogously, $X : W_1 \to V_1$. The equalities $\ker(X) \cap W_1 = 0$ and $\ker(Y) \cap V_1 = 0$ now say that $Y : V_1 \to W_1$ and $X : W_1 \to V_1$ are bijections. In particular,

$$Ye = fYe \quad \text{and} \quad Xf = eXf, \tag{2}$$

where $e = e(V_1, V_2)$ and $f = e(W_1, W_2)$.

Further, we have $A_s = eAe$ as the actions of these operators on both V_1 and V_2 coincide. Analogously $B_s = fBf$. Now from $A_s = eAe = eXYe$, $B_s = fBf = fYXf$ and (2) we derive $A_s = eXYe = eXfYe = eXffYe$ and $B_s = fYXf = fYeXf = fYeeXf$, implying, in particular, that A_s and B_s are primarily $M(n,\mathbb{F})$-conjugated.

The fact that Y induces a bijection from V_1 to W_1 implies that $\dim(V_1) = \dim(W_1)$ and thus $\dim(V_2) = \dim(W_2)$. Let Z be any matrix such that $\ker(Z) = V_1$ and $Z : V_2 \to W_2$ is a bijection, which exists since $\dim(V_2) = \dim(W_2)$. Consider the matrix $M = fYe+Z$. We have $M(V_1) = fYe(V_1) = W_1$ since $Z(V_1) = 0$ and $M(V_2) = Z(V_2) = W_2$ since $e(V_2) = 0$. Hence $fYe + Z \in \mathrm{GL}(n,\mathbb{F})$.

From the definition of Z we obtain $fZ = Ze = 0$, which implies

$$B_sM = fYeeXf(fYe + Z) = fYeeXffYe = (fYe + Z)eXffYe = MA_s.$$

Hence $B_s = MA_sM^{-1}$, which completes the proof. □

We would like to remark that, for an algebraically closed $\mathbb{F}$, Theorem 1 provides a criterion of $M(n,\mathbb{F})$-conjugacy in terms of Jordan normal forms: A and B are $M(n,\mathbb{F})$-conjugated if and only if their Jordan normal forms coincide up to a permutation of Jordan cells and a replacement of some nilpotent Jordan cells by some other nilpotent Jordan cells.

3 Nilpotent subsemigroups of $M(n,\mathbb{F})$

The results of this section are mostly known or can be easily deduced from the existing literature. However, we did not manage to single them out in the literature in one piece and in the form we present them. Our presentation emphasizes the similarity with the corresponding results for transformation semigroups, see [GK2, GK3, GK4, GM]. We include the proofs for the sake of completeness.

Lemma 3. *Let T be a maximal nilpotent subsemigroup of $M(n,\mathbb{F})$. Then T is, in fact, a subalgebra of $M(n,\mathbb{F})$.*

Proof. Let T be a maximal nilpotent subsemigroup of $M(n,\mathbb{F})$. Denote by T' the linear span of T inside $M(n,\mathbb{F})$. Then every element in T' is a finite linear combination of elements from T, and $T \subset T'$. Since matrix multiplication is bilinear and T is a semigroup, we obtain that T' is in fact a subsemigroup of $M(n,\mathbb{F})$. To complete the proof it is enough to show that T' is nilpotent, which follows immediately from the nilpotency of T and

$$\left(\sum_{i_1=1}^{m_1} \lambda_{i_1}^{(1)} a_{i_1}^{(1)}\right) \dots \left(\sum_{i_s=1}^{m_s} \lambda_{i_s}^{(s)} a_{i_s}^{(s)}\right) = \sum_{i_1=1}^{m_1} \cdots \sum_{i_s=1}^{m_s} \left(\lambda_{i_1}^{(1)} \dots \lambda_{i_s}^{(s)} a_{i_1}^{(1)} \dots a_{i_s}^{(s)}\right).$$

□

Nilpotent subalgebras of $M(n,\mathbb{F})$ are classical objects both in algebraic geometry and in representation theory and have a natural description in terms of flags in $\mathbb{F}^n$. By a *flag*, $\mathcal{F}$, in $\mathbb{F}^n$ of *length* $l(\mathcal{F}) = k$ we will mean a filtration, $\mathcal{F} : 0 = V_0 \subset V_1 \subset \cdots \subset V_k = \mathbb{F}^n$, of $\mathbb{F}^n$ by subspaces, such that $V_i \neq V_{i+1}$, $i = 1, \dots, k-1$. The tuple $(d_1, \dots, d_k)$, where

$d_i = \dim(V_i/V_{i-1})$, will be called the *signature* of $\mathcal{F}$ and will be denoted by $\mathrm{sig}(\mathcal{F})$. A basis, $\{v_1, \dots, v_n\}$, of $\mathbb{F}^n$ is said to be an *$\mathcal{F}$-basis* if $v_1, \dots, v_{\dim(V_i)}$ is a basis of V_i for all $i = 1, \dots, k$.

If S is a semigroup with 0 and $k \geq 2$, we denote by $N_k(S)$ the set of all nilpotent subsemigroups of S of nilpotency degree k. This is a partially ordered set with respect to inclusion and the set $\cup_{k\geq 2} N_k(S)$ is the set of all (non-zero) nilpotent subsemigroups of S, which is also partially ordered with respect to inclusion.

To each flag $\mathcal{F}$ in $\mathbb{F}^n$ of length k, we associate the nilpotent subsemigroup $\varphi(\mathcal{F}) = \{a \in M(n, \mathbb{F}) | a(V_i) \subset V_{i-1} \text{ for all } i\}$ of $M(n, \mathbb{F})$ of nilpotency degree k. To each nilpotent subsemigroup S of $M(n, \mathbb{F})$ of nilpotency degree k, we associate the flag $\psi(S)$ in $\mathbb{F}^n$ defined as follows: $0 \subset \langle S^{k-1}(\mathbb{F}^n)\rangle \subset \langle S^{k-2}(\mathbb{F}^n)\rangle \subset \dots \subset \langle S(\mathbb{F}^n)\rangle \subset \mathbb{F}^n$. The correctness of this definition follows from the following statement.

Proposition 4. *1. If $\mathcal{F}$ is a flag in $\mathbb{F}^n$ of length k, then $\varphi(\mathcal{F})$ is a nilpotent semigroup of $M(n, \mathbb{F})$ of nilpotency degree k.*

2. If S is a nilpotent subsemigroup of $M(n, \mathbb{F})$ of nilpotency degree k, then $\psi(S)$ is a flag of F^n.

Proof. Obviously, $\varphi(\mathcal{F})$ is a nilpotent semigroup and $\varphi(\mathcal{F})^k = 0$. Let $B = \{v_1, \dots, v_n\}$ be some $\mathcal{F}$-basis in $\mathbb{F}^n$. Consider the element $a \in M(n, \mathbb{F})$, defined on B as follows: $a(v_{\dim(V_i)}) = v_{\dim(V_{i-1})}$, $i = 2, \dots, k$, and $a(v_j) = 0$ otherwise. Then $a(V_i) \subset V_{i-1}$ by the definition and hence $a \in \varphi(\mathcal{F})$. But $a^{k-1}(v_n) = v_{\dim(V_1)}$ and hence $a^{k-1} \neq 0$. This means that $\mathrm{nd}(\varphi(\mathcal{F})) > k - 1$ and thus equals k. This proves the first statement.

Since $a_1 a_2 a_3 \dots a_{i+1} = (a_1 a_2) a_3 \dots a_{i+1}$ we have $S^{i+1} \subset S^i$ and thus obtain the inclusion $\langle S^{i+1}(\mathbb{F}^n)\rangle \subset \langle S^i(\mathbb{F}^n)\rangle$. Hence it is only left to prove that $\langle S^i(\mathbb{F}^n)\rangle \neq \langle S^{i+1}(\mathbb{F}^n)\rangle$. But $\langle S^{i+1}(\mathbb{F}^n)\rangle$ has a basis, consisting of elements from $S^{i+1}(\mathbb{F}^n)$. Further, $S^k = 0$ implies $S^{k-i-1}(S^{i+1}(\mathbb{F}^n)) = 0$ and, using the fact that the elements from S are linear operators on $\mathbb{F}^n$, we obtain that $S^{k-i-1}(\langle S^{i+1}(\mathbb{F}^n)\rangle) = 0$. To complete the proof it is now enough to show that $S^{k-i-1}(\langle S^i(\mathbb{F}^n)\rangle) \neq 0$. Since $S^i(\mathbb{F}^n) \subset \langle S^i(\mathbb{F}^n)\rangle$, it is even enough to show that $S^{k-i-1}(S^i(\mathbb{F}^n)) \neq 0$, which reduces to $S^{k-1}(\mathbb{F}^n) \neq 0$. The last inequality follows from the fact that $\mathrm{nd}(S) = k$ and therefore $S^{k-1} \neq 0$. □

Proposition 4 immediately implies the following classical results:

Corollary 5. *If S is a nilpotent subsemigroup of $M(n, \mathbb{F})$, then* $\mathrm{nd}(S) \leq n$.

Corollary 6. *A subsemigroup, $S \subset M(n, \mathbb{F})$, is nilpotent if and only if S consists of nilpotent elements.*

Lemma 7. *1. Let $i \in \{2, \dots, k\}$. Then for every $v \in V_i \setminus V_{i-1}$ and $w \in V_{i-1} \setminus V_{i-2}$ there exists $a \in \varphi(\mathcal{F})$ such that $a(v) = w$.*

2. $\varphi(\mathcal{F})^i(\mathbb{F}^n) = V_{k-i}$ for all $i = 1, \dots, k$.

3. $\varphi(\mathcal{F})$ is a subalgebra of $M(n, \mathbb{F})$.

Proof. We can certainly find an $\mathcal{F}$-basis B of V such that both v and w belong to B. The linear operator $a \in M(n, \mathbb{F})$, which sends v to w and annihilates all other elements of B, belongs to $\varphi(\mathcal{F})$, which proves the first statement. The second and the third statements are obvious. □

Let $\mathcal{F}$ be a flag in $\mathbb{F}^n$. We will say that $\operatorname{sig} \mathcal{F}$ is also the *signature* of the semigroup $\varphi(\mathcal{F})$. Let k be a positive integer, $k \geq 2$. A nilpotent subsemigroup, $S \subset M(n, \mathbb{F})$, is called *$k$-maximal* provided that S is a maximal element in $N_k(S)$.

Proposition 8. *$\varphi(\mathcal{F})$ is k-maximal with $k = l(\mathcal{F})$ and every k-maximal nilpotent subsemigroup of $M(n, \mathbb{F})$ equals $\varphi(\mathcal{F})$ for some flag, $\mathcal{F}$, of $\mathbb{F}^n$ of length k.*

Proof. Let S be a nilpotent semigroup of $M(n, \mathbb{F})$ of nilpotency degree k. From the definitions of φ and ψ it follows that $S \subset \varphi(\psi(S))$. Hence any k-maximal nilpotent subsemigroup of $M(n, \mathbb{F})$ has to be of the form $\varphi(\mathcal{F})$ for a flag, $\mathcal{F}$, of $\mathbb{F}^n$ of length k.

Now assume that $S = \varphi(\mathcal{F}) \subset T$, where $T \in N_k(M(n, \mathbb{F}))$. Then $S^i(\mathbb{F}^n) = V_{k-i} \subset T^i(\mathbb{F}^n)$ for all i and hence it is enough to prove that $S^i(\mathbb{F}^n) = T^i(\mathbb{F}^n)$. Assume that j is minimal such that this equality fails. This means that there exists $v \in S^{j-1}(\mathbb{F}^n) = T^{j-1}(\mathbb{F}^n)$ and $a \in T$ such that $a(v) \notin S^j(\mathbb{F}^n)$. Consider a new flag, $\mathcal{F}'$, which has the form

$$0 = V_0 \subset V_1 \subset \dots \subset V_{k-j} \subset \langle T^j(\mathbb{F}^k)\rangle \subset V_{k-j+1} \subset \dots V_k = \mathbb{F}^n.$$

For every $i = 0, 1, \dots, k-1$, $i \neq j-1$, pick $v_j \in S^i(\mathbb{F}^n) \setminus S^{i+1}(\mathbb{F}^n)$. Then the set

$$\{v_{k-1}, \dots, v_j, a(v), v, v_{j-2}, \dots, v_0\}$$

is linearly independent, as its components belong to the different subquotients of $\mathcal{F}'$, so we can extend this set to a basis, say B, of $\mathbb{F}^n$, coordinated with $\mathcal{F}'$. Now consider the elements a_i, $i \neq j-2, j-1$, defined as follows: $a_i(v_i) = v_{i+1}$ and $a_i(w) = 0$, $w \in B \setminus \{v_i\}$; and elements b, c, defined as follows: $b(v_{j-2}) = v$, $b(w) = 0$, $w \in B \setminus \{v_{j-2}\}$; $c(a(v)) = v_j$, $c(w) = 0$, $w \in B \setminus \{a(v)\}$. It follows immediately that b, c and all a_i are elements of $\varphi(\mathcal{F})$ and hence of T. But $a_{k-2} \dots a_j caba_{j-3} \dots a_1(v_1) = v_{k-1} \neq 0$ and hence $\operatorname{nd}(T) > k$, a contradiction. □

Corollary 9. *The maps φ and ψ are mutually inverse bijections between the set of k-maximal nilpotent subsemigroups of $M(n, \mathbb{F})$ and the set of all flags in $\mathbb{F}^n$ of length k.*

We will say that a flag $\mathcal{F}$ is a *consolidation* of $\mathcal{F}'$ provided that each V'_i equals V_j for some j. In particular, if $\mathcal{F}$ is a consolidation of $\mathcal{F}'$ then either $\mathcal{F} = \mathcal{F}'$ or $l(\mathcal{F}) > l(\mathcal{F}')$.

Proposition 10. *$\varphi(\mathcal{F}') \subset \varphi(\mathcal{F})$ if and only if $\mathcal{F}$ is a consolidation of $\mathcal{F}'$.*

Proof. The "if" part follows from the definition of φ. To prove the "only if" part we set $S = \varphi(\mathcal{F}')$, $T = \varphi(\mathcal{F})$ and actually have to show that for every i there is j such that $S^i(\mathbb{F}^n) = T^j(\mathbb{F}^n)$. Assume that this is not the case and let i be minimal with this property. From $S \subset T$ we have $S^i \subset T^i$. Let j be maximal such that $S^i(\mathbb{F}^n) \subset T^j(\mathbb{F}^n)$.

In particular, $S^i(\mathbb{F}^n) \neq T^j(\mathbb{F}^n)$ and $S^i(\mathbb{F}^n) \setminus T^{j+1}(\mathbb{F}^n) \neq \varnothing$. Note that our choice of i and j guarantees that $T^j(\mathbb{F}^n) \setminus S^i(\mathbb{F}^n) \subset S^{i-1}(\mathbb{F}^n) \setminus S^i(\mathbb{F}^n)$. Take $v \in T^j(\mathbb{F}^n) \setminus S^i(\mathbb{F}^n)$. By the choice of i and j, $T^{j+1}(\mathbb{F}^n) \cap S^i(\mathbb{F}^n)$ is a proper subspace of $S^i(\mathbb{F}^n)$. Since $S^{i+1}(\mathbb{F}^n)$ is a proper subspace of $S^i(\mathbb{F}^n)$ as well, we get that $(T^{j+1}(\mathbb{F}^n) \cap S^i(\mathbb{F}^n)) \cup S^{i+1}(\mathbb{F}^n) \neq S^i(\mathbb{F}^n)$ (a non-zero vector space is never a union of two proper subspaces). Take $w \in S^i(\mathbb{F}^n) \setminus ((T^{j+1}(\mathbb{F}^n) \cap S^i(\mathbb{F}^n)) \cup S^{i+1}(\mathbb{F}^n))$. By the first statement of Lemma 7, there exists $a \in S \subset T$ such that $a(v) = w$, but $a(T^j(\mathbb{F}^n)) \not\subset T^{j+1}(\mathbb{F}^n)$, as $v \in T^j(\mathbb{F}^n)$ and $w \notin T^{j+1}(\mathbb{F}^n)$ by the construction. This gives a contradiction and completes the proof. □

4 Isomorphism of maximal nilpotent subsemigroups

In the present section we give a description of the k-maximal nilpotent subsemigroups of $M(n, \mathbb{F})$ for a fixed k up to isomorphism.

Corollary 11. *Let $\mathbb{F}$ be infinite. Then every two maximal 2-nilpotent subsemigroups of $M(n, \mathbb{F})$ are isomorphic.*

Proof. Clearly, $|M(n, \mathbb{F})| = |\mathbb{F}|$ in case of infinite $\mathbb{F}$. Then, if $a \in \varphi(\mathcal{F})$ we get that $\lambda a \in \varphi(\mathcal{F})$ for all $\lambda \in \mathbb{F}$ and hence $|\mathbb{F}| \leq |\varphi(\mathcal{F})| \leq |M(n, \mathbb{F})| = |\mathbb{F}|$. Thus $|\varphi(\mathcal{F})| = |\mathbb{F}|$ and the statement follows. Now the statement follows from the fact that two semigroups, S and T, with zero multiplication (i.e $ab = 0$ for all a, b) are isomorphic if and only if $|S| = |T|$ since any bijection $S \setminus \{0\} \to T \setminus \{0\}$ extends to an isomorphism via $0 \mapsto 0$. □

It is clear that the last statement is completely wrong if we replace "semigroup" with "algebra". This shows the difference between these two theories. The main reason for this difference is that in the case of the "algebra" structure there is a nice finite invariant: dimension. This invariant does not work in the "semigroup" case because vector spaces of different finite dimensions over an infinite field still have the same cardinality.

Lemma 12. *Let $\mathbb{F} = \mathbb{F}_q$ be a finite field, $|\mathbb{F}_q| = q = p^l$, where p is a prime. Let $\mathcal{F}$ and $\mathcal{F}'$ be two flags in $\mathbb{F}^n$ of length 2. The semigroups $S = \varphi(\mathcal{F})$ and $T = \varphi(\mathcal{F}')$ are isomorphic if and only if $\{\dim(V_1), \dim(V_2/V_1)\} = \{\dim(V_1'), \dim(V_2'/V_1')\}$.*

Proof. As $\mathbb{F}$ is finite, all semigroups we consider will be finite as well, so we can try to calculate their cardinalities. So, we assume $\dim(V_1) = m$ and have $\dim(V_2/V_1) = n - m$. Let $B = \{v_1, \dots, v_n\}$ be an $\mathcal{F}$-basis of $\mathbb{F}^n$. An element, $a \in M(n, \mathbb{F})$, belongs to S if and only if $a(v_i) = 0$, $i = 1, \dots, m$, and $a(v_i) = \sum_{j=1}^m a_j v_j$, $a_j \in \mathbb{F}$, $i > m$. Hence $|S| = q^{m(n-m)}$.

If $\dim(V_1') = m'$, we get $|T| = q^{m'(n-m')}$. Further $|S| = |T|$ implies $q^{m(n-m)} = q^{m'(n-m')}$, which is equivalent to $m(n - m) = m'(n - m')$. The left hand side of the last equality is a quadratic polynomial in m, and the right hand side represents the value of the same polynomial in the point m'. This implies that there the equality holds if and only if $m = m'$ or $m = (n - m')$. Now everything again follows from the fact that two semigroups, S and T, with zero multiplication are isomorphic if and only if $|S| = |T|$. □

We switch to the case of arbitrary $\mathbb{F}$. The main result of this section is the following theorem.

Theorem 13. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two flags in $\mathbb{F}^n$ of length $r > 2$.*

1. *If $\mathbb{F}$ is infinite and $\mathrm{sig}(\mathcal{F}) = (k, 1, l)$, $k, l > 1$, then the semigroups $S = \varphi(\mathcal{F})$ and $T = \varphi(\mathcal{F}')$ are isomorphic if and only if $\mathrm{sig}(\mathcal{F}') = (k', 1, l')$, $k', l' > 1$.*

2. *If $\mathbb{F}$ is infinite and $\mathrm{sig}(\mathcal{F})$ is different from $(k, 1, l)$, $k, l > 1$, then the semigroups $S = \varphi(\mathcal{F})$ and $T = \varphi(\mathcal{F}')$ are isomorphic if and only if $\mathrm{sig}(\mathcal{F}) = \mathrm{sig}(\mathcal{F}')$.*

3. *If $\mathbb{F}$ is finite, then the semigroups $S = \varphi(\mathcal{F})$ and $T = \varphi(\mathcal{F}')$ are isomorphic if and only if $\mathrm{sig}(\mathcal{F}) = \mathrm{sig}(\mathcal{F}')$.*

To prove this theorem we will need more notation and several lemmas.

Let $\mathbb{F}$ be a field of an infinite cardinality γ and V be a vector-space over $\mathbb{F}$ with a fixed basis, $\{e_i | i = 1, \dots, n\}$. For $v = \sum_{i=1}^{n} a_i e_i$ we set $\hat{v} = (a_1, \dots, a_n)$ and denote by $\hat{V}$ the set of all $0 \neq v \in V$, such that $a_i = 1$ for the minimal i for which $a_i \neq 0$. For $v \in \hat{V}$ set $l_v = \{\lambda v | \lambda \in \mathbb{F}^*\}$ and we have that V is a disjoint union of $\{0\}$ and all l_v, $v \in \hat{V}$. Because of our choice of γ we have $|l_v| = |V| = \gamma$. Moreover, $|\hat{V}| = \gamma$ if $n > 1$.

Lemma 14. *Let $M = (m_{i,j})_{i=1,\dots,k}^{j=1,\dots,l}$ be a $k \times l$ matrix of rank 1 over $\mathbb{F}$. Then there exist unique $\lambda \in \mathbb{F}^*$, $v \in \hat{\mathbb{F}^k}$ and $w \in \hat{\mathbb{F}^l}$ such that $M = \lambda v w^t$.*

Proof. Let v' (resp. w') be a non-zero column (resp. row) of M, which exists, since M has rank 1. Then $v' = \lambda_1 v$ and $w' = \lambda_2 w$ for some $\lambda_1, \lambda_2 \in \mathbb{F}$, $v \in \hat{\mathbb{F}^k}$ and $w \in \hat{\mathbb{F}^l}$. Assume that i' (resp. j') is the number of the first non-zero coordinate of v (resp. w). Then, taking $\lambda = m_{i',j'}$, we obviously get $M = \lambda v w^t$. □

Proposition 15. *Assume that $\mathbb{F}$ is infinite. Let S and T be some 3-maximal nilpotent semigroups of signature $(k, 1, l)$ and $(m, 1, n)$ with $k, l, m, n > 1$ respectively. Then $S \simeq T$.*

Proof. Let $\mathcal{F}$ be the flag of $\mathbb{F}^{k+l+1}$, stabilized by S and $\mathcal{F}'$ be the flag of $\mathbb{F}^{m+n+1}$, stabilized by T. Choose also an $\mathcal{F}$-basis, $\{e_i\}$, in $\mathbb{F}^{k+l+1}$, and an $\mathcal{F}'$-basis, $\{f_j\}$, in $\mathbb{F}^{m+n+1}$. Then the elements of S are matrices of the form

$$\begin{pmatrix} 0 & v & X \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix},$$

where $v \in \mathbb{F}^k$, $w^t \in \mathbb{F}^l$ and X is a $k \times l$ matrix over $\mathbb{F}$. Analogously, the elements of T are matrices of the form

$$\begin{pmatrix} 0 & a & Y \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix},$$

where $a \in \mathbb{F}^m$, $b^t \in \mathbb{F}^n$ and Y is an $m \times n$ matrix over $\mathbb{F}$. We construct a bijection, $\varphi : S \to T$, in several steps.

First, we construct bijections $\psi_1 : \mathbb{F}^k \to \mathbb{F}^m$ and $\psi_2 : \mathbb{F}^l \to \mathbb{F}^n$ as follows: both ψ_1 and ψ_2 send 0 to 0; then, as $k, l, m, n > 1$, the sets $\hat{\mathbb{F}^i}$, $i = k, l, m, n$, have the same cardinality and we can consider some bijections $\psi_1' : \hat{\mathbb{F}^k} \to \hat{\mathbb{F}^m}$ and $\psi_2' : \hat{\mathbb{F}^l} \to \hat{\mathbb{F}^m}$; the latter uniquely extend to ψ_1 and ψ_2 by the properties $\psi_1(\lambda x) = \lambda\psi_1(x)$, $\lambda \in \mathbb{F}$, $x \in \mathbb{F}^k$, and $\psi_2(\lambda x) = \lambda\psi_2(x)$, $\lambda \in \mathbb{F}$, $x \in \mathbb{F}^l$.

Second, we decompose the set of all $k \times l$ (resp. $m \times n$) matrices over $\mathbb{F}$ into a disjoint union of three subsets: $\{0\}$, $M(1)$ and $M(2)$ (resp. $\{0\}$, $M(1)'$ and $M(2)'$), where the first set contains the zero matrix, the second one contains all matrices of rank 1 and the last one contains all other matrices. Since $\mathbb{F}$ is infinite, we have $|M(1)| = |M(2)| = |M(1)'| = |M(2)'|$. Let ψ_3 denote some bijection from $M(2)$ to $M(2)'$. The last bijection $\psi_4 : M(1) \to M(1)'$ is defined as follows: we take $M \in M(1)$ and use Lemma 14 to write $M = \lambda v w^t$ for some $\lambda \in \mathbb{F}$, $v \in \hat{\mathbb{F}^k}$ and $w \in \hat{\mathbb{F}^l}$; then we set $\psi_4(M) = \lambda\psi_1(v)\psi_2(w)^t$. Lemma 14 guarantees that ψ_4, as defined above, is a bijection. Let ψ_5 be the bijection between the sets of all $k \times l$ and $m \times n$ matrices, composed from ψ_3 and ψ_4.

Now we define $\varphi : S \to T$ as follows:

$$S \ni \begin{pmatrix} 0 & v & X \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \overset{\varphi}{\mapsto} \begin{pmatrix} 0 & \psi_1(v) & \psi_5(X) \\ 0 & 0 & \psi_2(w^t)^t \\ 0 & 0 & 0 \end{pmatrix} \in T.$$

Finally, we are going to prove that φ is a homomorphism. In S we have

$$\begin{pmatrix} 0 & v & X \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & v' & X' \\ 0 & 0 & w' \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & vw' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Applying φ we get

$$\begin{pmatrix} 0 & \psi_1(v) & \psi_5(X) \\ 0 & 0 & \psi_2(w^t)^t \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \psi_1(v') & \psi_5(X') \\ 0 & 0 & \psi_2(w'^t)^t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \psi_5(vw') \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the left-hand side equals

$$\begin{pmatrix} 0 & 0 & \psi_1(v)\psi_2(w'^t)^t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, we have to prove the equality $\psi_5(vw') = \psi_1(v)\psi_2(w'^t)^t$. This equality is an equality on $\{0\} \cup M(1)'$ and follows directly from the definition of ψ_4 (the $M(1)$-part of the map ψ_5). This completes the proof. □

Proposition 16. *Let T_1, T_2 be two isomorphic r-maximal nilpotent subsemigroups of the semigroup $M(n, \mathbb{F})$, $r \geq 3$, $\operatorname{sig}(T_1) = (i_1, i_2, \ldots, i_r)$ and $\operatorname{sig}(T_2) = (j_1, j_2, \ldots, j_r)$. Then $i_t = j_t$ for all $t = 2, \ldots, r-1$.*

To prove this statement we will need some notation. Recall that a reflexive and transitive relation, $<$, on a set, X, is called a preorder. A preorder, $<$, on X induces the equivalence $\sim$ on X defined by: $a \sim b$ if and only if $a < b$ and $b < a$. Then the relation $<$ becomes a partial order on the quotient set $X/\sim$. By the *height* (resp. *depth*) of the preorder $<$ we will understand the height (resp. depth) of the induced partial order on $X/\sim$.

Let $T = \varphi(\mathcal{F})$ be an r-maximal subsemigroup in $M(n, \mathbb{F})$ (here $\mathcal{F}$ is a flag of length r). For $A, B \in T$ we set $A \prec B$ provided that $AC = 0$ implies $BC = 0$ for all $C \in T$, and we set $A \ll B$ provided that $CA = 0$ implies $CB = 0$ for all $C \in T$. It is clear that both $\prec$ and $\ll$ are preorders.

Lemma 17. *1. $A \prec B$ if and only if $(V_{r-1} \cap \ker(A)) \subset (V_{r-1} \cap \ker(B))$.*

2. The set $M_0^{\prec}$ of all maximal elements with respect to $\prec$ consists of all elements $A \in T$ such that $\ker(A) \supset V_{r-1}$.

3. The set $M_i^{\prec}$ of all elements of depth i, $i = 1, 2$, with respect to $\prec$ consists of all elements A such that $\dim(\ker(A) \cap V_{r-1}) = \dim(V_{r-1}) - i$.

Proof. Let $A \prec B$ and suppose that there exists $v \in (V_{r-1} \cap \ker(A))$ such that $v \notin (V_{r-1} \cap \ker(B))$. Since $v \in V_{r-1}$, there exists $C \in T$ such that $\langle v \rangle = \mathrm{Im}(C)$. This guarantees $AC = 0$ and $BC \neq 0$, a contradiction. That $A \prec B$ follows from $(V_{r-1} \cap \ker(A)) \subset (V_{r-1} \cap \ker(B))$ is obvious. This proves the first statement and the second statement follows directly from the first one and Lemma 7.

Let $A \in M_1^{\prec}$. Then $\ker(A) \not\supset V_{r-1}$ by the second statement. Assume that $\dim(\ker(A) \cap V_{r-1}) < \dim(V_{r-1}) - 1$ and take arbitrary $0 \neq v \in V_{r-1} \setminus \ker(A)$. Choose an $\mathcal{F}$-basis, $\{v_i\}$, of $\mathbb{F}^n$, which contains v and a basis of $\ker(A) \cap V_{r-1}$, and consider the linear operator B defined as follows: $B(v) = 0$, $B(v_i) = A(v_i)$, $v_i \neq v$. It follows directly form the definition and the first statement that $B \in T$ and that $A \prec B$, moreover, $\dim(V_{r-1} \cap \ker(B)) = \dim(V_{r-1} \cap \ker(A)) + 1$. Hence $\ker(B) \not\supset V_{r-1}$ as $\dim(\ker(A) \cap V_{r-1}) < \dim(V_{r-1}) - 1$, and thus $B \notin M_0^{\prec}$ by the second statement. This implies that $A \notin M_1^{\prec}$, a contradiction. That all A such that $\dim(\ker(A) \cap V_{r-1}) = \dim(V_{r-1}) - 1$ belong to $M_1^{\prec}$ is obvious. This completes the proof for $M_1^{\prec}$ and the arguments for $M_2^{\prec}$ are similar. □

Lemma 18. *1. $A \ll B$ if and only if $((V \setminus V_1) \cap \mathrm{Im}(A)) \supset ((V \setminus V_1) \cap \mathrm{Im}(B))$.*

2. The set $M_0^{\ll}$ of all maximal elements with respect to $\ll$ consists of all elements A such that $\mathrm{Im}(B) \subset V_1$.

3. The set $M_i^{\ll}$ of all elements of depth i, $i = 1, 2$, with respect to $\ll$ consists of all elements A such that $\dim(\mathrm{Im}(A)) = \dim(\mathrm{Im}(A) \cap V_1) + i$.

Proof. Analogous to that of Lemma 17 with corresponding changes and consideration of images instead of kernels. □

The main difficulty in the study of nilpotent subsemigroups in $M(n,\mathbb{F})$ up to isomorphism is that usual ranks of matrices are not preserved under isomorphisms. To improve the situation, we introduce the notion of a *super rank* (denoted by SupR), which we define in the following way.

Denote $K_{1,0} = M_1^{\prec} \cap M_0^{\lll}$ and $K_{0,1} = M_0^{\prec} \cap M_1^{\lll}$. Consider the set $M_1^{\prec} \cap M_1^{\lll}$ and define the subset

$$K_{1,1} = \{A \in M_1^{\prec} \cap M_1^{\lll} \,:\, TAT \neq 0\}.$$

An element, $A \in M(n,\mathbb{F})$, is called *indecomposable* provided that A can not be decomposed into a product of two elements from T. For an indecomposable element, $A \in M(n,\mathbb{F})$, we will write $\mathrm{SupR}(A) = 1$ provided that $A \in K_{1,1} \cup K_{1,0} \cup K_{0,1}$.

Define $K_{2,0} = M_2^{\prec} \cap M_0^{\lll}$ and $K_{0,2} = M_0^{\prec} \cap M_2^{\lll}$. Consider the set $M_2^{\prec} \cap M_1^{\lll}$ and define the subset

$$K_{2,1} = \{A \in M_2^{\prec} \cap M_1^{\lll} \,:\, TAT \neq 0\}.$$

Consider the set $M_1^{\prec} \cap M_2^{\lll}$ and define the subset

$$K_{1,2} = \{A \in M_1^{\prec} \cap M_2^{\lll} \,:\, TAT \neq 0\}.$$

Consider the set $M_2^{\prec} \cap M_2^{\lll}$ and define the subset

$$K_{2,2} = \{A \in M_2^{\prec} \cap M_2^{\lll} \,:\, TAT \neq 0 \text{ and } TAT \neq TBT \text{ for all } B, \mathrm{SupR}(B) = 1\}.$$

For an indecomposable element, $A \in M(n,\mathbb{F})$, we will write $\mathrm{SupR}(A) = 2$ provided that $A \in K_{2,0} \cup K_{0,2} \cup K_{2,1} \cup K_{1,2} \cup K_{2,2}$.

Lemma 19. *Let T_1 and T_2 be two isomorphic semigroups of signatures $(i_1, \dots, i_r)$ and $(j_1, \dots, j_r)$ respectively and $\varphi : T_1 \to T_2$ be an isomorphism. Let $A \in T_1$ be an indecomposable element of super rank i, $i = 1, 2$. Then $\varphi(A)$ is an indecomposable element of super rank i, $i = 1, 2$, respectively.*

Proof. Follows from the fact that isomorphism preserves indecomposable elements and the sets $K_{u,v}$, where $0 \leq u, v \leq 2$. □

Recall that for a semigroup, S, the notation S^1 denotes either S provided that S has the identity or the semigroup $S \cup \{1\}$ in the opposite case. Our main aim in the following lemma is to describe the super ranks for indecomposable elements in terms of the multiplication.

Lemma 20. *Let $A \in T$ be an indecomposable element and $T^1AT^1 \neq \{A, 0\}$.*

1. $\mathrm{SupR}(A) = 1$ *if and only if there exists an element, $B \in T$, of (usual) rank* 1*, such that $CAD = CBD$ for all $C, D \in T^1$, where either $C \neq 1$ or $D \neq 1$.*

2. $\mathrm{SupR}(A) = 2$ *if and only if* $\mathrm{SupR}(A) \neq 1$ *and there exists an element, $B \in T$, of (usual) rank* 2*, such that $CAD = CBD$ for all $C, D \in T^1$, where either $C \neq 1$ or $D \neq 1$.*

Proof. We prove the first statement and the second one is analogous. Let $A = (a_{i,j})$ be an indecomposable element with $\mathrm{SupR}(A) = 1$. We start with the necessity.

Let $\{v_i\}$ be an $\mathcal{F}$-basis of $\mathbb{F}^n$. If $A \in K_{1,0}$ then $M \in M_1^{\prec}$ and $M \in M_0^{\ll}$. From Lemmas 17 and 18 we get $\dim(\ker(A) \cap V_{r-1}) = \dim(V_{r-1}) - 1$ and $\mathrm{Im}(A) \subset V_1$. Consider the element $B \in T$, defined as follows: $B(v_i) = A(v_i)$, $i \leq \dim(V_{r-1})$, and $B(v_i) = 0$ in other cases. From $\dim(\ker(A) \cap V_{r-1}) = \dim(V_{r-1}) - 1$ it follows that $rank(B) = 1$ and it is obvious that $CAD = CBD$ for all $C, D \in T^1$ such that either $C \neq 1$ or $D \neq 1$.

The case $A \in K_{0,1}$ can be handled by analogous argument. Hence we consider the remaining case $A \in K_{1,1}$. From Lemmas 17 and 18 we get $\dim(\ker(A) \cap V_{r-1}) = \dim(V_{r-1}) - 1$ and $\dim(\mathrm{Im}(A)) = \dim(\mathrm{Im}(A) \cap V_1) + 1$. Let $\{v_i\}$ be an $\mathcal{F}$-basis of $\mathbb{F}^n$. Consider the element $B \in T$, defined via $B(v_i) = A(v_i)$, $\dim(V_1) < i \leq \dim(V_{r-1})$, and $B(v_i) = 0$ in other cases. Already the condition $\dim(\ker(A) \cap V_{r-1}) = \dim(V_{r-1}) - 1$ guarantees that $rank(B) \leq 1$ and the condition $TAT \neq 0$ guarantees $rank(B) > 0$. Therefore $rank(B) = 1$ and one easily gets $CAD = CBD$ for all $C, D \in T^1$ such that either $C \neq 1$ or $D \neq 1$. This completes the proof of the necessity.

Let $A, B \in M(n, \mathbb{F})$ be such that $T^1AT^1 \neq \{A, 0\}$, $rank(B) = 1$ and $CAD = CBD$ for all $C, D \in T^1$ such that either $C \neq 1$ or $D \neq 1$. Then we have $\ker(A) \cap V_{r-1} = \ker(B) \cap V_{r-1}$ and $\mathrm{Im}(A) \setminus (\mathrm{Im}(A) \cap V_1) = \mathrm{Im}(B) \setminus (\mathrm{Im}(B) \cap V_1)$. Hence $\dim(\ker(A) \cap V_{r-1}) = \dim(\ker(B) \cap V_{r-1})$. Since $rank(B) = 1$, we get that either $\dim(\ker(A) \cap V_{r-1}) = \dim(V_{r-1})$ and hence $A \in M_0^{\prec}$ or $\dim(\ker(A) \cap V_{r-1}) = \dim(V_{r-1}) - 1$ and hence $A \in M_1^{\prec}$. Analogously one gets $A \in M_0^{\ll}$ or $A \in M_1^{\ll}$. If $A \in M_1^{\prec} \cap M_1^{\ll}$ then $T^1AT^1 \neq \{A, 0\}$ guarantees that $A \in K_{1,1}$. This completes the proof. $\square$

Proof of Proposition 16. Fix $1 < s < r$. For $i = 1, 2$ consider the following sets:

$$T_i(s) = \{A \in T_i \,:\, T_i^{s-2} A T_i^{r-s} \neq 0 \text{ and } \mathrm{SupR}(A) = 1\},$$
$$\tilde{T}_i(s) = T_i^{r-s} \cap \{B \in T_i \,:\, T^{s-1}B \neq 0\}.$$

Let $\varphi : T_1 \to T_2$ be an isomorphism. It follows from Lemma 19 that $\varphi(T_1(s)) = T_2(s)$. Denote by $u(i)$, $i = 1, 2$, the minimal positive integer for which there exists a subset, $T_i'(s) \subset T_i(s)$, such that $|T_i'(s)| = u(i)$ and for all $B \in \tilde{T}_i(s)$ there exists $C \in T_i'(s)$ such that $CB \neq 0$. It is clear that $u(1) = u(2)$ and to complete the proof it is enough to show that $u(1) = i_s$.

First we show that $u(1) \geq i_s$. Indeed, let $H = \{A_1, \ldots, A_j\} \subset T_i(s)$ and $j < i_s$. Using Lemma 20, we can find the elements $B_1, \ldots, B_j$ such that $rank(B_l) = 1$ for all l and $CA_lD = CB_lD$ for all $C, D \in T^1$, where either $C \neq 1$ or $D \neq 1$. But then the rank of arbitrary linear combination of B_l's does not exceed $j < i_s$. Since $\dim(V_s/V_{s-1}) = i_s > j$ we get that there exists $v \in V_s \setminus V_{s-1}$ such that $B_l(v) = 0$ for all l. This means that $A_lC = B_lC = 0$ for all l if $C \in T_i^{r-s}$ such that $\mathrm{Im}(C) = \langle v \rangle$. As $T^{s-1}C = 0$ and hence $C \in \tilde{T}_1(s)$, we get that H does not satisfy the necessary conditions.

Finally, one can construct a subset, $H \subset T_i(s)$, such that $|H| = i_s$ and for all $B \in \tilde{T}_i(s)$ there exists $C \in H$ satisfying $CB \neq 0$, in the following way: let $0 \neq v \in V_{s-1}$ and $\{v_1, \ldots, v_{i_s}\}$ be arbitrary elements from V_s, which are mapped onto a basis of V_s/V_{s-1}

under the canonical projection. Extend $\{v_1, \ldots, v_{i_s}\} \cup \{v\}$ to an $\mathcal{F}$-basis, $\mathcal{B}$ say, of $\mathbb{F}^n$. Let A_l, $l = 1, \ldots, i_s$, be elements of rank 1 in $M(n, \mathbb{F})$, defined via $A_l(v_l) = v$ and $A_l(w) = 0$ for all $w \in \mathcal{B} \setminus \{v_l\}$. It is obvious that the set $H = \{A_1, \ldots, A_{i_s}\} \subset T$ satisfies the necessary condition. This completes the proof. □

To proceed we will need the following lemma.

Lemma 21. *Let V and V' be two vector spaces over a field. Let $V(2)$ and $V'(2)$ denote the sets of all subspaces of V and V' respectively, whose dimensions do not exceed 2. If $V(2)$ and $V'(2)$ are isomorphic as posets with respect to inclusions then* $\mathrm{dim}V = \mathrm{dim}V'$.

Proof. It is enough to show that the poset of all subspaces of V (with respect to inclusions) can be reconstructed from $V(2)$. To characterize 3-dimensional subspaces we remark that every such a subspace is determined by some 2-dimensional subspace, α say, and some 1-dimensional subspace, l say, such that $l \not\subset \alpha$. In other words, every pair (α, l), $l \not\subset \alpha$, defines a 3-dimensional subspace, which we will denote by (α, l), abusing notation. A 1-dimensional subspace, m, is contained in (α, l) if and only if there exists a 1-dimensional subspace, k, contained in α, such that m is contained in the two-dimensional subspace, generated by k and l. Hence we can characterize all 1-dimensional subspaces in (α, l) only in terms of elements from $V(2)$. A 2-dimensional subspace k is contained in (α, l) if and only if all 1-dimensional subsets of k are contained in (α, l). A subspace k is contained in a subspace m if and only if all 1-dimensional subsets of k are contained in m. This implies that all subspaces of V of dimension at most 3 and their inclusions can be characterized only in terms of elements from $V(2)$ and their inclusions. Hence the poset $V(2)$ completely determines the poset $V(3)$ or all subspaces of V of dimension at most 3 with respect to inclusions. The proof is now easily completed by induction. □

Let $0 \neq A \in T$ be a decomposable element. We will say that the *super rank* $\mathrm{SupR}(A) = 1$ if there exists a decomposition, $A = A_1 A_2 \ldots A_l$, of A into a product of indecomposable elements and $j \in \{1, \ldots, l\}$ such that $\mathrm{SupR}(A_j) = 1$. We will say that the *super rank* $\mathrm{SupR}(A) = 2$ if $\mathrm{SupR}(A) \neq 1$ and there exists a decomposition, $A = A_1 A_2 \ldots A_l$, of A into a product of indecomposable elements and $j \in \{1, \ldots, l\}$ such that $\mathrm{SupR}(A_j) = 2$.

Lemma 22. *Let $A \in T$ be decomposable.*

1. *$rank(A) = 1$ if and only if* $\mathrm{SupR}(A) = 1$.
2. *$rank(A) = 2$ if and only if* $\mathrm{SupR}(A) = 2$.

Proof. Let $\mathrm{SupR}(A) = 1$, $A = A_1 A_2 \ldots A_l$ and $j \in \{1, \ldots, l\}$ be such that $\mathrm{SupR}(A_j) = 1$. Let B be the element of rank 1 such that $CA_jD = CBD$ for all $C, D \in T^1$, where either $C \neq 1$ or $D \neq 1$. Then $A = A_1 \ldots A_{j-1} B A_{j+1} \ldots A_l$ and hence $rank(A) \leq rank(B) = 1$. That $rank(A) = 1$ now follows from $A \neq 0$.

Conversely, let $rank(A) = 1$ and let $A = A_1 A_2 \ldots A_l$ be a decomposition of A into indecomposable elements from T. Let v be a non-zero element in $\mathrm{Im}(A)$ and $\{v_i\}$ be an $\mathcal{F}$-basis of $\mathbb{F}^n$, such that $v \in \{v_i\}$. Consider the linear operator B, defined as follows:

$B(v) = v$ and $B(v_i) = 0$, $v_i \neq v$. Then $A = A_1 A_2 \dots A_l = BA_1A_2 \dots A_l$ and $rank(B) = rank(BA_1) = 1$. Moreover, from the construction of B it follows immediately that $BA_1 \in T$. If BA_1 is indecomposable, we get $rank(BA_1) = 1$ and $T^1 BA_1 T^1 \ni A$ and hence does not coincide with $\{BA_1, 0\}$. This implies $\mathrm{SupR}(BA_1) = 1$. Otherwise we can substitute the element A by the element BA_1 and apply the same arguments. This procedure will certainly stop after not more than n steps, since T is a nilpotent semigroup of nilpotency degree $\leq n$ (see Corollary 5).

The proof of the second statement is analogous. □

Proof of Theorem 13. Let T_1 and T_2 be two r-maximal nilpotent subsemigroup in $M(n, \mathbb{F})$. Assume that $\mathrm{sig}(T_1) = (i_1, \dots, i_r)$ and $\mathcal{F} : 0 = V_0 \subset V_1 \subset \dots$ be the corresponding flag in $\mathbb{F}^n$. Assume that $\mathrm{sig}(T_2) = (j_1, \dots, j_r)$ and $\mathcal{F} : 0 = W_0 \subset W_1 \subset \dots$ be the corresponding flag in $\mathbb{F}^n$. The strategy of our proof is to show that all i_s can be characterized in terms of some invariants, preserved under any isomorphism. If $r > 3$ and $1 < s < r$, this was done in Proposition 16. Hence here we have to consider only remaining cases.

Case 1. $i_1, i_r, \sum_{s=2}^{r-1} i_s \geq 2$. Then it is easy to see that for every $V' \subset V_1$, $\dim(V') = 2$, there exists a decomposable element, A, of rank 2, such that $\mathrm{Im}(A) = V'$. Let $\varphi : T_1 \to T_2$ be an isomorphism. Denote by $\mathrm{Ann}(T_1)$ the two-sided annihilator of T_1. First we remark that the inequality $\sum_{s=2}^{r-1} i_s \geq 2$ implies that for a fixed $i = 1, 2$ and for $A, B \in \mathrm{Ann}(T_1)$, $rank(A) = rank(B) = i$, the condition $\mathrm{Im}(A) = \mathrm{Im}(B)$ is equivalent to the existence of C, $rank(C) = i$, such that $A = CD_1$ and $B = CD_2$ for some D_1 and D_2. In particular, for this C we will have $\mathrm{Im}(C) = \mathrm{Im}(A) = \mathrm{Im}(B)$. Hence, $\mathrm{Im}(A) = \mathrm{Im}(B)$ is equivalent to $\mathrm{Im}(\varphi(A)) = \mathrm{Im}(\varphi(B))$. Now from Lemmas 22 and 19 it follows that $\dim(\mathrm{Im}(\varphi(A))) = \dim(\mathrm{Im}(A))$.

Let A, B be two elements from $\mathrm{Ann}(T_1)$, $rank(A) = 1$, $rank(B) = 2$. Then it is easy to see that $\mathrm{Im}(A) \subset \mathrm{Im}(B)$ is equivalent to the condition that for arbitrary decomposition $B = CD_1$, with C indecomposable and $\mathrm{SupR}(C) = 2$, there exists D_2 such that $A = CD_2$. This implies that $\mathrm{Im}(A) \subset \mathrm{Im}(B)$ is equivalent to $\mathrm{Im}(\varphi(A)) \subset \mathrm{Im}(\varphi(B))$. Hence φ induces an isomorphism from the partially ordered set of all subsets of dimension ≤ 2 in V_1 to the partially ordered set of all subsets of dimension ≤ 2 in W_1. Now Lemma 21 guarantees $\dim(V_1) = \dim(W_1)$.

Case 2. $\sum_{s=2}^{r-1} i_s \geq 2$ and $i_1 = 1$. This situation is obviously characterized by the fact that $K_{2,0} = \varnothing$. Hence $i_1 = 1$ implies $j_1 = 1$. Moreover, going to the opposite semigroup, we also get $i_r = 1$ implies $j_r = 1$ in the case $\sum_{s=2}^{r-1} i_s \geq 2$.

Case 3. $r = 3$, $i_2 = 1$ and $i_1 = 1$. This case is characterized by the fact that there exists an indecomposable element, A, such that $\mathrm{SupR}(A) = 1$ and AT contains all decomposable elements. Hence in this case we get $j_1 = 1$. Analogously $i_3 = 1$ implies $j_3 = 1$.

Case 4. $r = 3$, $i_2 = 1$, $i_1, i_3 > 1$ and $\mathbb{F}$ is infinite. From previous cases we get that $j_1, j_3 > 1$ as well. By Proposition 15, $T_2 \simeq T_1$ for arbitrary $j_1, j_3 > 1$.

Case 5. $r = 3$, $i_2 = 1$, $i_1, i_3 > 1$ and $|\mathbb{F}| = q = p^l < \infty$. One easily calculates that the number of decomposable elements in $\mathrm{Ann}(T_1)$ ($i_1 \times i_3$-matrices of rank ≤ 1) equals $q^{i_1+i_3-1}$. Hence, the number of indecomposable elements in $\mathrm{Ann}(T_1)$ ($i_1 \times i_3$-matrices of rank > 1) equals $q^{i_1+i_3} - q^{i_1+i_3-1}$. Hence $T_1 \simeq T_2$ implies $q^{i_1+i_3} - q^{i_1+i_3-1} = q^{j_1+j_3} - q^{j_1+j_3-1}$ and from

$i_1 + i_3 = j_1 + j_3 = n - 1$ we get $i_1 i_3 = j_1 j_3$. Further, one can easily see that the number of elements in the right annihilator of T_1 equals $q^{i_1(i_2+i_3)}$. From $T_1 \simeq T_2$ we get $q^{i_1(i_2+i_3)} = q^{j_1(j_2+j_3)}$ and, finally, from $i_2 = j_2$ and from $i_1 i_3 = j_1 j_3$ we derive $i_1 = j_1$. This also implies $i_3 = j_3$ and completes the proof. □

Careful analysis of the proof above shows that, in fact, the following statement is true.

Corollary 23. *Let $r > 3$ and T_1, T_2 be some r-maximal nilpotent semigroups (possibly in different $M(i, \mathbb{F})$). Then $T_1 \simeq T_2$ if and only if* $\mathrm{sig}(T_1) = \mathrm{sig}(T_2)$.

5 Appendix: Isolated and completely isolated subsemigroups (in the case of a finite $\mathbb{F}$)

For $i = 0, \dots, n$ we denote by D_i the set of all $A \in M(n, \mathbb{F})$ such that the rank of A equals i. The sets D_i, $i = 0, \dots, n$, constitute an exhaustive list of $\mathcal{D}$-classes in $M(n, \mathbb{F})$, see [Ok2, Chapter 2]. For $i = 0, \dots, n$ we also denote by I_i the two-sided ideal $D_0 \cup \dots \cup D_i$ of $M(n, \mathbb{F})$. We start with the following lemma, which we will often use in the sequel.

Lemma 24. *For every positive integer k, $1 \leq k \leq n - 1$, the ideal I_k is generated as a semigroup by D_k.*

Proof. Let $S = \langle D_k \rangle$. Since $I_k = \cup_{j \leq k} D_j$, by inductive arguments it is enough to show that $D_{k-1} \subset S$. Let $A \in D_{k-1}$, $V_1 = \mathrm{Im}(A)$, $V_2 = \ker(A)$. Then $\dim(\ker(A)) \geq 2$. Let $v_1, \dots, v_i$ be a basis of $\ker(A)$, $v_{i+1}, \dots, v_n$ a basis of $\mathrm{Im}(A)$, and $w_{i+1}, \dots, w_n$ be arbitrary such that $v_j = A(w_j)$, $j > i$. Note that this implies that $w_{i+1}, \dots, w_n$ are linearly independent and $v_1, \dots, v_i, w_{i+1}, \dots, w_n$ form the basis of $\mathbb{F}^n$. Choose further two arbitrary elements w, u, such that $\dim(\langle w, u, V_1 \rangle) = \dim(V_1) + 2$. Note that $i = n - k + 2 \geq 2$ by choice of k. Define the linear operator B_1 as follows: $B_1(v_l) = 0$, $l = 1, \dots, i - 1$; $B_1(v_i) = w$; $B_1(w_l) = v_l$, $l > i$. Let B_2 be a linear operator of rank k, satisfying $B_2(u) = u$, $B_2(w) = 0$, $B_2(v_l) = v_l$, $l > i$. Clearly such B_2 exists and $B_2 B_1 = A$. This completes the proof. □

Recall that a subsemigroup, T, of a semigroups, S, is called *completely isolated* provided that $xy \in T$ implies $x \in T$ or $y \in T$ for arbitrary $x, y \in S$. Further, T is called *isolated* provided that $x^n \in T$ implies $x \in T$ for every $x \in S$.

Proposition 25. *Assume that $\mathbb{F}$ is finite. Then the only completely isolated subsemigroups of $M(n, \mathbb{F})$ are $M(n, \mathbb{F})$,* $\mathrm{GL}(n, \mathbb{F})$ *and $I_{n-1} = M(n, \mathbb{F}) \setminus \mathrm{GL}(n, \mathbb{F})$.*

Proof. It is obvious that the subsemigroups $M(n, \mathbb{F})$, $\mathrm{GL}(n, \mathbb{F})$ and I_{n-1} are completely isolated. Let S be a completely isolated subsemigroup of $M(n, \mathbb{F})$. The semigroup $S \cap \mathrm{GL}(n, \mathbb{F})$ is finite and hence must contain an idempotent provided that it is not empty. Since $\mathrm{GL}(n, \mathbb{F})$ contains only one idempotent we get that either $S \supset \mathrm{GL}(n, \mathbb{F})$ or $S \cap \mathrm{GL}(n, \mathbb{F}) = \varnothing$.

Assume that $S \cap I_{n-1} \neq \varnothing$. By Lemma 24, I_{n-1} is generated by D_{n-1}. Hence every element from $S \cap I_{n-1}$ can be written as a product of elements from D_{n-1}. Since S is completely isolated, we conclude that $S \cap D_{n-1}$ is not empty. Let $A \in S \cap D_{n-1}$. For every $B \in D_{n-1}$ there exist $C, D \in \mathrm{GL}(n, \mathbb{F})$ such that $A = CBD$. If $S \supset \mathrm{GL}(n, \mathbb{F})$ then $B = C^{-1}AD^{-1} \in S$. Otherwise $B \in S$ because S is completely isolated. Thus $S \supset D_{n-1}$. This means that $S \supset I_{n-1} = M(n, \mathbb{F}) \setminus \mathrm{GL}(n, \mathbb{F})$ by Lemma 24 and the statement follows. □

Let $\mathcal{A}$ and $\mathcal{B}$ denote some non-empty sets of $n-1$-dimensional and 1-dimensional subspaces in $\mathbb{F}^n$ respectively. Assume that $V_1 \not\subset V_2$ for every $V_1 \in \mathcal{B}$ and every $V_2 \in \mathcal{A}$. Set

$$S(\mathcal{A}, \mathcal{B}) = \{A \in M(n, \mathbb{F}) \,:\, \mathrm{Im}(A) \in \mathcal{A}, \ker(A) \in \mathcal{B}\}.$$

Lemma 26. *For every $\mathcal{A}$ and $\mathcal{B}$ as above the set $S(\mathcal{A}, \mathcal{B})$ is an isolated subsemigroup in $M(n, \mathbb{F})$.*

Proof. Let $A, B \in S(\mathcal{A}, \mathcal{B})$. Since $\mathrm{Im}(B) \cap \ker(A) = 0$, we get that $\ker(AB) = \ker(B)$ and $\mathrm{Im}(AB) = \mathrm{Im}(A)$. This means that $AB \in S(\mathcal{A}, \mathcal{B})$ and thus $S(\mathcal{A}, \mathcal{B})$ is a subsemigroup in $M(n, \mathbb{F})$.

Let $A \in M(n, \mathbb{F})$ be such that $A^k \in S(\mathcal{A}, \mathcal{B})$ for some positive integer k. Then $rank(A^k) = n-1$ and hence $rank(A) = n-1$. This implies $\ker(A) = \ker(A^k) \in \mathcal{B}$ and $\mathrm{Im}(A) = \mathrm{Im}(A^k) \in \mathcal{A}$. Therefore $A \in S(\mathcal{A}, \mathcal{B})$, which completes the proof. □

For any decomposition $\mathbb{F}^n = V_1 \oplus V_2$, we denote by $e(V_1, V_2)$ the uniquely defined idempotent from $M(n, \mathbb{F})$, which represents the linear operator of projection on V_1 along V_2. Note that the set of all possible $e(V_1, V_2)$ constitute an exhaustive list of idempotents in $M(n, \mathbb{F})$.

Lemma 27. *Let $\mathbb{F}$ be finite and S be an isolated semigroup in $M(n, \mathbb{F})$. Assume that one of the following conditions is satisfied:*

1. *S contains 0.*
2. *S contains two idempotents $e(V_1, V_2), e(V_1', V_2')$ of rank $n-1$, such that $V_2 \subset V_1'$.*
3. *S contains some idempotent of rank $\leq n-2$.*

Then $S \supset I_{n-1}$.

Proof. First we consider the case when S contains 0. Since S is isolated and contains 0, it must contain all nilpotent matrices. Let V_1 and V_2 be two non-trivial subspaces in $\mathbb{F}^n$ such that $\mathbb{F}^n = V_1 \oplus V_2$. Choose a basis, $v_1, \dots, v_i$, of V_1 and a basis, $v_{i+1}, \dots, v_n$, of V_2, and consider the linear operators A and B defined as follows: $A(v_j) = v_{j-1}$, $j = 2, \dots, i+1$, $A(v_j) = 0$ otherwise; $B(v_j) = v_{j+1}$, $j = 1, \dots, i$, $B(v_j) = 0$ otherwise. Clearly, both A and B are nilpotent and hence are contained in S. Thus $S \ni AB = e(V_1, V_2)$, which means that S contains all non-invertible idempotents. Since for a finite $\mathbb{F}$ some power of every element in I_{n-1} is an idempotent, we finally get that $S \supset I_{n-1}$.

Now we consider the case when S contains two idempotents $e(V_1, V_2), e(V_1', V_2')$ of rank $n-1$, such that $V_2 \subset V_1'$. Because of the first part of the lemma proved above it is enough to show that S contains 0. Let $v_1, \dots, v_{n-1}$ be a basis of V_1', such that v_{n-1} is a basis of V_2, and let v_n be a basis of V_2'. Consider the linear operator A defined as follows: $A(v_i) = v_{i+1}$, $i < n-1$, $A(v_{n-1}) = v_1$ and $A(v_n) = 0$. Then $A^{n-1} = e(V_1', V_2') \in S$ and hence $A \in S$, since S is isolated. Therefore $B_i = A^i e(V_1, V_2) A^{n-1-i} \in S$ for every $i = 1, \dots, n-1$. But $B_i(v_j) = v_j$, $j \neq n, i$, and $B_i(v_n) = B_i(v_i) = 0$. Hence $0 = B_1 B_2 \dots B_{n-1} \in S$.

Finally, we consider the case when S contains some idempotent of rank $\leq n-2$. Because of the second part of the lemma proved above it is enough to show that S contains two idempotents, say e, f, of rank $n-1$, such that $\ker(e) \in \mathrm{Im}(f)$. Let $e(V_1, V_2) \in S$ be a non-invertible idempotent and $\dim(V_2) \geq 2$. Choose a basis, $v_1, \dots, v_i$, of V_1 and a basis $v_{i+1}, \dots, v_n$, of V_2, and consider the linear operators A and B defined as follows: $A(v_j) = v_j$, $j \leq i$, $A(v_{i+1}) = 0$, $A(v_j) = v_{j-1}$, $j > i+1$; $B(v_j) = v_j$, $j \leq i$, $B(v_n) = 0$, $B(v_j) = v_{j+1}$, $j = i+1, \dots, n-1$. Then $A^{\dim(V_2)} = B^{\dim(V_2)} = e(V_1, V_2) \in S$ and thus $A, B \in S$ as S is isolated. But this means that $AB, BA \in S$. But both AB and BA are idempotents of rank $n-1$ and by construction $\ker(AB) = \langle v_n \rangle \subset \mathrm{Im}(BA)$. This completes the proof. □

Theorem 28. *Assume that $\mathbb{F}$ is finite. Then the only isolated subsemigroups of the semigroup $M(n, \mathbb{F})$ are $M(n, \mathbb{F})$, $\mathrm{GL}(n, \mathbb{F})$, I_{n-1} and all semigroups $S(\mathcal{A}, \mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are some non-empty sets of $n-1$-dimensional and 1-dimensional subspaces in $\mathbb{F}^n$ respectively, such that $V_1 \not\subset V_2$ for every $V_1 \in \mathcal{B}$ and every $V_2 \in \mathcal{A}$.*

Proof. Since every completely isolated subsemigroup is isolated, it follows from Lemma 26 that all the semigroups listed above are indeed isolated. Let now S be an isolated semigroup of $M(n, \mathbb{F})$. By the same arguments as in Proposition 25, we get that either $S \supset \mathrm{GL}(n, \mathbb{F})$ or $S \cap \mathrm{GL}(n, \mathbb{F}) = \varnothing$. We consider these two cases separately.

Case 1. $S \supset \mathrm{GL}(n, \mathbb{F})$. If $S = \mathrm{GL}(n, \mathbb{F})$, we are done. So, we can assume that $S \cap I_{n-1} \neq \varnothing$. Then $S \cap D_k \neq \varnothing$ for some k and hence $S \supset D_k$ since $S \supset \mathrm{GL}(n, \mathbb{F})$. Lemma 24 now implies $S \supset I_k$, in particular, $S \ni 0$. Thus $S \supset I_{n-1}$ by Lemma 27 and we get $S = M(n, \mathbb{F})$ in this case.

Case 2. $S \cap \mathrm{GL}(n, \mathbb{F}) = \varnothing$. Then S contains a non-invertible idempotent. If S contains an idempotent of rank $\leq n-2$, then $S \supset I_{n-1}$ according to Lemma 27 and thus $S = I_{n-1}$. Hence we can now assume that all idempotents of S have rank $n-1$. Let $E(S) = \{e_1, \dots, e_m\}$ and set $\mathcal{A} = \{\mathrm{Im}(e_1), \dots, \mathrm{Im}(e_m)\}$, $\mathcal{B} = \{\ker(e_1), \dots, \ker(e_m)\}$. From Lemma 27 it follows that $\ker(e_j) \not\subset \mathrm{Im}(e_{j'})$ for all j, j'. Let $V_1 \in \mathcal{A}$ and $e_\alpha \in E(S)$ be such that $V_1 = \mathrm{Im}(e_\alpha)$. Let $V_2 \in \mathcal{B}$ and $e_\beta \in E(S)$ be such that $V_2 = \ker(e_\beta)$. Then $\mathrm{Im}((e_\alpha e_\beta)^i) = V_1$ and $\ker((e_\alpha e_\beta)^i) = V_2$ for all i, in particular, for that i, for which the element $(e_\alpha e_\beta)^i$ belongs to $E(S)$. This proves that $E(S) \ni e(V_1, V_2)$ for all $V_1 \in \mathcal{A}$ and all $V_2 \in \mathcal{B}$. Take now arbitrary $A \in S(\mathcal{A}, \mathcal{B})$. Since $A^i \in E(S(\mathcal{A}, \mathcal{B})) = E(S)$ for some i, we get that $A \in S$ and hence $S \supset S(\mathcal{A}, \mathcal{B})$. On the other hand, the fact that $S(\mathcal{A}, \mathcal{B})$ is isolated says that $A \notin S(\mathcal{A}, \mathcal{B})$ implies that the corresponding idempotent A^i can not belong to $E(S) = E(S(\mathcal{A}, \mathcal{B}))$ since either $\mathrm{Im}(A^i) \notin \mathcal{A}$ or $\ker(A^i) \notin \mathcal{B}$. From this we derive $S = S(\mathcal{A}, \mathcal{B})$ and complete the proof. □

Acknowledgments

This paper was essentially written during the visits of the first author to Uppsala University, which were supported by The Royal Swedish Academy of Sciences and The Swedish Institute. The financial support of The Academy. The Swedish Institute and the hospitality of Uppsala University are gratefully acknowledged. We also would like to thank Prof. O. Ganyushkin for useful conversations. We are in debt to Prof. T. Ekholm and Prof. O. Viro for their help regarding the proof of Lemma 21. We thank Prof. J. Okniński and Prof. O. Artemovych for their comments on the first version of the paper. We are especially in debt to the referee for many very valuable comments.

References

[CP] *A. Clifford, G. Preston*, The algebraic theory of semigroups. Vol. I. and II. Mathematical Surveys, No. 7 American Mathematical Society, Providence, R.I. 1961.

[GK1] *O. G. Ganyushkin, T. V. Kormysheva*, The chain decomposition of partial permutations and classes of conjugate elements of the semigroup $\mathcal{IS}_n$, Visnyk of Kyiv University, 1993, no. 2, 10-18.

[GK2] *O. G. Ganyushkin, T. V. Kormysheva*, The structure of nilpotent subsemigroups of a finite inverse symmetric semigroup. (Ukrainian), Dopov. Nats. Akad. Nauk Ukrainy 1995, no. 1, 8–10.

[GK3] *O. G. Ganyushkin, T. V. Kormysheva*, On nilpotent subsemigroups of a finite symmetric inverse semigroup. (Russian) Mat. Zametki 56 (1994), no. 3,29–35, 157 translation in Math. Notes 56 (1994), no. 3-4, 896–899 (1995).

[GK4] *O. G. Ganyushkin, T. V. Kormysheva*, Isolated and nilpotent subsemigroups of a finite inverse symmetric semigroup. (Ukrainian) Dopov./ Dokl. Akad. Nauk Ukrainy 1993, no. 9, 5–9.

[GM] *O. G. Ganyushkin, V. Mazorchuk*, On the structure of $\mathcal{IO}_n$, Semigroup Forum 66 (2003), no. 3, 455–483.

[GTS] *O. G. Ganyushkin, S. G. Temnikov, H. M. Shafranova*, Groups of automorphisms for maximal nilpotent subsemigroups of the semigroup IS(M). (Ukrainian), Mat. Stud. 13 (2000), no. 1, 11–22.

[Hi] *P. Higgins*, Techniques of semigroup theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

[Ho] *J. Howie*, Fundamentals of semigroup theory. London Mathematical Society Monographs. New Series, 12. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[Ku] *G. Kudryavtseva*, On conjugacy in regular epigroups. Preprint math GR/0605698.

[KM] *G. Kudryavtseva, V. Mazorchuk*, On conjugation in some transformation and Brauer-type semigroups, to appear in Publ. Math. Debrecen.

[La] *M. Lawson*, Inverse semigroups. The theory of partial symmetries. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[Li] *S. Lipscomb*, Symmetric inverse semigroups. Mathematical Surveys and Monographs, 46. American Mathematical Society, Providence, RI, 1996.

[Ma] *A. I. Maltsev*, Nilpotent semigroups. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki 4 (1953), 107–111.

[MT] *V. Mazorchuk, G. Tsyaputa*, Isolated subsemigroups in the variants of $\mathcal{T}_n$, Preprint math.GR/0503489.

[Ok1] *J. Okniński*, Nilpotent semigroups of matrices, Math. Proc. Cambridge Philos. Soc. 120 (1996), no. 4, 617-630.

[Ok2] *J. Okniński*, Semigroups of matrices. Series in Algebra, 6. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[Sh1] *H. Shafranova*, Nilpotent subsemigroups of transformation semigroups, Ph.D. Thesis, Kyiv University, 2000.

[Sh2] *H. Shafranova*, Maximal nilpotent subsemigroups of the semigroup $\mathcal{IS}(\mathbb{N})$, Visn., Ser. Fiz.-Mat. Nauky, Kyiv. Univ. Im. Tarasa Shevchenka 1997, No.4 (1997), 98–103.

[Shev] *L. N. Shevrin*, On nilsemigroups, Uspehi Mat. Nauk. 14, No. 5 (1959), 216-217.

V.M.: Department of Mathematics, Uppsala University, SE 471 06, Uppsala, SWEDEN, e-mail: *mazor@math.uu.se*

G.K.: Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 64, Volodymyrska st., 01033, Kyiv, UKRAINE,
e-mail: *akudr@univ.kiev.ua*