On the radical of $\mathcal{FP}^+(S_n)$

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Abstract

We prove that the semigroups $\mathfrak{B}_n$ of all binary relations and the factor-power $\mathcal{FP}^+(S_n)$ of the symmetric group can be asymptotically approximated by nilpotent semigroups. Further, we show that almost all elements of these semigroups satisfy the equation $x^2 = 0$, where 0 denotes the full binary relation. All these facts are obtained from a careful study of the radical $\mathfrak{R}_n$ in $\mathcal{FP}^+(S_n)$. Along this study we also derive some corollaries for doubly stochastic matrices.

1 Introduction and the main result

Let T be a semigroup of (possibly partial) transformations on the set M. The Boolean $\mathcal{B}(T)$ inherits from T the natural structure of a semigroup. Moreover, the equivalence relation $\sim$ on $\mathcal{B}(T)$, defined as follows: $A \sim B$ if and only if $m^A = m^B$ for all $m \in M$, appears to be a congruence on $\mathcal{B}(T)$. The corresponding quotient $\mathcal{B}(T)/_\sim$ is usually denoted by $\mathcal{FP}(T, M)$. The equivalence class of the empty set is always the isolated zero in the semigroup $\mathcal{FP}(T, M)$, so we can form the semigroup $\mathcal{FP}^+(T, M) = \mathcal{FP}(T, M) \setminus \{\overline{\varnothing}\}$, which is called *the factor power of* (T, M). Extending the action of the semigroup T on M to the Boolean $\mathcal{B}(M)$, we get a natural action of $\mathcal{FP}^+(T, M)$ on $\mathcal{B}(M)$.

This construction appeared first in [GM1] and was later studied in [GM2, GM3, M1, M2] with the special emphasize on the case, when $T = S_n$, the full finite symmetric group with the natural action on the set $N = \{1, 2, \ldots, n\}$. In particular, it was shown that $\mathcal{FP}^+(S_n)$ asymptotically approximates the semigroup $\mathfrak{B}_n$ of all binary relations on N, moreover, $\mathcal{FP}^+(S_n)$ has a very nice inner description inside $\mathfrak{B}_n$. The semigroup $\mathcal{FP}^+(S_n)$ appears naturally also as the quotient of the semigroup Ω_n of doubly stochastic real matrices. Several classes of subsemigroups, automorphisms and Green relations for $\mathcal{FP}^+(S_n)$ have been already described.

In the present paper we continue the study of the semigroup $\mathcal{FP}^+(S_n)$ with the emphasize on its *radical* $\mathfrak{R}_n$, which is the intersection of all maximal nilpotent subsemigroups in $\mathcal{FP}^+(S_n)$ (see [GM3]). Alternatively, $\mathfrak{R}_n$ consists of all $\tau \in \mathcal{FP}^+(S_n)$, such that $|\tau(A)| > |A|$ for every proper subset $A \subset N$. The semigroup $\mathfrak{R}_n$ is nilpotent of nilpotency degree n ([GM3, Theorem 6]). The main result of the present paper is the following theorem, which seems to be quite surprising:

Theorem 1. $\mathfrak{R}_n$ *asymptotically approximates* $\mathcal{FP}^+(S_n)$ *and* $\mathfrak{B}_n$, *that is,*

$$\lim_{n \to \infty} \frac{|\mathfrak{R}_n|}{|\mathcal{FP}^+(S_n)|} = \lim_{n \to \infty} \frac{|\mathfrak{R}_n|}{|\mathfrak{B}_n|} = 1.$$

In other words, the semigroup $\mathfrak{B}_n$ is asymptotically approximated by a nilpotent semigroup. If one recalls, see [McS, S], that arbitrary semigroup is isomorphic to a transitive semigroup of binary relations, then our results seem to be quite parallel with and somehow explain the main result in [SYT], where it was obtained that 99% of all semigroups of order 8 are nilpotent.

The paper is organized as follows: we prove our main theorem in Section 2. In Section 3 we discuss some corollaries, analogous results and other applications of the radical $\mathfrak{R}_n$. In particular, we derive a result about a kind of "row-column symmetric" behavior of doubly stochastic matrices. Finally, we formulate several conjectures and open problems in Section 4.

2   Proof of the main result

We have a natural chain of embeddings $\mathfrak{R}_n \subset \mathcal{FP}^+(S_n) \subset \mathfrak{B}_n$. As we have already mentioned in the Introduction, [GM2, Theorem 6] states that the semigroup $\mathcal{FP}^+(S_n)$ asymptotically approximates $\mathfrak{B}_n$. Hence, it is enough to prove that the radical $\mathfrak{R}_n$ asymptotically approximates $\mathcal{FP}^+(S_n)$. Actually, we are not going to calculate the necessary limit but rather would like to show that

$$\lim_{n \to \infty} \frac{|\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n|}{|\mathcal{FP}^+(S_n)|} = 0,$$

which would clearly imply the necessary statement.

Our aim now is to find an effective upper bound for the number of elements in $\mathcal{FP}^+(S_n)$, which do not belong to the radical. From the definition of the radical, an element, τ, belongs to $\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$ if and only if there exists $A \subset N$, $A \neq \varnothing, N$, such that $|\tau(A)| = |A|$. To get a convenient formula, we even will leave $\mathcal{FP}^+(S_n)$ and make the estimate in $\mathfrak{B}_n$. In the next lemma for $A \subset N$ and $\varphi \in \mathfrak{B}_n$ we denote

$$\varphi(A) = \{b \in N \ : \ \text{there exists } a \in A \text{ such that } a\varphi b\}.$$

We call a binary relation, $\varphi \in \mathfrak{B}_n$, *strongly invariant* provided that there exist $A, B \subset N$, $0 < |A| = |B| < n$, satisfying $\varphi(A) \subset B$ and $\varphi(\overline{A}) \subset \overline{B}$.

Lemma 1. *The number of strongly invariant binary relations* $\varphi \in \mathfrak{B}_n$ *is less or equal than*

$$d_n = \sum_{i=1}^{n-1} \binom{n}{i}^2 2^{i^2 + (n-i)^2}.$$

Proof. We denote $i = |A|$. Then we can choose A in $\binom{n}{i}$ different ways and B in $\binom{n}{i}$ different ways. Every binary relation φ, which we count, is, by assumptions, a subset of $(A \times B) \cup (\overline{A} \times \overline{B})$. But $|(A \times B) \cup (\overline{A} \times \overline{B})| = i^2 + (n-i)^2$ and to complete the proof one has to apply the multiplication rule and sum up over all i. $\square$

Corollary 1. $|\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n| \leq d_n.$

Proof. Let $\tau \in \mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$ and $A \subset N$, $A \neq \varnothing, N$, such that $|\tau(A)| = |A|$. Then we have $s(A) = A$ for every permutation $s \in \tau$ (here τ is considered as the maximal element in the corresponding equivalence class of $\mathcal{B}(S_n)$), and hence $s(\overline{A}) = \overline{A}$. This implies $\tau(\overline{A}) = \overline{A}$. Therefore τ is a strongly invariant binary relation and the statement follows from lemma 1. $\square$

Denote

$$d_n' = \binom{n}{1}^2 2^{1^2+(n-1)^2} + \binom{n}{n-1}^2 2^{(n-1)^2+1^2} = 2n^2 2^{n^2-2n+2}$$

and $d_n'' = d_n - d_n'$.

Now we can compute the limit $\lim\limits_{n\to\infty} \dfrac{d_n}{2^{n^2}}$, using $\binom{n}{k} \leq 2^n$ and the fact that $i^2 + (n-i)^2 \leq 2^2 + (n-2)^2 = n^2 - 4n + 8$ for all $n \geq 4$ and $2 \leq i \leq n-2$:

$$0 \leq \lim_{n\to\infty} \frac{d_n}{2^{n^2}} = \lim_{n\to\infty} \frac{d_n'}{2^{n^2}} + \lim_{n\to\infty} \frac{d_n''}{2^{n^2}} \leq 2 \lim_{n\to\infty} \frac{n^2 2^{n^2-2n+2}}{2^{n^2}} +$$

$$+ \lim_{n\to\infty} \frac{\sum_{i=2}^{n-2} \binom{n}{i}^2 2^{i^2+(n-i)^2}}{2^{n^2}} \leq \lim_{n\to\infty} n^2 2^{3-2n} + \lim_{n\to\infty} n 2^{2n} 2^{8-4n} = 0.$$

Hence

$$0 \leq \lim_{n\to\infty} \frac{|\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n|}{|\mathcal{FP}^+(S_n)|} \leq \lim_{n\to\infty} \frac{d_n |\mathfrak{B}_n|}{|\mathcal{FP}^+(S_n)| |\mathfrak{B}_n|} = \lim_{n\to\infty} \frac{d_n}{2^{n^2}} \lim_{n\to\infty} \frac{|\mathfrak{B}_n|}{|\mathcal{FP}^+(S_n)|} = 0.$$

This completes the proof of Theorem 1.

3 Corollaries, applications and some other facts for the radical $\mathfrak{R}_n$

Let $B_n \subset A_n$, $n \in \mathbb{N}$, be two families of sets. We say that *almost all* elements of A_n belong B_n if $|B_n|/|A_n| \to 1$, $n \to \infty$ (we refer the reader to [K] for the corresponding terminology in graph theory). The following statement contains several direct corollaries from Theorem 1.

Corollary 2. *1. $\mathfrak{B}_n$ is asymptotically approximated by a nilpotent semigroup.*

2. $\mathcal{FP}^+(S_n)$ is asymptotically approximated by a nilpotent semigroup.

3. Almost all elements in $\mathfrak{B}_n$ are nilpotent with respect to the full binary relation 0.

4. Almost all elements in $\mathcal{FP}^+(S_n)$ are nilpotent.

5. Almost all elements in $\mathcal{FP}^+(S_n)$ are contained in the radical.

6. The transitive closure of almost all oriented graphs coincides with the full oriented graph (with loops).

7. Almost all oriented graphs are connected.

8. Almost all oriented graphs are strongly connected.

Quite amazing property of the radical $\mathfrak{R}_n$ obtained above is a good motivation to study $\mathfrak{R}_n$ more detailed. We start this study with one more asymptotic property of the radical. We recall that the element 0 in $\mathcal{FP}^+(S_n)$ corresponds is the full binary relation $N \times N$ in $\mathfrak{B}_n$.

Proposition 1. *Almost all elements of $\mathfrak{R}_n$ (and hence of $\mathcal{FP}^+(S_n)$ and $\mathfrak{B}_n$) satisfy equation $x^2 = 0$.*

Proof. Because of Theorem 1 and [GM2, Theorem 6], it is enough to prove the statement for $\mathfrak{B}_n$. We try to estimate the number of those binary relations φ for which $\varphi^2 \neq 0$. If $\varphi^2 \neq 0$, then there exist $x, y \in N$ (possibly $x = y$) such that φ^2 does not contain (x, y). First we assume $x \neq y$. That φ^2 does not contain (x, y) means that for every $z \in N$ the elements (x, z) and (z, y) can not be contained in φ together, that is either both (x, z) and (z, y) do not belong to φ or (x, z) does not belong to φ or (z, y) does not belong to φ. Hence, the number of such φ does not exceed $3^n \cdot 2^{n^2 - 2n}$.

If $x = y$, then φ can not contain (x, x) and for every $z \in N$ either both (x, z) and (z, x) do not belong to φ or (x, z) does not belong to φ or (z, x) does not belong to φ. Hence, the number of such φ does not exceed $3^{n-1} \cdot 2^{n^2 - 2n + 1}$. Therefore the number of all binary relations φ, satisfying $\varphi^2 \neq 0$, does not exceed $\binom{n}{2} \cdot 3^n \cdot 2^{n^2 - 2n} + n \cdot 3^{n-1} \cdot 2^{n^2 - 2n + 1}$. Dividing with $|\mathfrak{B}_n| = 2^{n^2}$ we get

$$\frac{n(n-1) \cdot 3^n + n \cdot 3^{n-1} \cdot 4}{2^{2n+1}} = \frac{3^n(n(n-1) + \frac{4}{3}n)}{2^{2n+1}}.$$

Since $3^3 < 2^5$, we can rewrite and estimate the last expression in the following way:

$$\frac{3^n(n(n-1) + \frac{4}{3}n)}{2^{2n+1}} = \frac{3^n}{2^{5n/3}} \cdot \frac{(n(n-1) + \frac{4}{3}n)}{2 \cdot 2^{n/3}} < \frac{(n(n-1) + \frac{4}{3}n)}{2 \cdot 2^{n/3}} \to 0, \quad n \to \infty.$$

$\square$

4

Since $\mathfrak{R}_n$ is a two-sided ideal of $\mathcal{FP}^+(S_n)$, we can consider the corresponding Rees congruence $\rho = \rho_{\mathfrak{R}_n}$ and the quotient $\mathcal{FP}^+(S_n)_\rho$ modulo this congruence. The semigroup $\mathcal{FP}^+(S_n)_\rho$ is quite small in comparison with $\mathcal{FP}^+(S_n)$. Indeed, from Theorem 1 it follows immediately that

$$\lim_{n\to\infty} \frac{|\mathcal{FP}^+(S_n)_\rho|}{|\mathcal{FP}^+(S_n)|} = 0.$$

The natural embedding $S_n \subset S_{n+1}$, which fixes $n + 1$, extends to the natural embedding $\mathcal{FP}^+(S_n) \subset \mathcal{FP}^+(S_{n+1})$. It is obvious that the image of $\mathcal{FP}^+(S_n)$ under this embedding does not intersect $\mathfrak{R}_{n+1}$ and hence $|\mathcal{FP}^+(S_n)_\rho| > |\mathcal{FP}^+(S_n)|$. Hence the semigroup $\mathcal{FP}^+(S_n)_\rho$ grows quicker than the radical of the previous factor power. At the same time most of properties can be easily translated from $\mathcal{FP}^+(S_n)$ to $\mathcal{FP}^+(S_n)_\rho$ (using the identification of non-zero elements from $\mathcal{FP}^+(S_n)_\rho$ with the corresponding elements in $\mathcal{FP}^+(S_n)$). In particular, we get the following.

Theorem 2. *1. The idempotents in $\mathcal{FP}^+(S_n)_\rho$ are equivalence relations.*

2. $\mathcal{FP}^+(S_n)_\rho$ and $\mathcal{FP}^+(S_n)$ have the same maximal subgroups.

3. The Green relations on $\mathcal{FP}^+(S_n)_\rho$ are induced from $\mathcal{FP}^+(S_n)$.

4. There is a natural bijection between the nilpotent subsemigroups in $\mathcal{FP}^+(S_n)_\rho$ and the nilpotent subsemigroups in $\mathcal{FP}^+(S_n)$, which contain $\mathfrak{R}_n$. This bijection preserves inclusion, in particular, for $n > 2$ there is a natural bijection between the maximal nilpotent subsemigroups in $\mathcal{FP}^+(S_n)_\rho$ and the maximal nilpotent subsemigroups in $\mathcal{FP}^+(S_n)$.

5. All automorphisms of $\mathcal{FP}^+(S_n)_\rho$ are inner.

Proof. The first statement follows from the fact that $\mathfrak{R}_n$ is nilpotent and hence contains exactly one idempotent, which is 0. The second statement follows from the observation that all elements of $\mathfrak{R}_n$ are nilpotent and hence $\mathfrak{R}_n$ does not intersect any maximal subgroup, corresponding to a non-zero idempotent.

The third statement follows from the fact that $\mathcal{FP}^+(S_n)_\rho$ is a Rees factor of $\mathcal{FP}^+(S_n)$.

The forth statement follows from the definition of $\mathfrak{R}_n$ as the intersection of all maximal nilpotent subsemigroups of $\mathcal{FP}^+(S_n)$.

The last statement can be obtained *mutatis mutandis* from [M1] if one remarks that all arguments used in [M1] can actually be applied to the set $\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$. $\square$

We refer the reader to [M2] and [GM3] for the description of Green relations and maximal nilpotent subsemigroups of $\mathcal{FP}^+(S_n)$ respectively. By Theorem 2, the same statements hold for the semigroup $\mathcal{FP}^+(S_n)_\rho$ as well.

We would like to finish this section with one more property of the radical and an application to the doubly stochastic matrices. Recall that there exists the canonical anti-involution $*$ on $\mathcal{FP}^+(S_n)$, defined for the elements $\tau \in \mathcal{FP}^+(S_n)$, $\tau = \overline{A} \in \mathcal{B}(S_n)/_\sim$, as follows: $(\overline{A})^* = \overline{\{a^{-1} : a \in A\}}$.

Lemma 2. $\mathfrak{R}_n^* = \mathfrak{R}_n$.

Proof. Since $*$ is an anti-involution, it is bijective on $\mathcal{FP}^+(S_n)$, and hence it is enough to prove that $(\mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n)^* = \mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$. Let $\tau = \overline{A} \in \mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$. Then there exists a proper subset, $T \subset N$, such that $|\overline{A}(T)| = |T|$. Set $T' = \tau(T)$. Then for every $a \in A$ we have $a(T) = T'$ and hence $a^{-1}(T') = T$. Therefore $(\overline{A})^*(T') = T$. In particular, $|(\overline{A})^*(T')| = |T'|$ and thus $(\overline{A})^* \in \mathcal{FP}^+(S_n) \setminus \mathfrak{R}_n$. This completes the proof. $\square$

Recall that an $n \times n$ real matrix, M, is called *doubly stochastic* provided that it has non-negative entries and in every row and in every column the sum of all elements is equal to 1. The set of all doubly stochastic matrices is a semigroup under the usual matrix multiplication. We say that two doubly stochastic matrices are equivalent if their entries are equal to zero simultaneously. It was shown in [GM2, Theorem 2] that this equivalence is, in fact, a congruence and that the corresponding quotient semigroup is isomorphic to $\mathcal{FP}^+(S_n)$. We say that a doubly stochastic matrix, M, *satisfies C-condition*, if for every positive integer k and every $1 \leq j \leq n$ the number of zeros in the j-th column in the matrix M^k is either zero or strictly less than the number of zeros in the j-th column in the matrix M^{k-1}. Analogously we define *R-condition* with respect to the rows of M.

Lemma 3. *A doubly stochastic matrix, M, belongs to the equivalence class, corresponding to some element in $\mathfrak{R}_n$, if and only if M satisfies C-condition.*

Proof. This follows from the fact that the natural action of $\mathcal{FP}^+(S_n)$ on N is coordinated with the natural action of doubly stochastic matrices on vectors from $\mathbb{R}_+^n$. $\square$

Corollary 3. *A doubly stochastic matrix, M, satisfies C-condition if and only if it satisfies R-condition.*

Proof. According to Lemma 3, the matrix M satisfies C-condition if and only if it belongs to the equivalence class, corresponding to an element from $\mathfrak{R}_n$. The anti-involution $*$ naturally extends to the transposition of doubly stochastic matrices. The transposition of matrices interchange rows and columns and preserves classes, which correspond to $\mathfrak{R}_n$, by Lemma 2. Now the statement follows from Lemma 3, applied to M^t. $\square$

4　Problems and conjectures

We would like to finish the paper with a list of several problems and conjectures related to the radical $\mathfrak{R}_n$ and the results of this paper. The first conjecture is very natural and is motivated by Theorem 1 and the results from [SYT]. For a positive integer, n, let us denote by a_n the number of isomorphism classes of semigroups with n elements, and by b_n the number or the isomorphism classes of nilpotent semigroups with n elements. We also denote by a_n' the total number of semigroups on N, and by b_n' the total number of nilpotent semigroups on N of nilpotency degree at most 3. Clearly $b_n < a_n$ and $b_n' < a_n'$.

Conjecture 1. $\lim\limits_{n \to \infty} \dfrac{b_n}{a_n} = 1$. *In other words, almost all finite semigroups are nilpotent.*

Conjecture 1 is an "up-to-isomorphism" version of the following conjecture, mentioned in [SYT] with the reference to [KRS], where a stronger statement (for nilpotent semigroups of nilpotency degree 3) is formulated as a theorem, but the proof is only outlined.

Conjecture 2. $\lim\limits_{n\to\infty} \dfrac{b_n'}{a_n'} = 1$.

By [SYT], the ratio $\tilde{b}_8/\tilde{a}_8$, where the corresponding numbers are counted up to isomorphism and anti-isomorphism, is approximately $\frac{99}{100}$. Further, each finite semigroup S is a subsemigroup in some $\mathfrak{B}_n$. One can hardly expect that the classical representation of S in $\mathfrak{B}_n$ will be related to the radical. However, as it was shown in [McS], every finite semigroup is isomorphic to a transitive semigroup of binary relations on a finite set. Further the transitive semigroups are usually contained in the radical and thus it is natural to expect that almost all S will be represented inside $\mathfrak{R}_n$ and therefore will be nilpotent.

The positive answer to Conjecture 1 would clearly imply the positive answer to the following statement, which however can be viewed as a separate claim.

Conjecture 3. *Let b_n'' denote the number of the isomorphism classes of semigroups with n elements having exactly 1 idempotent. Then* $\lim\limits_{n\to\infty} \dfrac{b_n''}{a_n} = 1$.

The next problem is a natural continuation of the previous one. Conjecture 1 states that almost all semigroups are nilpotent and a motivation for this is that $\mathfrak{B}_n$ contains a very big nilpotent subsemigroup $\mathfrak{R}_n$. It is then natural to expect that nilpotent subsemigroups can be represented inside $\mathfrak{R}_n$.

Problem 1. *Which finite nilpotent subsemigroups embeds into $\mathfrak{R}_n$ for some n?*

The following conjecture represents our point of view on this problem, however, we do not have so much evidence for it as in the case of Conjecture 1.

Conjecture 4. *All finite nilpotent subsemigroups embed into $\mathfrak{R}_n$.*

Now several problems and conjectures about $\mathfrak{R}_n$. The semigroup $\mathfrak{R}_n$ is nilpotent and hence it has trivial (1-element) classes for all Green relations. However, inside $\mathcal{FP}^+(S_n)$ the picture is quite different. Of course, $\mathfrak{R}_n$, as a two-sided ideal, is still closed under Green relations, but now it is possible that even $\mathcal{H}$-classes in $\mathfrak{R}_n$ are quite big. Here are the Green classes for the semigroup $\mathcal{FP}^+(S_3)$ (we set $N = \{1, 2, 3\}$, $\mathcal{H}_{k,l}^i = \mathcal{L}_k^i \cap \mathcal{R}_l^i$, and remark that $\mathcal{J} = \mathcal{D}$):

$$
\mathcal{D}^1 = \mathcal{R}_1^1 = \mathcal{L}_1^1 = \mathcal{H}_{1,1}^1 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \right.
$$
$$
\left. \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\};
$$

$\mathcal{D}^2:$

	$\mathcal{R}_1^2$	$\mathcal{R}_2^2$	$\mathcal{R}_2^2$
$\mathcal{L}_1^2$	$\left\{\begin{pmatrix}1&2&3\\1,2&1,2&3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,2&3&1,2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\3&1,2&1,2\end{pmatrix}\right\}$
$\mathcal{L}_2^2$	$\left\{\begin{pmatrix}1&2&3\\2,3&2,3&1\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\2,3&1&2,3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1&2,3&2,3\end{pmatrix}\right\}$
$\mathcal{L}_3^2$	$\left\{\begin{pmatrix}1&2&3\\1,3&1,3&2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,3&2&1,3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\2&1,3&1,3\end{pmatrix}\right\}$

$\mathcal{D}^3:$

	$\mathcal{R}_1^3$	$\mathcal{R}_2^3$	$\mathcal{R}_2^3$
$\mathcal{L}_1^3$	$\left\{\begin{pmatrix}1&2&3\\1,2&N&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,2&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&N&1,2\end{pmatrix}\right\}$
$\mathcal{L}_2^3$	$\left\{\begin{pmatrix}1&2&3\\2,3&N&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&2,3&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&N&2,3\end{pmatrix}\right\}$
$\mathcal{L}_3^3$	$\left\{\begin{pmatrix}1&2&3\\1,3&N&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,3&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&N&1,3\end{pmatrix}\right\}$

$$\mathcal{D}^4 = \mathcal{R}_1^4 = \mathcal{L}_1^4 = \mathcal{H}_{1,1}^4 = \left\{\begin{pmatrix}1&2&3\\1,2&2,3&3,1\end{pmatrix}, \begin{pmatrix}1&2&3\\2,3&1,2&3,1\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1&2&3\\3,1&2,3&1,2\end{pmatrix}, \begin{pmatrix}1&2&3\\1,2&3,1&2,3\end{pmatrix}, \begin{pmatrix}1&2&3\\2,3&3,1&1,2\end{pmatrix}, \begin{pmatrix}1&2&3\\3,1&1,2&2,3\end{pmatrix}\right\};$$

$\mathcal{D}^5:$

	$\mathcal{R}_1^5$	$\mathcal{R}_2^5$	$\mathcal{R}_2^5$
$\mathcal{L}_1^5$	$\left\{\begin{pmatrix}1&2&3\\1,2&2,3&N\end{pmatrix}, \begin{pmatrix}1&2&3\\2,3&1,2&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,2&N&2,3\end{pmatrix}, \begin{pmatrix}1&2&3\\2,3&N&1,2\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,2&2,3\end{pmatrix}, \begin{pmatrix}1&2&3\\N&2,3&1,2\end{pmatrix}\right\}$
$\mathcal{L}_2^5$	$\left\{\begin{pmatrix}1&2&3\\1,3&1,2&N\end{pmatrix}, \begin{pmatrix}1&2&3\\1,2&1,3&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,3&N&1,2\end{pmatrix}, \begin{pmatrix}1&2&3\\1,2&N&1,3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,3&1,2\end{pmatrix}, \begin{pmatrix}1&2&3\\N&1,2&1,3\end{pmatrix}\right\}$
$\mathcal{L}_3^5$	$\left\{\begin{pmatrix}1&2&3\\1,3&2,3&N\end{pmatrix}, \begin{pmatrix}1&2&3\\2,3&1,3&N\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\1,3&N&2,3\end{pmatrix}, \begin{pmatrix}1&2&3\\2,3&N&1,3\end{pmatrix}\right\}$	$\left\{\begin{pmatrix}1&2&3\\N&1,3&2,3\end{pmatrix}, \begin{pmatrix}1&2&3\\N&2,3&1,3\end{pmatrix}\right\}$

$$\mathcal{D}^6 = \mathcal{R}_1^6 = \mathcal{L}_1^6 = \mathcal{H}_{1,1}^6 = \left\{\begin{pmatrix}1&2&3\\N&N&N\end{pmatrix}\right\};$$

For $\mathcal{FP}^+(S_3)$ we haver $\mathfrak{R}_3 = \mathcal{D}^3 \cup \mathcal{D}^4 \cup \mathcal{D}^5 \cup \mathcal{D}^6$ and we see that even the $\mathcal{H}$-classes of $\mathfrak{R}_3$, viewed as a semigroup in $\mathcal{FP}^+(S_3)$, can be non-trivial. Actually, it is easy to see that in this case we have only 10 elements in $\mathfrak{R}_3$ forming trivial $\mathcal{H}$-classes. One can also remark that all elements from $\mathfrak{R}^3$ satisfy $x^2 = 0$.

Problem 2. *Let c_n denote the number of those elements in $\mathfrak{R}_n$, which form trivial $\mathcal{H}$-classes in $\mathcal{FP}^+(S_n)$. Find the asymptotic $\frac{c_n}{|\mathfrak{R}_n|}$, $n \to \infty$.*

The following question is closely related to this problem.

Problem 3. *Let τ be an element of $\mathcal{FP}^+(S_n)$ (in particular, of $\mathfrak{R}_n$). Find the cardinality of the $\mathcal{H}$-class (resp. $\mathcal{L}$-class, $\mathcal{R}$-class or $\mathcal{D}$-class) containing the element τ.*

The next conjecture is natural after Proposition 1, where it was shown that almost all elements in $\mathfrak{R}_n$ satisfy $x^2 = 0$.

Conjecture 5. *Almost all pairs (x, y) of elements from $\mathfrak{R}_n$ satisfy $xy = 0$.*

Certainly, the elements satisfying $x^2 = 0$ do not form a semigroup of nilpotency degree 2. Denote by f_n the maximal cardinality among all nilpotent subsemigroups in $\mathfrak{R}_n$ of nilpotency degree 2.

Problem 4. *Find the asymptotic $\frac{f_n}{|\mathfrak{R}_n|}$, $n \to \infty$.*

Our last collection of problems is related to the study of the so-called *cross-sections* for semigroups. Let S be a semigroup and $\sim$ an equivalence relation on S. A $\sim$-*cross-section* is a subsemigroup, T, of S, containing exactly one element from every equivalence class with respect to $\sim$. The most important are cross-sections with respect to the Green relations, see [CR, P] and references therein.

Problem 5. *Describe all $\mathcal{H}-$ and $\mathcal{D}-$cross-sections of $\mathcal{FP}^+(S_n)$.*

Problem 6. *Denote by $\sim_\mathcal{H}$ and $\sim_\mathcal{D}$ the equivalence relations on $\mathfrak{R}_n$, obtained by restricting to $\mathfrak{R}_n$ the Green relations $\mathcal{H}$ and $\mathcal{D}$ on $\mathcal{FP}^+(S_n)$ respectively. Describe all $\sim_\mathcal{H}-$ and $\sim_\mathcal{D}-$cross-sections of $\mathfrak{R}_n$.*

Connected to these problems is the following general question:

Problem 7. *Describe all congruences ρ on $\mathcal{FP}^+(S_n)$ (resp. $\mathfrak{R}_n$) and determine for which ρ there exist $\rho-$cross-sections (retracts) in $\mathcal{FP}^+(S_n)$ (resp. $\mathfrak{R}_n$).*

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