The full finite inverse symmetric semigroup $\mathcal{IS}_n$

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Abstract

In this paper we have collected several combinatorial results about the full finite inverse symmetric semigroup $\mathcal{IS}_n$ including those obtained in the last decade. This includes the description of automorphisms and endomorphisms, Green's relations, presentation and also the description of some classes of subsemigroups, e.g. isolated, completely isolated and nilpotent.

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1 Introduction

The full symmetric group $\mathbb{S}_n$ of all one-to-one maps on an n-element set is one of the most classical objects in the Group Theory. Going from groups to semigroups there are several ways to extend $\mathbb{S}_n$ to a bigger semigroup. The most straightforward one is to consider the set of all (non necessarily one-to-one) maps. This gives us the full transformation semigroup $\mathcal{T}_n$. However, it was first noted by Wagner in [Wa] that, using the notion of a *partially defined map*, one gets a more symmetric object. Following Wagner, the full symmetric inverse semigroup $\mathcal{IS}_n$ is defined as the set of all partial one-to-one maps on an n-element set. Combining these two approaches together, i.e. considering all (not necessarily one-to-one) partial maps, we get the semigroup $\mathcal{PT}_n$ of all partial transformations.

Although $\mathcal{IS}_n$ was defined in 1952, the first systematic monograph about it appeared only in 1996 ([LI]) and so far it still remains the only available source one can read about the definition and basic properties of $\mathcal{IS}_n$ in a detailed systematic way. In its turn this monograph contains only some basic facts about $\mathcal{IS}_n$ (and also $\mathcal{T}_n$ and $\mathcal{PT}_n$) unified by a suitable notation for representing the elements of $\mathcal{IS}_n$. This does not go much farther than presentation of $\mathcal{IS}_n$, canonical form of the elements, description of the conjugacy classes and centralizers and the definition of the alternating semigroups.

The primary purpose of this paper was to write a more extended monograph to include variety of combinatorial results about the structure of $\mathcal{IS}_n$, obtained in the last decade. In particular, we will describe automorphisms and endomorphisms of $\mathcal{IS}_n$ and several classes of subsemigroups, namely isolated, completely isolated and nilpotent subsemigroups. The study of the last ones will lead us to several interesting combinatorial results, when we try to calculate the cardinals of maximal nilpotent subsemigroups of $\mathcal{IS}_n$.

2 Notation

We will denote by $\mathbb{N}$ the set of positive integers. For $n \in \mathbb{N}$ we denote by $\mathcal{N}_n$ the set $\{1, 2, \dots, n\}$.

Let S be a set. A *partial transformation* of S is a map, $f : A \to S$, where $A \subset S$. The subset A is called the *domain* of f and is denoted by $\mathrm{dom}(f)$. The set $f(A) = \{f(x) | x \in A\}$ is called the *range* of f and is denoted by $\mathrm{ran}(f)$. A partial transformation, f, is called a *partial bijection* if $f(x) \neq f(y)$ for all $x \neq y \in \mathrm{dom}(f)$. Any partial bijection f is in fact a usual bijection between $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$. For a partial bijection, f, on S by f^{-1} we will mean the partial bijection defined as follows: $\mathrm{dom}(f^{-1}) = \mathrm{ran}(f)$ and for $y \in \mathrm{ran}(f)$ the value $f^{-1}(y)$ equals the (unique) element $x \in \mathrm{dom}(f)$ such that $f(x) = y$. For a partial bijection we obviously have $|\,\mathrm{ran}(f)| = |\,\mathrm{dom}(f)|$ and this common value is called the *rank* of f and is denoted by $\mathrm{rank}(f)$.

Among all partial transformations we distinguish two very special ones. The first one is the element, whose domain is $\varnothing$. We will denote this transformation by $\mathbf{0}$. By definition, $\mathbf{0}$ is a partial bijection. Another important element is the identity map $\mathbf{1} : S \to S$, defined by $\mathbf{1}(x) = x$ for any $x \in S$. This is a bijection (and hence a partial bijection) on S.

Let f be a partial transformation of S and $A \subset S$. We set $f(A) = \{f(x) | x \in A \cap \mathrm{dom}(f)\}$, in particular, $f(A) = \varnothing$ if $A \cap \mathrm{dom}(f) = \varnothing$. We also set $f^{-1}(A) = \{x \in \mathrm{dom}(f) | f(x) \in A\}$.

If S is a set, $\mathbb{S}(S)$ will denote the *symmetric group* on S, i.e. the group of all bijections on S under the operation of composition of maps. We set $\mathbb{S}_n = \mathbb{S}(\mathcal{N}_n)$ and will denote by $\mathbb{A}_n$ the subgroup of even permutations.

If n is a non-negative integer, we denote by $n!$ the product $1 \cdot 2 \cdot \dots \cdot n$ $(0! := 1)$. For $0 \leq k \leq n$, $k \in \mathbb{Z}$, by $\binom{n}{k}$ we denote the binomial coefficient $\frac{n!}{k!(n-k)!}$. By B_n we denote the n-th Bell number, which is the number of (unordered) decompositions of an n-element set into subsets.

3 Basic notions from the abstract theory of semigroups

Recall that a *semigroup* is a set, S, together with an associative binary operation, say $* : S \times S \to S$. The associativity means $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$. Such semigroup is denoted by $(S, *)$ or simply by S if the operation is clear from the context. If $x, y \in S$ and $x = y * z$ or $x = z * y$ for some $z \in S$ we will write $x \leq y$ and call $\leq$ the *divisability order* on S. Clearly $\leq$ is a partial pre-order.

Let $(S,*)$ be a semigroup. An element, $x \in S$, is called the *identity* element provided $x*s = s*x = s$ for every $s \in S$. Not every semigroup has an identity element, but if so, such element is unique. If one wants to emphasize that S has an identity one says that S is a *monoid.* An element, $x \in S$, is called the *zero* element provided $x*s = s*x = x$ for every $s \in S$. Not every semigroup has a zero element, but if so, such element is unique. An element, $x \in S$, is called a *left zero* (resp. *right zero*), if $x*s = x$ (resp. $s*x = x$) for any $s \in S$.

The difference between semigroups and monoids is seen in the definition of homomorphisms. If $(S_1,*_1)$ and $(S_2,*_2)$ are two semigroups, a *homomorphism of semigroups* is a map, $f: S_1 \to S_2$, such that $f(x *_1 y) = f(x) *_2 f(y)$ for all $x,y \in S_1$. While if $(S_1,*_1)$ and $(S_2,*_2)$ are two monoids with identities $\mathbf{1}_1$ and $\mathbf{1}_2$ respectively, a *homomorphism of monoids* is a map, $f: S_1 \to S_2$, such that $f(x *_1 y) = f(x) *_2 f(y)$ for all $x,y \in S_1$ and $f(\mathbf{1}_1) = \mathbf{1}_2$. Hence not any homomorphism $f: S_1 \to S_2$ as semigroups will be a homomorphism of monoids.

An element, $x \in S$, is called *idempotent* if $x*x = x$. The set of all idempotents of S is denoted by $E(S)$.

Let $x \in S$. The element $y \in S$ is called an *inverse* element for x provided $x*y*x = x$. The semigroup S is called *regular* if there exists an inverse element for any $x \in S$ and *inverse* if for any $x \in S$ there exist the unique $y \in S$ such that x is inverse for y and y is inverse for x, in other words if each $x \in S$ uniquely defines a pair of inverse to each other elements. If $(S,*)$ is inverse, for each element $x \in S$ the corresponding unique element y, forming with x the pair described above, is denoted by $x^{\#}$. One has $(x^{\#})^{\#} = x$ and $(x*y)^{\#} = y^{\#} * x^{\#}$. The last equality is not straightforward. Its proof uses the characterization of inverse semigroups as regular semigroups with commuting idempotents. If $(S,*)$ is an inverse semigroup then there is a canonical partial order on S defined as follows: $x\omega y$ if and only if $x^{\#} * x = y^{\#} * x$ if and only if $x * x^{\#} = y * x^{\#}$. For inverse S the set $E(S)$ with respect to both ω and $\leq$ is a lower semi-lattice.

Assume that $(S,*)$ has the identity element 1. The *Green's relations* $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ on $(S,*)$ are defined as binary relations in the following way: $x\mathcal{L}y$ if and only if $S*x = S*y$; $x\mathcal{R}y$ if and only if $x*S = y*S$; $x\mathcal{J}y$ if and only if $S*x*S = S*y*S$ for any $x,y \in S$ and $\mathcal{H} = \mathcal{L} \bigwedge \mathcal{R}$, $\mathcal{D} = \mathcal{L} \bigvee \mathcal{R}$. If S is *periodic* (i.e. each element generates a finite subsemigroup) then necessarily $\mathcal{J} = \mathcal{D}$. Further, $x\mathcal{L}y$ if and only if there exist $u,v \in S$ such that $x = u*y$ and $y = v*x$; $x\mathcal{R}y$ if and only if there exist $u,v \in S$ such that $x = y*u$ and $y = x*v$; $x\mathcal{D}y$ if and only if there exists $z \in S$ such that $x\mathcal{L}z$ and $z\mathcal{R}y$.

A subset, I, of a semigroup, $(S,*)$, is called a *left ideal* (resp. *right ideal*) provided $S*I \subset I$ (resp. $I*S \subset I$). A subset, which is both left and right ideal is called a *two-sided ideal* or simply *ideal.*

If $x \in S$ for $n \in \mathbb{N}$ we define recursively $x^1 = x$ and $x^{n+1} = x^n * x$. If $(S,*)$ has the identity element 1 we set $x^0 = 1$.

Assume that $(S,*)$ has the zero element 0. An element, $x \in S$, is called *nilpotent* of *nilpotency degree* $k \in \mathbb{Z}_+$ provided $x^k = 0$ and $x^{k-1} \neq 0$. The semigroup $(S,*)$ is called *nilpotent* of *nilpotency degree* $k \in \mathbb{Z}_+$ provided $x_1 * x_2 * \cdots * x_k = 0$ for all $x_1,\ldots,x_k \in S$ and there exists $y_1,\ldots,y_{k-1} \in S$ such that $y_1 * y_2 * \cdots * y_{k-1} \neq 0$. If S is finite, its nilpotency is equivalent to the nilpotency of all elements.

If $x \in S$ then the *centralizer* of x in S is the set $C_S(x) = C(x) = \{y \in S | xy = yx\}$. The *center* of S is the set $Z(S) = \cap_{x \in S} C(x)$. $Z(S)$ can be empty, but if it is non-empty, it is a commutative (abelian) subsemigroup of S.

If $x \in S$, by $\langle x \rangle$ we will denote the subsemigroup of S, generated by x, i.e. $\langle x \rangle = \{x^k | k \in \mathbb{N}\}$. This semigroup coincides with the minimal subsemigroup of S, containing x, or, in other words, with the intersection of all subsemigroups of S, containing x. The number of elements in $\langle x \rangle$ is called the *order* of x and is denoted by $\mathfrak{o}(x)$. If S is an inverse semigroup, by $\langle x \rangle_{inv}$ we will denote the minimal inverse subsemigroup of S, containing x, i.e. the intersection of all inverse subsemigroups of S, containing x. The number of elements in $\langle x \rangle_{inv}$ is called the *inverse order* of x and is denoted by $\mathfrak{o}_{inv}(x)$.

Assume that S is finite. Then so is $\langle x \rangle$ for any $x \in S$. Hence $x^r = x^s$ for some minimal positive integers $r < s$. In this case r is called the *index* of x and $m = s - r$ is called the *period* of x. The index and the period of x are denoted $\mathfrak{i}(x)$ and $\mathfrak{p}(x)$ respectively. Obviously $\mathfrak{i}(x) + \mathfrak{p}(x) = \mathfrak{o}(x) + 1$ and the semigroup $K_x = \{x^r, x^{r+1}, \dots, x^{s-1}\}$ is the cyclic group of order m.

Let $(S, *)$ be a semigroup with 1 and G be its maximal subgroup of invertible elements. The elements $x, y \in S$ are said to be *G-conjugated* provided there exists $g \in G$ such that $x = g^{-1}yg$. This will be denoted $x \sim_G y$. The binary relation $\sim_G$ is an equivalence relation on S.

The elements $x, y \in S$ are said to be *primary S-conjugated* provided there exists $u, v \in S$ such that $x = u * v$ and $y = v * u$. The binary relation $\sim_{pS}$ of primal S-conjugation is reflexive and symmetric, but not transitive in general. We denote by $\sim_S$ the transitive closure of this relation, i.e. the minimal transitive relation containing $\sim_{pS}$. If $x \sim_S y$, the elements x and y will be called *S-conjugated*. The relation $\sim_S$ is an equivalence relation on S. Both $\sim_G$ and $\sim_B$ generalize the notion of conjugated elements in a group and hence coincide if S is a group. In the general case there is only the following inclusion: $\sim_G \subset \sim_S$.

Let $(S, *)$ be a semigroup. An equivalence relation, $\sim$, on S is called *left stable* (resp. *right stable*) if $x \sim y$ implies $s * x \sim s * y$ (resp. $x * s \sim y * s$) for any $x, y, s \in S$. A relation, which is both left and right stable is called *stable* or *congruence* on S. If $\sim$ is a congruence, then $*$ induces an associative operation on $S/\sim$ via $\overline{a} * \overline{b} = \overline{a * b}$, where $\overline{x}$ denotes the equivalence class of $x \in S$. Moreover, the canonical map $(S, *) \to (S/\sim, *)$ is a homomorphism. Conversely, if $\varphi : (S, *) \to (T, \star)$ is a homomorphism of semigroups, then the relation $x \sim y$ if and only if $\varphi(x) = \varphi(y)$ is a congruence on S. Hence it is usually said that congruences are "kernels" of homomorphisms of semigroups.

If $(S, *)$ is a semigroup and T is a subsemigroup of S, then T is called *completely isolated* in S if $a * b \in T$ implies $a \in T$ or $b \in T$. T is called *isolated* if $a^k \in T$ implies $a \in T$. Obviously, any completely isolated subsemigroup is isolated. Converse is not true in general.

4 Definition of $\mathcal{IS}_n$ and cyclic decomposition of elements

For a set, S, let $\mathcal{IS}(S)$ denote the set of all partial bijections on S. If $f, g \in S$ then the map $f \circ g : \mathrm{dom}(g) \cap g^{-1}(\mathrm{dom}(f)) \to S$, defined by $f \circ g(x) = f(g(x))$, $x \in \mathrm{dom}(g) \cap g^{-1}(\mathrm{dom}(f))$, is a partial transformation. We note that, by definition, $\mathrm{dom}(f\circ g) \subset \mathrm{dom}(g)$ and $\mathrm{ran}(f \circ g) \subset \mathrm{ran}(f)$. As g is injective on $\mathrm{dom}(g) \cap g^{-1}(\mathrm{dom}(f))$ and f is injective on $\mathrm{ran}(g) \cap \mathrm{dom}(f)$, $f \circ g$ is injective on $\mathrm{dom}(g) \cap g^{-1}(\mathrm{dom}(f))$ and hence is a partial bijection. Further, if $f, g, h \in \mathcal{IS}(S)$ the partial transformation $f \circ (g \circ h)$ is defined on $\mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g)) \cap (g \circ h)^{-1}(\mathrm{dom}(f))$ and the partial transformation $(f \circ g) \circ h$ is defined on $\mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g) \cap g^{-1}(\mathrm{dom}(f))) = \mathrm{dom}(h) \cap h^{-1}(\mathrm{dom}(g)) \cap (g \circ h)^{-1}(\mathrm{dom}(f)) = A$. Moreover, for $x \in A$ we have $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x)$ because of the associativity of the composition of usual maps. This means that the operation $\circ$ defined on $\mathcal{IS}(S)$ above is associative and hence $(\mathcal{IS}(S), \circ)$ is a semigroup. This semigroup is called the *full inverse symmetric semigroup* on S and is usually denoted simply by $\mathcal{IS}(S)$. The operation $\circ$ is certainly called the *composition* of partial bijections. If $S = \mathcal{N}_n$, $\mathcal{IS}(\mathcal{N}_n)$ is called the *full inverse symmetric semigroup* of rank n and is denoted by $\mathcal{IS}_n$. The last semigroup will be the main object of our interest in this paper.

The element $\mathbf{0}$ is the zero element in $\mathcal{IS}_n$ since $\mathrm{dom}(\mathbf{0} \circ f) = \mathrm{dom}(f \circ \mathbf{0}) = \varnothing$ and hence $\mathbf{0} \circ f = f \circ \mathbf{0} = \mathbf{0}$ for any $f \in \mathcal{IS}_n$. The identity transformation $\mathbf{1} \in \mathcal{IS}_n$ satisfies $\mathbf{1} \circ f = f \circ \mathbf{1} = f$ for any $f \in \mathcal{IS}_n$ and therefore is the identity element of $\mathcal{IS}_n$.

One of the main conditions to be able to work with a semigroup is to have a convenient way to write down the elements of this semigroup and to perform the operation for the given two elements. Recall that for S_n there are two standard ways to write down the elements: the first one is to present them as permutations

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

This way is very comfortable for multiplying elements. The second one is to present them as products of commuting cycles, e.g $(124)(36)(5) \in S_6$ (the so-called cyclic decomposition). Both these ways have natural analogs for $\mathcal{IS}_n$.

First we describe the permutation form of partial bijections. Let $f \in \mathcal{IS}_n$. We associate with f the following *partial permutation* of $\mathcal{N}_n$:

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix},$$

where $a_i = f(i)$ if $i \in \mathrm{dom}(f)$ and $a_i = \varnothing$ otherwise. Thus the symbol $\varnothing$ in the lower row means that the partial permutation is not defined on the element from $\mathcal{N}_n$ standing above. Certainly, one can choose any symbol to illustrate this phenomena (e.g. $-$ is chosen in [LI]). Our choice is based on the equality $f(\{i\}) = \varnothing$, $i \notin \mathrm{dom}(f)$.

As an example, we can now list all the elements of $\mathcal{IS}_2$:

$$\left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & \varnothing \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & \varnothing \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \varnothing & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \varnothing & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \varnothing & \varnothing \end{pmatrix} \right\}.$$

The multiplication of elements from $\mathcal{IS}_n$ in this notation can be done in the same way as for S_n. We want only to emphasize that in the paper we will multiply the elements from right to left, as it is usually done for transformations. For example

$$\left(\begin{array}{cccc}1&2&3&4\\2&\varnothing&\varnothing&3\end{array}\right)\circ\left(\begin{array}{cccc}1&2&3&4\\\varnothing&2&4&1\end{array}\right)=\left(\begin{array}{cccc}1&2&3&4\\\varnothing&\varnothing&3&2\end{array}\right).$$

Now we want to introduced an analogue of cyclic decomposition for elements from $\mathcal{IS}_n$. For this we have to introduce two classes of elements first. Let $A=\{x_1,x_2,\dots,x_k\}\subset\mathcal{N}_n$ be an ordered subset. Denote by $(x_1,\dots,x_k)$ the unique element $f\in\mathcal{IS}_n$ such that $f(x_i)=x_{i+1}$, $i=1,2,\dots,k-1$, $f(x_k)=x_1$, and $f(x)=x$, $x\notin A$. Clearly $(x_1,\dots,x_k)\in S_n$. We will call $(x_1,\dots,x_k)$ a *cycle* of length k. The set A will be called the *support* of $(x_1,\dots,x_k)$. Now assume that $A\neq\varnothing$ and denote by $[x_1,\dots,x_k]$ the unique element $f\in\mathcal{IS}_n$ such that $f(x_i)=x_{i+1}$, $i=1,2,\dots,k-1$, $f(x_k)=\varnothing$, and $f(x)=x$, $x\notin A$. Obviously $[x_1,\dots,x_k]\in\mathcal{IS}_n\setminus S_n$. We will call $[x_1,\dots,x_k]$ a *chain* of length k and A the *support* of $[x_1,\dots,x_k]$. We note that $(x_1,\dots,x_k)=(x_2,\dots,x_k,x_1)$ but $[x_1,\dots,x_k]\neq[x_2,\dots,x_k,x_1]$ if $k>1$.

It is convenient to have a common name for both cycles and chains, so an element, $f\in\mathcal{IS}_n$, which is either cycle or chain will be called a *connected* element. We start with the following trivial observation.

Lemma 4.1. *Let f and g be connected elements of $\mathcal{IS}_n$ with supports A and B correspondingly. If $A\cap B=\varnothing$ then $f\circ g=g\circ f$.*

Proof. This is true because f and g are transformations which actually act on disjoint sets. $\square$

Theorem 4.1. *Any element of $\mathcal{IS}_n$ decomposes into a product of connected elements with pairwise disjoint supports. This product is unique up to a permutation of factors provided the union of all supports in the product equals $\mathcal{N}_n$.*

Proof. Fix an element, $f\in\mathcal{IS}_n$. We associate with f a finite oriented graph, Γ_f, the so-called *graph of the action of f on $\mathcal{N}_n$*. The set of vertices of Γ_f is $\mathcal{N}_n$. For $x,y\in\mathcal{N}_n$ we draw an arrow from x to y if and only if $f(x)=y$. Consider the connected components of Γ_f. Let A be one of them. As f is a partial transformation, for any $x\in A$ there exists at most one arrow starting in x. As f is injective, for any $x\in A$ there exists at most one arrow terminating in x.

There are two possibilities. The first one is that for any $x\in A$ there is an arrow terminating in x. This means that $f^{-1}(x)$ is defined for any $x\in A$. So, take any $x_0\in A$ and consider the set $\{x_i=f^{-i}(x)|i\in\mathbb{Z}_+\}$. This set is finite, hence there exist $i,j\in\mathbb{Z}_+$, $i<j$, such that $f^{-i}(x)=f^{-j}(x)$. Applying f^j we get $f^{j-i}(x)=x$. Set $y_i=f^{i-1}(x)$, $i=1,\dots,j-i$, and we will have $f(y_i)=y_{i+1}$, $i=1,\dots,j-i-1$, and $f(y_{j-i})=y_1$. In particular, $\{y_1,\dots,y_{j-i}\}$ is a connected component of Γ_f, hence coincides with A. Denote by f_A the cycle $(y_1,\dots,y_{j-i})$

The second possibility is that there exists $x \in A$ such that there are no arrows terminating in x. Consider the set $B = \{y_1 = x\} \cup \{y_{i+1} = f^i(x) | i \in \mathbb{N} \text{ and } f^{i-1}(x) \in \mathrm{dom}(f)\}$. This set is finite, hence either there exist $i, j \in \mathbb{N}$, $i < j$ such that $y_i = y_j$ or there exists $i \in \mathbb{Z}_+$ such that $y_i \notin \mathrm{dom}(f)$. The first possibility is impossible by injectivity of f and hence there is some $i \in \mathbb{N}$ such that $y_i \notin \mathrm{dom}(f)$. As in the first case, as B is a connected component of Γ_f, we get $B = A$. Denote by f_A the chain $[y_1, \dots, y_i]$.

If $A \neq B$ are two connected components we obviously have $A \cap B = \varnothing$ and hence $f_A f_B = f_B f_A$ by Lemma 4.1. From $\cup_A A = \mathcal{N}_n$ we also have $\prod_A f_A = f$ (in both cases A runs through the set of all connected components of Γ_f). This gives us the desired decomposition.

The uniqueness statement is trivial and left to the reader. $\square$

Each (unique only up to permutations of factors) decomposition, given by Theorem 4.1 will be called the *chain decomposition* of the elements of $\mathcal{IS}_n$. From now on, if nothing is stated, the notation

$$f = (a_1, \dots, a_k) \dots (b_1, \dots, b_l)[c_1, \dots, c_m] \dots [d_1, \dots, d_s] \tag{1}$$

means a chain decomposition of $f \in \mathcal{IS}_n$. For f let l_i (resp. m_i) be the number of cycles (resp. chains) of length i in the chain decomposition. We will write $ct(f) = (l_1, \dots, l_n)$ and $cht(f) = (m_1, \dots, m_n)$ for the *cyclic type* and the *chain type* of f respectively. We remark that sometimes the cycles of length 1 are omitted in the decomposition (1).

Theorem 4.1 justifies the name "connected" for cycles and chains, as these elements come from the connected components of the graph of action. As an illustration of the above theorem we give a chain decomposition of all elements from $\mathcal{IS}_2$ (omitting cycles of length 1):

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \mathbf{1}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (1,2), \begin{pmatrix} 1 & 2 \\ 1 & \varnothing \end{pmatrix} = [2], \begin{pmatrix} 1 & 2 \\ 2 & \varnothing \end{pmatrix} = [12],$$
$$\begin{pmatrix} 1 & 2 \\ \varnothing & 2 \end{pmatrix} = [1], \begin{pmatrix} 1 & 2 \\ \varnothing & 1 \end{pmatrix} = [21], \begin{pmatrix} 1 & 2 \\ \varnothing & \varnothing \end{pmatrix} = [1][2].$$

5 Idempotents and maximal subgroups in $\mathcal{IS}_n$

We proceed with the description of the idempotents of $\mathcal{IS}_n$. If $A \subset \mathcal{N}_n$ we denote by ε_A the element of $\mathcal{IS}_n$ defined as follows: $\varepsilon_A(x) = x$, $x \in A$, and $\varepsilon_A(x) = \varnothing$, $x \notin A$. In particular, $\varepsilon_{\mathcal{N}_n} = \mathbf{1}$ and $\varepsilon_\varnothing = \mathbf{0}$. Clearly $\varepsilon_A \circ \varepsilon_A = \varepsilon_A$ and hence ε_A is an idempotent of $\mathcal{IS}_n$. It is not difficult to find out that ε_A, $A \subset \mathcal{N}_n$, exhaust all idempotents of $\mathcal{IS}_n$. For $x \in \mathcal{N}_n$ we set $\varepsilon(x) = \varepsilon_{\mathcal{N}_n \setminus \{x\}}$.

Lemma 5.1. *If $f \in \mathcal{IS}_n$ such that $f \circ f = f$ then $f = \varepsilon_{\mathrm{dom}(f)}$.*

Proof. Let $x \in \mathrm{dom}(f)$ and $f(x) = y$. From $f \circ f = f$ we get $f(f(x)) = f(x) = y$, in particular, $f(x) = y \in \mathrm{dom}(f)$ and $f(y) = f(x)$. As f is a partial bijection, we have $x = y$, which implies $f = \varepsilon_{dom(f)}$. $\square$

Lemma 5.2. *Let $A, B \subset \mathcal{N}_n$. Then $\varepsilon_A \circ \varepsilon_B = \varepsilon_{A\cap B}$. In particular, $E(S)$ is a commutative subsemigroup of $\mathcal{IS}_n$.*

Proof. If $x \in A \cap B$ we have $\varepsilon_A \circ \varepsilon_B(x) = \varepsilon_A(\varepsilon_B(x)) = \varepsilon_A(x) = x$. If $x \in B \setminus A$ we have $\varepsilon_A \circ \varepsilon_B(x) = \varepsilon_A(\varepsilon_B(x)) = \varepsilon_A(x) = \varnothing$ and if $x \notin B$ we have $\varepsilon_A \circ \varepsilon_B(x) = \varepsilon_A(\varepsilon_B(x)) = \varepsilon_A(\varnothing) = \varnothing$. Thus $\varepsilon_A \circ \varepsilon_B = \varepsilon_{A\cap B}$ and the lemma follows. $\square$

In each semigroup there is a natural bijection between idempotents and maximal subgroups. The above description of idempotents makes the description of maximal subgroups of $\mathcal{IS}_n$ quite easy.

Lemma 5.3. *The maximal subgroup $G(A)$ of $\mathcal{IS}_n$, which corresponds to ε_A is $\mathbb{S}(A)$. In particular, the group $\mathbb{S}_n$ of invertible elements is the unique maximal subgroup of $\mathcal{IS}_n$ of cardinality $n!$.*

Proof. Clearly $\mathbb{S}(A) \subset G(A)$. If $f \in G(A)$ then from $\varepsilon_A \circ f \circ \varepsilon_A = f$ we get $\mathrm{dom}(f), \mathrm{ran}(f) \subset A$. But $f^k = \varepsilon_A$ for some $k \in \mathbb{N}$ implies that $\mathrm{dom}(\varepsilon_A) = A \subset \mathrm{dom}(f)$. Hence $\mathrm{dom}(f) = A$, thus $\mathrm{rank}(f) = |A|$ and $\mathrm{ran}(f) = A$. This means $f \in \mathbb{S}(A)$. $\square$

6 Centralizers of elements in $\mathcal{IS}_n$

Our next step in study of $\mathcal{IS}_n$ is to describe the centralizers of elements in this semigroup (compare with [LI, Chapters 3 and 4]). Assume that we have fixed an element, $f = (a_1, \dots, a_k) \dots (b_1, \dots, b_l)[c_1, \dots, c_m] \dots [d_1, \dots, d_s] \in \mathcal{IS}_n$. Here we require that for any $x \in \mathcal{N}_n$ such that $f(x) = x$ the cycle (x) appears once in the decomposition. Set $A = A(f) = \{a_1, \dots, a_k, \dots, b_1, \dots, b_l\}$ and $B = B(f) = \mathcal{N}_n \setminus A$.

A sequence, $x_1, \dots, x_t$, will be called a *chain* of f provided $f(x_i) = x_{i+1}$, $i = 1, \dots, t-1$, and $f(x_t) = \varnothing$. In general, $[x_1, \dots, x_t]$ does not necessary occur as a factor in the chain decomposition of f. But from the proof of Theorem 4.1 it follows that any *maximal chain*, i.e. which is not a proper subsequence of any other chain, does occur in this decomposition.

Lemma 6.1. 1. *If $f \in \mathcal{IS}_n$ such that $f \circ g = g \circ f$ then $g(A) \subset A$ and $g(B) \subset B$.*

2. *The centralizer $C(f')$ of the element $f' = (a_1, \dots, a_k) \dots (b_1, \dots, b_l)$ in $\mathcal{IS}(A)$ equals $C_1 \circ C$, where C_1 is the centralizer of f' in $\mathbb{S}(A)$ and $C = \langle \varepsilon_A, \varepsilon_{\{a_1,\dots,a_k\}}, \dots, \varepsilon_{\{b_1,\dots,b_l\}} \rangle$.*

3. *The centralizer $C(f'')$ of the element $f'' = [c_1, \dots, c_m] \dots [d_1, \dots, d_s]$ in $\mathcal{IS}(B)$ consists of all elements of the form*

$$\left[\begin{array}{cccccccccccc} c_1 & \dots & c_i & c_{i+1} & \dots & c_m & \dots & d_1 & \dots & d_j & d_{j+1} & \dots & d_s \\ c'_1 & \dots & c'_i & \varnothing & \dots & \varnothing & \dots & d'_1 & \dots & d'_j & \varnothing & \dots & \varnothing \end{array}\right],$$

where the sequences $c'_1, \dots, c'_i$; $\dots$; $d'_1, \dots, d'_j$ are chains of f'' (not necessarily maximal).

Proof. Take $x \in A$. Without loss of generality we can assume $x = a_1$. If $g(a_1) \neq \varnothing$, from $g(a_1) = g(f(a_k)) = f(g(a_k))$ we get $g(a_k) \neq \varnothing$ and hence by induction $g(a_i) \neq \varnothing$ for all $i = 1, \dots, k$. Moreover, $f(g(a_1)) = g(a_2)$, ... , $f(g(a_k)) = g(a_1)$. Hence $(g(a_1), \dots, g(a_k))$ is a cycle containing in the chain decomposition of f. This means $\{g(a_1), \dots, g(a_k)\} \subset A$.

Now assume $g(\{c_1, \dots, c_m\}) \neq \varnothing$ and let i be the maximal index such that $g(c_i) \neq \varnothing$. By the same arguments as above we get $g(c_{i-1}) \neq \varnothing$, ... , $g(c_1) \neq \varnothing$ and $f(g(c_1)) = g(c_2)$, ... , $f(g(c_{i-1})) = g(c_i)$. Moreover, from $\varnothing = g(c_{i+1}) = g(f(c_i)) = f(g(c_i))$ it follows that $\{g(c_1), \dots, g(c_i)\}$ is a chain of f.

Combining the two paragraphs above we get all three statements of the lemma. $\square$

Lemma 6.1 immediately implies the following description of $C(f)$.

Theorem 6.1. *Let f be as above. Then $C_{\mathcal{IS}_n}(f) = (C_1 \circ C) \oplus C_2$, where $C_1 \circ C$ is the centralizer of f' and C_2 is the centralizer of f'', described in Lemma 6.1.*

Proof. Follows from Lemma 6.1. $\square$

7 The structure of $\mathcal{IS}_n$ as an inverse semigroup

In this section we want to justify the word "inverse" in the name of $\mathcal{IS}_n$.

Proposition 7.1. *$\mathcal{IS}_n$ is an inverse semigroup and $f^{\#} = f^{-1}$ for all $f \in \mathcal{IS}_n$.*

Proof. Directly from definition of f^{-1} we get $f \circ f^{-1} \circ f = f$ and $f^{-1} \circ f \circ f^{-1} = f^{-1}$. Hence $\mathcal{IS}_n$ is regular and f, f^{-1} is a pair of inverse to each other elements in $\mathcal{IS}_n$. Assume that $g \in \mathcal{IS}_n$ such that $f \circ g \circ f = f$ and $g \circ f \circ g = g$. Let $x \in \mathrm{dom}(f)$. Then $f(x) \in \mathcal{N}_n$ and from the first equality we have $f(g(f(x))) \in \mathcal{N}_n$. In particular, $\mathrm{ran}(f) \subset \mathrm{dom}(g)$ and $\mathrm{rank}(f) \leq \mathrm{rank}(g)$. From the second equality we analogously get $\mathrm{ran}(g) \subset \mathrm{dom}(f)$ and $\mathrm{rank}(g) \leq \mathrm{rank}(f)$. Hence $\mathrm{rank}(f) = \mathrm{rank}(g)$ and thus $\mathrm{ran}(f) = \mathrm{dom}(g)$, $\mathrm{ran}(g) = \mathrm{dom}(f)$. So, we can consider $f : \mathrm{dom}(f) \to \mathrm{ran}(f)$ and $g : \mathrm{ran}(f) \to \mathrm{dom}(f)$. Multiplying now $f \circ g \circ f = f$ with f^{-1} from the left and from the right we get $g = f^{-1}$. This completes the proof. $\square$

It will be also very useful to know how to calculate f^{-1} from the known f. If f is given as a partial permutation,

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix},$$

then in $f^{-1} = f$ the rows should be replaced, leaving all $a_i \in \mathcal{N}_n$ and completing the upper row with the rest of elements, which will be mapped to $\varnothing$. For example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \varnothing & 2 & 1 & \varnothing & 3 \end{pmatrix}^{\#} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & \varnothing & \varnothing \end{pmatrix}.$$

In cyclic notation we have:

$$((a_1,\dots,a_k)\dots(b_1,\dots,b_l)[c_1,\dots,c_m]\dots[d_1,\dots,d_s])^{\#} = \\ (a_k,\dots,a_1)\dots(b_l,\dots,b_1)[c_m,\dots,c_1]\dots[d_s,\dots,d_1].$$

8 The (inverse) order of elements in $\mathcal{IS}_n$

Denote $P(n)=\max_{f\in\mathcal{IS}_n}\mathfrak{o}(f)$ and $PI(n)=\max_{f\in\mathcal{IS}_n}\mathfrak{o}_{inv}(f)$. The aim of this section is to obtain asymptotic formulae for $P(n)$ and $PI(n)$. We start with the description of $\mathfrak{i}(f)$ and $\mathfrak{p}(f)$ for $f\in\mathcal{IS}_n$, which we will need.

Lemma 8.1. *Let $f\in\mathcal{IS}_n$.*

1. *If $f\in\mathbb{S}_n$ then $\mathfrak{i}(f)=1$. Otherwise $\mathfrak{i}(f)$ equals the length of the longest chain in f.*
2. *$\mathfrak{p}(f)$ equals the least common multiple of the lengths of cycles, contained in the cyclic decomposition of f ($\mathfrak{p}(f)=1$ if this decomposition does not contain any cycles).*

Proof. For $f\in\mathbb{S}_n$ the first statement is obvious. Let

$$f=(a_1,\dots,a_k)\dots(b_1,\dots,b_l)[c_1,\dots,c_m]\dots[d_1,\dots,d_s]\in\mathcal{IS}_n,$$

where all cycles of length 1 appear ones, and N be the length of the longest chain of f. Set $A=\{a_1,\dots,a_k,\dots,b_1,\dots,b_l\}$ and $B=\mathcal{N}_n\setminus A$. Then for $k\geq N$ we have $f^k(x)=\varnothing$ for any $x\in B$ while for $m<N$ there exists an element, x, of the longest chain such that $f^m(x)\neq\varnothing$. Hence $f^k\neq f^m$ for any $k\geq N$ and $m<N$ implying $\mathfrak{i}(f)\geq N$. Let $\hat{N}$ denote the least common multiple of the lengths of cycles, containing in the cyclic decomposition of f ($\hat{N}=1$ if this decomposition does not contain any cycles). Then $f^{N\hat{N}}(f^N)^{\hat{N}}=\varepsilon_A$ and hence $f^{k+N\hat{N}}=f^k$ for all $k\geq N$. This means $\mathfrak{i}(f)\leq N$ and thus $\mathfrak{i}(f)=N$. This proves the first part. The second part is left to the reader. □

Lemma 8.2. *Let $f\in\mathcal{IS}_n$ then*

1. *$|\langle f\rangle|=\mathfrak{i}(f)+\mathfrak{p}(f)-1$;*
2. *$|\langle f\rangle_{inv}|\leq 2(\mathfrak{i}(f)+1)^3+\mathfrak{p}(f)$.*

Proof. The first statement is true for any element of any semigroup, see Section 3. To prove the second statement we have to construct $T=\langle f\rangle_{inv}$. Of course T contains f and $g=f^{-1}=f^{\#}$ and all possible products of f and g. Let $\hat{T}$ be a subsemigroup of $\mathcal{IS}_n$, generated by f and g. From $(a*b)^{\#}=b^{\#}*a^{\#}$ it follows that $\hat{T}$ is closed under $\#$ and hence coincides with T. So, our problem reduces to the question: how many different elements does $\hat{T}$ contain?

First of all we show that any $x\in\hat{T}$ can written as $a(i,j,k)=f^i\circ g^j\circ f^k$ or $b(i,j,k)=g^i\circ f^j\circ g^k$, $i,j,k\in\mathbb{Z}_+$. Indeed, this is enough to prove for $x(i,j)=f\circ g^i\circ f^j\circ g$, $i,j\in\mathbb{N}$.

If $i=1$ (resp. $j=1$) we have $x(i,j)=f^j\circ g$ (resp. $f\circ g^i$). If $i,j>1$, assume $i\le j$. We have $x(i,j)=(f\circ g)\circ(g^{i-1}\circ f^{i-1})\circ f^{j-i+1}\circ g$. Being idempotents of an inverse semigroup, the elements $f\circ g$ and $g^{i-1}\circ f^{i-1}$ commute and we get $x(i,j)=g^{i-1}\circ f^j\circ g$. The case $i>j$ can be treated analogously from the right side.

Let A and B be as in Lemma 8.1. Then $f(A)\subset A$, $f(B)\subset B$, $g(A)\subset A$ and $g(B)\subset B$ and hence $x(A)\subset A$ and $x(B)\subset B$ for any $x\in\hat{T}$. If one of i,j,k is greater than $\mathfrak{i}(f)$ we immediately get that $a(i,j,k)(B)=\varnothing$ and $b(i,j,k)(B)=\varnothing$ as this holds for some factor of $a(i,j,k)$, resp. $b(i,j,k)$. Hence, modulo f^i, $i>\mathfrak{i}(f)$, $\hat{T}$ contains only $a(i,j,k)$ and $b(i,j,k)$ for $i,j,k\le\mathfrak{i}(f)$. Therefore $|\hat{T}|$ is not greater than $2(\mathfrak{i}(f)+1)^3$ plus the number of different f^i, $i>\mathfrak{i}(f)$, which, as we know, is not greater than $\mathfrak{p}(f)$. We are done. □

Theorem 8.1. *$\ln(P(n))\sim\ln(PI(n))\sim\sqrt{n\ln(n)}$ for $n\to\infty$.*

Proof. Denote by $C(n)$ the maximal order of elements in $\mathbb{S}_n$. For any $f\in\mathcal{IS}_n$ we have obvious inequalities $\mathfrak{i}(f)\le n$ and $\mathfrak{p}(x)\le C(n)$. Hence, by Lemma 8.2,

$$C(n)\le P(n)\le PI(n)\le C(n)+2(\mathfrak{i}(f)+1)^3\le C(n)+2(n+1)^3.$$

Applying $\ln(x)$ and taking into account $\ln(x+y)<\ln(x)+\ln(y)$, $x,y>2$, we get $\ln(C(n))\le\ln(P(n))\le\ln(PI(n))\le\ln(C(n))+3\ln(n+1)+\ln(2)$. Using the known asymptotic for $C(n)$: $\ln(C(n))\sim\sqrt{n\ln(n)}$ for $n\to\infty$, see e.g. [La], we get the necessary statement. □

9 Conjugated elements

Proposition 9.1. *Let $f,g\in\mathcal{IS}_n$. Then $f\sim_{\mathbb{S}_n}g$ if and only if $ct(f)=ct(g)$ and $cht(f)=cht(g)$ if and only if Γ_f and Γ_g are isomorphic.*

Proof. If $f=h^{-1}\circ g\circ h$, $h\in\mathbb{S}_n$ and

$$f=(a_1,\dots,a_k)\dots(b_1,\dots,b_l)[c_1,\dots,c_m]\dots[d_1,\dots,d_s]$$

is a chain decomposition of f, then

$$g=(h(a_1),\dots,h(a_k))\dots(h(b_1),\dots,h(b_l))[h(c_1),\dots,h(c_m)]\dots[h(d_1),\dots,h(d_s)]$$

is a chain decomposition of g. Conversely, if

$$f=(a_1,\dots,a_k)\dots(b_1,\dots,b_l)[c_1,\dots,c_m]\dots[d_1,\dots,d_s]$$

is a chain decomposition of f and

$$g=(a'_1,\dots,a'_k)\dots(b'_1,\dots,b'_l)[c'_1,\dots,c'_m]\dots[d'_1,\dots,d'_s]$$

the element

$$h=\begin{pmatrix} a_1 & \dots & a_k & \dots & b_1 & \dots & b_l & c_1 & \dots & c_m & \dots & d_1 & \dots & d_s\\ a'_1 & \dots & a'_k & \dots & b'_1 & \dots & b'_l & c'_1 & \dots & c'_m & \dots & d'_1 & \dots & d'_s\end{pmatrix}$$

gives $f=h^{-1}\circ g\circ h$. The equivalence of last two conditions is obvious. □

Theorem 9.1. *Let $f, g \in \mathcal{IS}_n$. Then $f \sim_{\mathcal{IS}_n} g$ if and only if $ct(f) = ct(g)$.*

Proof. First we prove that $ct(f) = ct(g)$ implies $f \sim_S g$. As the first step we note that $[a_1, a_2, \dots, a_k] \sim_{\mathcal{IS}_n} [a_1, a_2, \dots, a_{k-1}][a_k]$ as

$$[a_1, a_2, \dots, a_k] = [a_1, a_2, \dots, a_k] \circ [a_k], \quad [a_1, a_2, \dots, a_{k-1}][a_k] = [a_k] \circ [a_1, a_2, \dots, a_k].$$

If we now recall that for $f, g \in \mathbb{S}_n$ the relation $f \sim_{\mathcal{IS}_n} g$ holds if and only if $ct(f) = ct(g)$, the induction arguments give us that for the elements

$$f = (a_1, \dots, a_k) \dots (b_1, \dots, b_l)[c_1, \dots, c_m] \dots [d_1, \dots, d_s]$$

and

$$f' = (a_1, \dots, a_k) \dots (b_1, \dots, b_l)[c_1] \dots [c_m] \dots [d_1] \dots [d_s]$$

there holds $f \sim_{\mathcal{IS}_n} f'$. Now if $ct(f) = ct(g)$ we have $f \sim_{\mathcal{IS}_n} f'$ and $g \sim_{\mathcal{IS}_n} g'$. But $ct(g') = ct(f')$ and $cht(g') = cht(f')$. Hence $f' \sim_{\mathbb{S}_n} g'$ by Proposition 9.1. By transitivity of $\sim_{\mathcal{IS}_n}$ we finally get $f \sim_{\mathcal{IS}_n} g$.

To prove the necessity it is enough to consider $f, g \in \mathcal{IS}_n$ such that $f \sim_{p\mathcal{IS}_n} g$. We start with the statement that $f \sim_{p\mathcal{IS}_n} g$ implies $f^k \sim_{p\mathcal{IS}_n} g^k$ for any $k \in \mathbb{N}$. Indeed, if $f = v \circ u$ and $g = u \circ v$ then $f^k = v \circ ((u \circ v)^{k-1} \circ u)$ and $g^k = ((u \circ v)^{k-1} \circ u) \circ v$.

Hence, we can assume that all maximal chains of f and g have length one. So, $f = v \circ u$, $g = u \circ v$. Denote $A = \mathrm{dom}(f)$, $B = \mathrm{dom}(u) \setminus A$, $C = \mathrm{dom}(g)$, $D = \mathrm{dom}(v) \setminus C$, $E = \mathcal{N}_n \setminus \mathrm{dom}(v)$, $C_1 = u(A) \cap C$, $A_1 = u^{-1}(C_1)$, $D_1 = u(A) \cap D$, $A_2 = u^{-1}(D_1)$ and $C_2 = C \setminus C_1$.

As $v(u(B)) = f(B) = \varnothing$, we get $u(B) \subset E$. Moreover, from $A = f(A) = v(u(A))$ it follows that $u(A) \subset \mathrm{dom}(v) = C \cup D$, $u(A) = C_1 \cup D_1$ and $v(C_1 \cup D_1) = A$.

Consider now $v(C_2)$. As $C = C_1 \cup C_2$ and $g(C) = u(v(C))$, we have $v(C_2) \subset \mathrm{dom}(f)$ and from injectivity of v we get $A \cap v(C_2) = v(C_1 \cup D_1) \cap v(C_2) = v((C_1 \cup D_1) \cap C_2) = v(\varnothing)$.

Hence $v(C_2) \subset B$ and $u(v(C_2)) \subset u(B) \subset E$. But, from the other had, $u(v(C_2)) = g(C_2) \subset C$. As $E \cap C = \varnothing$, we get $u(v(C_2)) = \varnothing$. Therefore $v(C_2) = \varnothing$ and $C_2 = \varnothing$. So, we have

$$|\,\mathrm{dom}(g)| = |C_1 \cup C_2| = |C_1| \leq |C_1| + |D_1| = |A_1| + |A_2| = |\,\mathrm{dom}(f)|.$$

Switching f and g we also get $|\,\mathrm{dom}(f)| \leq |\,\mathrm{dom}(g)|$ hence $|\,\mathrm{dom}(f)| = |\,\mathrm{dom}(g)|$ and thus $D_1 = \varnothing$, $u(\mathrm{dom}(f)) = \mathrm{dom}(g)$ and $v(\mathrm{dom}(g)) = \mathrm{dom}(f)$. If we now fix a bijection between $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$ we reduce our question to the same question for the symmetric group $\mathbb{S}_{|\,\mathrm{dom}(f)|}$, where the answer is positive because of the known description of conjugacy classes. This completes the proof. $\square$

10 Green's relations

The analysis of Green's relations on a semigroup is one of the most basic questions. As $\mathcal{IS}_n$ is finite we automatically get $\mathcal{J} = \mathcal{D}$ and hence have to concentrate us only on the study of $\mathcal{L}$ and $\mathcal{R}$. Moreover, the antiinvolution $\#$ on $\mathcal{IS}_n$ transfers left ideals to right ideals and vise versa, so it will be enough to get a description only for, say $\mathcal{L}$.

Theorem 10.1. *Let $f, g \in \mathcal{IS}_n$. Then*

1. *$f\mathcal{L}g$ if and only if $\mathrm{dom}(f) = \mathrm{dom}(g)$;*

2. *$f\mathcal{R}g$ if and only if $\mathrm{ran}(f) = \mathrm{ran}(g)$;*

3. *$f\mathcal{H}g$ if and only if $\mathrm{ran}(f) = \mathrm{ran}(g)$ and $\mathrm{dom}(f) = \mathrm{dom}(g)$;*

4. *$f\mathcal{D}g$ if and only if $\mathrm{rank}(f) = \mathrm{rank}(g)$.*

5. *$f\mathcal{J}g$ if and only if $\mathrm{rank}(f) = \mathrm{rank}(g)$.*

Proof. Recall that $f\mathcal{L}g$ if and only if there exist $x, y \in \mathcal{IS}_n$ such that $f = x \circ g$ and $g = y \circ f$. In particular, from the first equality we get $\mathrm{dom}(f) \subset \mathrm{dom}(g)$ and from the second $\mathrm{dom}(g) \subset \mathrm{dom}(f)$. Hence $f\mathcal{L}g$ implies $\mathrm{dom}(f) = \mathrm{dom}(g)$. Conversely, if $A = \mathrm{dom}(f) = \mathrm{dom}(g)$ we write $A = \{x_1, \dots, x_k\}$ and consider any $\pi \in \mathbb{S}_n$ such that $\pi(f(x_i)) = g(x_i)$, $i = 1, \dots, k$. Such π exists as $|\,\mathrm{ran}(f)| = |\,\mathrm{dom}(f)| = |\,\mathrm{dom}(g)| = |\,\mathrm{ran}(g)|$. We have $g = \pi \circ f$ and $f = \pi^{-1} \circ g$. This implies $f\mathcal{L}g$ as mentioned above.

The antiinvolution $\#$ interchanges domains and ranges and hence the second statement follows from the first one after applying $\#$.

As $\mathcal{H} = \mathcal{L} \bigwedge \mathcal{R}$, the third statement follows from the first and the second ones.

Recall that $f\mathcal{D}g$ if and only if there exists $h \in \mathcal{IS}_n$ such that $f\mathcal{L}h$ and $h\mathcal{R}g$. From the first and the second statements we get $\mathrm{dom}(f) = \mathrm{dom}(h)$ and $\mathrm{ran}(h) = \mathrm{ran}(g)$. Hence $\mathrm{rank}(f) = |\,\mathrm{dom}(f)| = |\,\mathrm{dom}(h)| = |\,\mathrm{ran}(h)| = |\,\mathrm{ran}(g)| = \mathrm{rank}(g)$. Conversely, let $\mathrm{rank}(f) = \mathrm{rank}(g)$. Then $|\,\mathrm{dom}(f)| = \mathrm{rank}(f) = \mathrm{rank}(g) = |\,\mathrm{ran}(g)|$ and we can consider a partial bijection, $h : \mathrm{dom}(f) \to \mathrm{ran}(g)$. Again applying the first two statements we have $h\mathcal{L}f$ and $h\mathcal{R}g$ and hence $f\mathcal{D}g$.

The last statement is now trivial. □

Corollary 10.1. *Let $f, g \in \mathcal{IS}_n$. Then*

1. *$f\mathcal{L}g$ if and only if there exists $x \in \mathbb{S}_n$ such that $f = x \circ g$;*

2. *$f\mathcal{R}g$ if and only if there exists $x \in \mathbb{S}_n$ such that $f = g \circ x$;*

3. *$f\mathcal{D}g$ if and only if there exist $x, y \in \mathbb{S}_n$ such that $f = x \circ g \circ y$.*

Proof. Exercise. □

As a corollary we get a description of two-sided ideals of $\mathcal{IS}_n$.

Corollary 10.2. *$\mathcal{IS}_n$ has exactly $n+1$ ideals, $I_i = \{f \in \mathcal{IS}_n |\, \mathrm{rank}(f) \leq i\}$, $i = 0, 1, \dots, n$. Moreover, $I_i \subset I_{i+1}$ for all i and hence the poset of ideals is linear.*

Proof. It is enough to show that $T = \mathcal{IS}_n \circ f \circ \mathcal{IS}_n = I_{\mathrm{rank}(f)}$. By Theorem 10.1, T contains all element of $\mathcal{IS}_n$ of rank $\mathrm{rank}(f)$. If $\mathrm{rank}(f) = 0$ then $f = \mathbf{0}$ and $T = \{\mathbf{0}\}$. If $\mathrm{rank}(f) > 0$ then by induction in $t = \mathrm{rank}(f)$ it will be enough to show that T contains an element of rank $\mathrm{rank}(f) - 1$. Fix $x \in \mathrm{dom}(f)$, then $\mathrm{dom}(f \circ \varepsilon(x)) = \mathrm{dom}(f) \setminus \{x\}$ and hence $\mathrm{rank}(f \circ \varepsilon(x)) = \mathrm{rank}(f) - 1$. Certainly, $f \circ \varepsilon(x) \in T$. This completes the proof. □

11 Generators and presentation

Lemma 11.1. *$\mathcal{IS}_n$ is generated by $\mathbb{S}_n$ and any element of rank $n-1$.*

Proof. Let $f \in \mathcal{IS}_n$ such that $\mathrm{rank}(f) = n - 1$ and let T denote the semigroup of $\mathcal{IS}_n$, generated by $\mathbb{S}_n$ and f. We claim that T contains all elements of rank $n - 1$. Indeed, if $\mathrm{rank}(g) = n - 1$ then by Theorem 10.1 $g\mathcal{D}f$ and hence $g = x \circ f \circ y$ for some $x, y \in \mathbb{S}_n$ by Corollary 10.1. This implies $g \in T$. From the proof of Corollary 10.2 it now follows that T contains an element of rank $n - 2$. Applying the same arguments as above we complete the proof by induction. □

Theorem 11.1. *$\mathcal{IS}_n$ is isomorphic to the inverse monoid M, generated by the set of self-inverse generators $X = \{s_1, \dots, s_{n-1}, e\}$ subject to the relations*

$$s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = \mathbf{1}, \quad 1 \le i, j < n, \quad |i - j| > 1; \tag{2}$$

$$e^2 = e, (es_1)^2 = (es_1)^3, \quad \textit{and} \quad es_i = s_i e, \quad 1 < i < n. \tag{3}$$

Proof. Let $\varphi : X \to \mathcal{IS}_n$ be the map defined by $\varphi(s_i) = (i, i+1)$, $1 \le i < n$, and $\varphi(e) = \varepsilon_{\{1\}}$. The relations $(i, i+1)^2 = ((i, i+1)(i+1, i+2))^3 = ((i, i+1)(j, j+1))^2 = \mathbf{1}$ for $1 \le i, j < n$, $|i - j| > 1$ are Coxeter relations for $\mathbb{S}_n$. That $\varepsilon_{\{1\}}^2 = \varepsilon_{\{1\}}$ and $\varepsilon_{\{1\}} \circ (i, i+1) = (i, i+1) \circ \varepsilon_{\{1\}}$ is obvious. Finally, $x = \varepsilon_{\{1\}} \circ (1, 2) = [2, 1]$ and hence $x^2 = [1][2] = x^3$. This shows that φ is a homomorphism of monoids. Moreover, the image of φ contains $\mathbb{S}_n$ and $\varepsilon_{\{1\}}$ and hence coincides with $\mathcal{IS}_n$ by Lemma 11.1. Therefore, φ is an epimorphism and to complete the proof we have only to show that $|M| \le |\mathcal{IS}_n|$.

We start with introducing new element. Inductively define $e_i \in M$ as follows: $e_1 = e$ and $e_i = s_i e_{i-1} s_i$. It is straightforward that $e_i e_i = e_i$. But we want to obtain some additional relations:

First we claim that $(es_1)^2 = (s_1 e)^2 = (s_1 e)^3$. Indeed, as X is an inverse monoid, its idempotents commute and hence $s_1 e s_1 e = e_2 e = e e_2 = e s_1 e s_1$. The equality $(s_1 e)^2 = (s_1 e)^3$ follows from $(es_1)^2 = (es_1)^3$ by applying $\#$.

Next we claim that $s_i e_i = e_{i+1} s_i$, $1 \le i < n$; $e_i s_i = s_i e_{i+1}$, $1 \le i < n$; $e_i s_{i+1} = s_{i+1} e_i$, $1 \le i < n$; $s_i e_j = e_j s_i$, $|i - j| > 1$. Indeed, the first follows from $s_i e_i = s_i e_i s_i s_i = e_{i+1} s_i$ and the second one is similar. If we write $e_i = s_{i-1} \dots s_1 e s_1 \dots s_{i-1}$, we see that s_{i+1} commutes with all multiplicands and thus s_{i+1} commutes with e_i. This proves the third relation and analogous arguments prove the last one for $j > i$. If $j < i$ we have $s_j e_i = s_j (s_{i-1} \dots s_1 e s_1 \dots s_{i-1}) = s_{i-1} \dots s_{j+2} s_j e_{j+2} s_{j+2} \dots s_{i-1}$ and hence it is sufficient to consider the case $i = j + 2$. In this case we have $s_j e_{j+2} = s_j s_{j+1} s_j e_j s_j s_{j+1} = s_{j+1} s_j s_{j+1} e_j s_j s_{j+1} = s_{j+1} s_j e_j s_{j+1} s_j s_{j+1} = s_{j+1} s_j e_j s_j s_{j+1} s_j = e_{j+2} e_j$ as desired.

Now it follows that in any $w \in M$ we can collect all e_i on the right-hand side or on the left-hand side and thus w can be written as $s_{i_1} \dots s_{i_q} e_{j_1} \dots e_{j_p}$ or as $e_{j'_1} \dots e_{j'_p} s_{i'_1} \dots s_{i'_q}$.

For $i > j$ set

$$s_{i,j+1} = s_{j+1,i} = s_i s_{i-1} \dots s_{j+1} s_j s_{j+1} \dots s_{i-1} s_i = s_j s_{j+1} \dots s_{i-1} s_i s_{i-1} \dots s_{j+1} s_j$$

(the last is a relation in $\mathbb{S}_n$) and $s_{i+1,i} = s_{i,i+1} = s_i$. It is straightforward that $s_{i,j} e_j s_{i,j} = e_i$. Finally we claim that $s_{i,j} e_i e_j = e_i e_j s_{i,j} = e_i e_j$. Indeed, the first equality easily follows from the above rule of moving e'_s on the right, so we will prove only the second one. Start with $s_1 e s_1 e = e s_1 e s_1$, which implies $s_1 e s_1 e s_1 = e s_1 e$; but $(s_1 e)^2 = (s_1 e)^3$ and hence $s_1 e s_1 e = (s_1 e s_1 e s_1) e = (e s_1 e) e = e s_1 e$. So, $e s_{1,k} e = e s_{k-1} \dots s_2 s_1 s_2 \dots s_{k-1} = s_{k-1} \dots s_2 (e s_1 e) s_2 \dots s_{k-1} = s_{k-1} \dots s_2 (s_1 e s_1 e) s_2 \dots s_{k-1} = e_k e = e e_k = e_k e_1 = e_1 e_k$. Now $e_1 e_2 s_{1,2} = e_2 e s_1 = s_1 e s_1 e s_1 = e s_1 e = s_1 e s_1 e = e_2 e_1$. Conjugating first with $s_{1,i}$ and then with $s_{2,j}$ we get the desired equality.

Let $T \subset \mathcal{N}_n$ and $e_T = \prod_{i \in T} e_i$. Let $\mathbb{S}_n / \mathbb{S}(T)$ denote the set of left cosets. Then for any $\pi \in \mathbb{S}_n / \mathbb{S}(T)$ and any $x, y \in \pi$ from the previous paragraph we derive $x e_T = y e_T$. Hence the number of elements from M, which can be presented in the form $w e_T$, $w \in \mathbb{S}_n$ is less or equal $|\mathbb{S}_n / \mathbb{S}(T)| = n!/(n - |T|)!$. So, $|M| \leq \sum_{i=0}^{n} \binom{n}{i} \frac{n!}{(n-i)!} = |\mathcal{IS}_n|$ (see facts in Section 23). This completes the proof. □

12 Isolated and completely isolated subsemigroups

Proposition 12.1. *The only completely isolated subsemigroups of $\mathcal{IS}_n$ are $\mathcal{IS}_n$, $\mathbb{S}_n$ and $\mathcal{IS}_n \setminus \mathbb{S}_n$.*

Proof. As for a finite semigroup the product of non-invertible elements is not invertible itself, the subsemigroups $\mathcal{IS}_n$, $\mathbb{S}_n$ and $\mathcal{IS}_n \setminus \mathbb{S}_n$ of $\mathcal{IS}_n$ are completely isolated. Conversely, let T be a completely isolated subsemigroup of $\mathcal{IS}_n$. Assume first that $T \cap \mathbb{S}_n \neq \varnothing$ and thus contains an element, say a. Then T contains $\mathbf{1}$ as a power of a and then it contains any $b \in \mathbb{S}_n$ because $b^k = \mathbf{1}$ for some k and T is isolated. Therefore we have either $T \cap \mathbb{S}_n = \varnothing$ or $\mathbb{S}_n \subset T$. Let $S = \mathcal{IS}_n \setminus \mathbb{S}_n$ and assume that $T \cap S \neq \varnothing$ and thus contains an element, say c. Then T contains a non-invertible idempotent, say e. As any non-invertible idempotent of $\mathcal{IS}_n$ is a product of $e(x)$'s, we get, using the complete isolatedness, that T contains some $\varepsilon(x)$, $x \in \mathcal{N}_n$. Take any $y \in \mathcal{N}_n$ and write $\varepsilon(x) = (x, y) \circ \varepsilon(y) \circ (x, y)$. If $\mathbb{S}_n \subset T$ we get that T contains $\varepsilon(y)$ as it is a semigroup. Otherwise it contains $\varepsilon(y)$ as it is completely isolated. This implies that T contains all non-invertible idempotents. Now, as some power of each $d \in S$ is a non-invertible idempotent, $S \subset T$ as T is completely isolated. We get either $S \subset T$ or $S \cap T = \varnothing$. Altogether, we have only three possibilities for T and they are precisely those, listed in the formulation. □

For $x \in \mathcal{N}_n$ let $S(x)$ denote the maximal subgroup of $\mathcal{IS}_n$, associated with idempotent $\varepsilon(x)$. In other words, $S(x)$ consists of all elements f from $\mathcal{IS}_n$ satisfying the following conditions: $\mathrm{rank}(f) = n - 1$, $f(\mathcal{N}_n \setminus \{x\}) = \mathcal{N}_n \setminus \{x\}$. Equivalently, $S(x)$ consists of all elements f such that $f^k = \varepsilon(x)$ for some k.

Lemma 12.1. *All $S(x)$ are isolated subsemigroups in $\mathcal{IS}_n$.*

Proof. Let $f^k \in S(x)$ for some k. As $\mathrm{rank}(f^k) = n-1$, we have $\mathrm{rank}(f) = n-1$ as well, moreover, from $\mathrm{dom}(f^k) \subset \mathrm{dom}(f)$ and $|\mathrm{dom}(f^k)| = |\mathrm{dom}(f)|$ we get $\mathrm{dom}(f^k) = \mathrm{dom}(f)$. As k is arbitrary, we can assume that f^k is an idempotent and hence $f^k = \varepsilon(x)$. From $\mathrm{ran}(f^k) \subset \mathrm{ran}(f)$ and $|\mathrm{ran}(f^k)| = |\mathrm{ran}(f)|$ we get $\mathrm{ran}(f^k) = \mathrm{ran}(f)$. This means that $\mathrm{dom}(f) = \mathrm{ran}(f) = \mathcal{N}_n \setminus \{x\}$ and the statement follows. □

Theorem 12.1. *The only isolated subsemigroups of $\mathcal{IS}_n$ are $\mathcal{IS}_n$, $\mathbb{S}_n$, $\mathcal{IS}_n \setminus \mathbb{S}_n$ and $S(x)$, $x \in \mathcal{N}_n$.*

Proof. Clearly all listed subsemigroups are isolated. So, let T be an isolated subsemigroup of $\mathcal{IS}_n$. As in the proof of Proposition 12.1 one gets that either $\mathbb{S}_n \subset T$ or $\mathbb{S}_n \cap T = \varnothing$.

First we consider the case $\mathbb{S}_n \subset T$. If $T \neq \mathbb{S}_n$ then there exists $a \in T \setminus \mathbb{S}_n$ and hence T contains a non-invertible idempotent, say ε_A, as a power of a. Write $\mathcal{N}_n \setminus A = \{x_1, \dots, x_k\}$ and consider $f = [x_1, \dots, x_k]$, $g = [x_k, \dots, x_1]$. As $f^k = g^k = \varepsilon_A$, we have $f, g \in T$ and hence $f \circ g = \varepsilon(x_k)$ belongs T as well. But $\mathbb{S}_n \subset T$ and thus T contains all $\varepsilon(x)$, therefore all non-invertible idempotents and, finally, all non-invertible elements, as T is isolated. In this case $T = \mathcal{IS}_n$.

Now assume that $\mathbb{S}_n \cap T = \varnothing$ and T contains a non-invertible element, say a. As in the previous paragraph we see, that T contains some $\varepsilon(x)$. If $\varepsilon(x)$ is the only idempotent of T, then T contains $S(x)$. Hence it coincides with $S(x)$ since any element outside $S(x)$ will come with other idempotent in T. The only case left is when T contains some other idempotent, which, as it is easy to see, is equivalent to the condition that T contains $\varepsilon(x) = [x]$ and $\varepsilon(y) = [y]$ for $x \neq y$. It will suffice to prove that T contains all $\varepsilon(z)$, $z \in \mathcal{N}_n$, because in this case it will again contain all non-invertible idempotents and hence coincide with $\mathcal{IS}_n \setminus \mathbb{S}_n$. Let $z \neq x, y$. Let $f = [x]$ and $g = (x, z)[y]$. $g^2 = [y]$ and hence $f, g \in T$. But $(f \circ g)^2 = [x][y][z] \in T$. Consider $f' = [x, y, z]$ and $g' = [z, y, x]$. We have $(f')^2 = (g')^2 = [x][y][z] \in T$. Hence $f', g' \in T$. This means that $g' \circ f' = \varepsilon(z) \in T$. This completes the proof. □

13 Description of maximal nilpotent subsemigroups of $\mathcal{IS}_n$

Let $S \subset \mathcal{IS}_n$ be a nilpotent semigroup. Then S has a zero element, say e, but there are no reasons why e should coincide with the global zero element $\mathbf{0}$. So, the study of nilpotent subsemigroups of $\mathcal{IS}_n$ naturally divides into two parts: the study of those, whose zero coincides with $\mathbf{0}$ and the study of all other. Our first step here is to show that the second problem reduces to the first one, however n can be changed during this reduction process.

Lemma 13.1. *Let S be a nilpotent subsemigroup of $\mathcal{IS}_n$ with zero element e_A, $A \subset \mathcal{N}_n$. Then $\{x \in \mathcal{N}_n | \pi(x) = x\} = A$ holds for any $\pi \in S$. In particular, S is a nilpotent subsemigroup of $\mathcal{IS}(\mathcal{N}_n \setminus A)$ with usual zero.*

Proof. Let $\pi \in S$, $x \in A$ and $\pi(x) = y$. Then $e_A \circ \pi(x) = e_A(x) = x = e_A(\pi(x)) = e_A(y)$. Hence $x = y$. From the other hand, if $x \notin A$ and $\pi(x) = x$ we get $\pi^l(x) = x$ for any $l \in \mathbb{N}$. But S is finite and e_A is the only idempotent of S. Hence $\pi^l = e_A$ for some l and we get $e_A(x) = x$, which contradicts the definition of e_A. The second statement follows from the first one. $\square$

From now on we assume that the zero element in all nilpotent semigroups we consider is $\mathbf{0}$. From the above proof it also follows that $\pi(x) \neq x$ for any element π from any nilpotent subsemigroup of $\mathcal{IS}_n$.

Our study of nilpotent subsemigroups will be based on the following construction, which connects nilpotent semigroups with partial orders on $\mathcal{N}_n$. Let $T \subset \mathcal{IS}_n$ be a nilpotent semigroup. Define the binary relation $\varphi(T) =<_T$ on $\mathcal{N}_n$: $x <_T y$ if and only if there exists $\pi \in T$ such that $\pi(y) = x$. As T is a subsemigroup, the relation $<_T$ is transitive. As $\pi(x) \neq x$ for any $\pi \in T$, $<_T$ is anti-reflexive. Combining last thing with the fact that T is a subsemigroup we also get that $<_T$ is any-symmetric and thus is a partial order on $\mathcal{N}_n$. Conversely, if $<$ is a partial order on $\mathcal{N}_n$, we can consider the set of elements $\psi(<) = Mon(<) = \{\pi \in \mathcal{IS}_n | \pi(x) < x \text{ for all } x \in \mathcal{N}_n\}$. This set is not empty as $\mathbf{0} \in Mon(<)$. Let $\pi, \tau \in T_<$ and $x \in \mathcal{N}_n$ such that $\pi(\tau(x)) \in \mathcal{N}_n$. Then $\pi(\tau(x)) < \tau(x) < x$ by transitivity of $<$ and hence $Mon(<)$ is a subsemigroup of $\mathcal{IS}_n$. The transitivity of $<$ also forces $\pi(x) \neq x$ for any $\pi \in Mon(<)$ and hence $Mon(<)$ has the unique idempotent, namely $\mathbf{0}$, which is a zero element of $Mon(<)$. This means that $Mon(<)$ is a nilpotent subsemigroup of $\mathcal{IS}_n$, whose zero coincides with $\mathbf{0}$.

A partial order, $<_1$, on a set, S, is said to be a *consolidation* of another partial order, $<_2$, if $x <_2 y$ implies $x <_1 y$. It is obvious that any non-linear order on S admits a non-trivial consolidation. For a partial order, $<$, on S call the *depth* of $<$ the maximal length of a chain $x_1 < x_2 < \dots < x_i$ in S and denote it by $d(<)$. It is also obvious that if a partial order, $<$, of depth k does not admit any non-trivial consolidation of depth k, then $<$ is a linearly ordered decomposition of S into k non-empty subsets.

Lemma 13.2. *1. Let $T_1 \subset T_2$ be two nilpotent subsemigroups of $\mathcal{IS}_n$. Then $<_{T_2}$ is a consolidation of $<_{T_1}$.*

2. Let $<_1$ and $<_2$ be two partial orders on $\mathcal{N}_n$ and $<_1$ be a consolidation of $<_2$. Then $Mon(<_2) \subset Mon(<_1)$. Moreover, if $<_1$ and $<_2$ are different, then $Mon(<_2) \neq Mon(<_1)$.

Proof. All statements but the last one are trivial. The last statement follows from the fact that $[i, j]\varepsilon_{\{i,j\}} \in \mathcal{IS}_n$. $\square$

Lemma 13.3. *1. Let T be a nilpotent subsemigroups of $\mathcal{IS}_n$. Then $T \subset \psi(\varphi(T))$.*

2. Let $<$ be a partial order on $\mathcal{N}_n$. Then $\varphi(\psi(<))$ equals $<$.

Proof. Obvious from the definition. $\square$

Corollary 13.1. *Let $<$ be a partial order on $\mathcal{N}_n$. If T is a nilpotent subsemigroups of $\mathcal{IS}_n$ such that $<_T$ equals $<$ then $T \subset Mon(<)$.*

Lemma 13.4. 1. *Let $<$ be a partial order on $\mathcal{N}_n$. Then the nilpotency degree of the semigroup $Mon(<)$ equals $d(<)$.*

2. *Let T be a nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree k. Then $d(<_T) = k$.*

Proof. To prove the first statement we take any $\pi = \pi_1 \circ \cdots \circ \pi_k \in Mon(<)$ and $x \in \mathcal{N}_n$. As $d(<) = k$, we get that $\pi(x)$ is undefined and thus $\pi = 0$. This means that the nilpotency degree of $Mon(<)$ is less or equal k. Let $x_1 < \dots x_k$ be a maximal chain. Then $\tau_i = [x_i, x_{i-1}] \in Mon(<)$ for all $1 < i \leq k$ and $\tau = \tau_1 \circ \dots \tau_{k-1}(x_k) = x_1$. Hence the nilpotency degree of $Mon(<)$ is greater than $k-1$.

To prove the second statement we consider $0 \neq \pi = \pi_1 \circ \cdots \circ \pi_{k-1} \in T$. Then there exists $x \in \mathcal{N}_n$ such that $\pi(x)$ is defined and we get $\pi(x) <_T \pi_2 \circ \dots \pi_{k-1}(x) <_T \cdots <_T x$. Thus $d(<_T) \geq k$. If $x_1 <_T \cdots <_T x_l$ and $\pi_i \in T$ are chosen such that $\pi_i(x_i) = x_{i-1}$, $1 < i \leq k$ then $\pi_1 \circ \cdots \circ \pi_l \neq 0$ and hence $l \leq k$. Hence $d(<_T) \leq k$. □

We have to note that $T \neq \psi(\varphi(T))$ in general. As an example consider $T = \langle [4321] \rangle \subset \mathcal{IS}_4$. Then $<_T$ is the natural order on $\{1,2,3,4\}$. Then $[43][21] \in Mon(<_T)$ but $[43][21] \notin T$. Now we are ready to prove the main result of this section.

Theorem 13.1. *The maps ψ and φ are mutually inverse bijections between the set of maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree k and linearly ordered decomposition of $\mathcal{N}_n$ into k non-empty subsets.*

Proof. Let T be a maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree k. By Lemma 13.4, φ sends any nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree k to a partial order on $\mathcal{N}_n$ of depth k and ψ does the opposite. But if T is maximal of nilpotency degree k, then, by Lemma 13.2, the order $<_T$ does not admit any non-trivial consolidation of depth k and hence defines a linearly ordered decomposition of $\mathcal{N}_n$ into k non-empty subsets. Conversely if $<$ is such order, then any nilpotent subsemigroup L of $\mathcal{IS}_n$, containing $Mon(<)$ defines a non-trivial consolidation $<_L$ of $<$. But $<$ is a maximal consolidation of depth k, hence $d(<_L) > k$ and the nilpotency degree of L is also greater than k. This completes the proof. □

Corollary 13.2. *The maps ψ and φ are mutually inverse bijections between the set of maximal nilpotent subsemigroups of $\mathcal{IS}_n$ and linear orders on $\mathcal{N}_n$. In particular, there exists precisely $n!$ maximal nilpotent subsemigroups in $\mathcal{IS}_n$ and they all have nilpotency degree n.*

Theorem 13.1 justifies the following notion. Let T be a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree k and $\rho = (M_1, \dots, M_k)$ be the ordered decomposition of $\mathcal{N}_n$ into non-empty subsets, which corresponds to $<_T$. The vector $(|M_1|, \dots, |M_k|) \in \mathbb{N}^k$ will be called the *type* of T and will be denoted by $\mathrm{type}(T)$. We will also use the same notation for $\mathrm{type}(\rho)$.

It is easy to see that the statement of Theorem 13.1 will remain true if one substitutes $\mathcal{IS}_n$ with any subsemigroup of $\mathcal{IS}_n$ containing the ideal I_1.

If we have a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree $k <$, it can be properly contained in nilpotent semigroups of bigger nilpotency degree. In fact, as we will see this is always the case. Knowing all maximal nilpotent subsemigroups of a fixed nilpotency degree it is natural to ask when a given maximal nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree k is contained in a given maximal nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree $> k$.

Theorem 13.2. *The maximal nilpotent subsemigroup T_1 in $\mathcal{IS}_n$ of nilpotency degree k, which corresponds to an ordered decomposition, $\rho_1 = (N_1^1, \dots, N_k^1)$, is contained in the maximal nilpotent subsemigroup T_2 in $\mathcal{IS}_n$ of nilpotency degree $m > k$, which corresponds to an ordered decomposition, $\rho_2 = (N_1^2, \dots, N_m^2)$, if and only if each block of ρ_1 is a union of some neighbor blocks of ρ_2 and the linear order on $\{N_1^1, \dots, N_k^1\}$ is induced from that on $\{N_1^2, \dots, N_m^2\}$.*

Proof. The sufficiency of the conditions is obvious, so we will prove only the necessity part. Let $T_1 \subset T_2$. If two elements, say x and y, containing in the same block of ρ_2, would belong to different blocks of ρ_1, then either $[x,y]\varepsilon_{\{x,y\}}$ or $[y,x]\varepsilon_{\{x,y\}}$ would belong to T_1 but not to T_2. Hence each block of ρ_2 is contained in a block of ρ_1. Now assume that two blocks, N_i^2 and N_j^2, are contained in one block N_r^1 and that some N_p^2, $i < p < j$, is not contained in N_r^1. Then N_p^2 is contained in some N_q^1, $q \neq r$. If $q < r$, we choose in N_p^2 and N_i^2 elements x and y correspondingly. Then T_1 contains $\pi = [x,y]\varepsilon_{\{x,y\}}$, but π does not belong to T_2 and we get a contradiction. Analogous arguments yield contradiction in the case $q > r$ proving that each N_r^1 is a union of neighbor blocks of ρ_2.

The statement about the orderings of $\{N_1^1, \dots, N_k^1\}$ and $\{N_1^1, \dots, N_m^1\}$ is left to the reader. □

Corollary 13.3. *The poset of all maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of all nilpotency degrees (with respect to inclusions) is a lower semi-lattice.*

Proof. Follows immediately from Theorem 13.2. □

14 Isomorphism and inclusions of maximal nilpotent subsemigroups of $\mathcal{IS}_n$

Having described all maximal nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree k, there arise a natural question: given two such semigroups, T_1 and T_2, how to find out if they are isomorphic or not. This will be the content of this section.

Lemma 14.1. *Let S and T be two maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree k. If $\mathrm{type}(S) = \mathrm{type}(T)$ then there exists $\pi \in \mathbb{S}_n$ such that $S = \pi^{-1} \circ T \circ \pi$, in particular, S and T are isomorphic.*

Proof. Let $(M_1, \dots, M_k)$ and $(N_1, \dots, N_k)$ be the ordered decompositions of $\mathcal{N}_n$, which correspond to $<_S$ and $<_T$. As $\text{type}(S) = \text{type}(T)$, we have $|M_i| = |N_i|$ for all $1 \le i \le k$ and hence there exists $\pi \in \mathbb{S}_n$, such that $\pi(M_i) = N_i$. It is obvious that $\pi^{-1} \circ T \circ \pi \subset S$ and $\pi \circ T \circ \pi^{-1} \subset T$ and hence the bijection $x \mapsto \pi^{-1} \circ x \circ \pi$ is an isomorphism. $\square$

For any vector, $(v_1, \dots, v_k)$, we set $(v_1, \dots, v_k)^\# = (v_k, \dots, v_1)$.

Lemma 14.2. *If S is a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree k then $S^\#$ is a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree k as well and* $\text{type}(S) = \text{type}(S^\#)^\#$.

Proof. Left to the reader. $\square$

Lemma 14.3. *Let S and T be two maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree k. If* $\text{type}(S) = \text{type}(T)^\#$ *then there exists $\pi \in \mathbb{S}_n$ such that $S^\#$ is isomorphic to $\pi^{-1} \circ T \circ \pi$, in particular, S and T are antiisomorphic.*

Proof. Follows from Lemmas 14.1 and 14.2. $\square$

Our goal is to reverse the statement of Lemma 14.1. It happens that we should consider two different cases: $k = 2$ and $K > 2$. The reason is that each nilpotent semigroup of nilpotency degree 2 is the so-called semigroup with zero multiplication, i.e. $x * y = 0$ for any x, y. Obviously, two such semigroups are isomorphic if and only if they have the same number of elements and, additionally, for such semigroups the notions of isomorphism and ani-isomorphism coincide.

Lemma 14.4. *If two maximal nilpotent subsemigroups S and T of nilpotency degree 2 in $\mathcal{IS}_n$ are isomorphic then* $\text{type}(S) = \text{type}(T)$ *or* $\text{type}(S) = \text{type}(T^\#)$.

Proof. As we remarked above, $S \simeq T$ if and only if $|S| = |T|$. So, to prove the lemma we will calculate $|S|$ for $\text{type}(S) = (m, n-m)$. Let the corresponding decomposition of $\mathcal{N}_n$ be M_1, M_2. So, $\pi \in S$ if and only if $\text{dom}(\pi) \subset M_2$ and $\text{ran}(\pi) \subset M_1$. Using elementary combinatorics, we obtain $|S| = \displaystyle\sum_{i=0}^{\min(m,n-m)} \binom{m}{i}\binom{n-m}{i} i! = f(m)$.

Now we claim $f(m_1) \ne f(m_2)$ if $1 \le m_1 < m_2 \le n/2$. For this we prove that $f(m) < f(m+1)$, $m+1 \le n/2$. This is equivalent to

$$\sum_{i=0}^{m} \binom{m}{i}\binom{n-m}{i} i! < \sum_{i=0}^{m+1} \binom{m+1}{i}\binom{n-m-1}{i} i!.$$

Now it is enough to prove that $\binom{m}{i}\binom{n-m}{i} i! < \binom{m+1}{i}\binom{n-m-1}{i} i!$ and this is equivalent to $\dfrac{n-m}{n-m-i} < \dfrac{m+1}{m+1-i}$. The last reduces to $m+1 < n-m$, which is true as $m+1 \le n/2$. This completes the proof. $\square$

So, the case $k = 2$ is completed and we can move to the case $k > 2$.

Lemma 14.5. *Let ρ_1 and ρ_2 be the decompositions of $\mathcal{N}_n$ of the types (n_1^1, n_2^1, n_3^1) and (n_1^2, n_2^2, n_3^3) correspondingly. If $T_1 = Mon(<_{\rho_1})$ is isomorphic (resp. antiisomorphic) to $T_2 = Mon(<_{\rho_2})$ then* $\text{type}(\rho_1) = \text{type}(\rho_2)$ *(resp.* $\text{type}(\rho_1) = \text{type}(\rho_2)^{\#}$*).*

Proof. Let $\rho_1 = (M_1^1, M_2^1, M_3^1)$ and $\rho_2 = (M_1^2, M_2^2, M_3^2)$. Clearly, because of Lemma 14.2 it is enough to consider the case $T_1 \simeq T_2$. Denote by L_j (resp. R_j) the set $\{x \in T_j | x \circ \pi = \mathbf{0}$ for any $\pi \in T_j\}$ (resp. $\{x \in T_j | \pi \circ x = \mathbf{0}$ for any $\pi \in T_j\}$), $j = 1, 2$. It is easy to see that $L_j = Mon(<_{\rho_j^1})$ and $R_j = Mon(<_{\rho_j^2})$, where $\rho_j^1 = (M_1^j \cup M_2^j, M_3^j)$ and $\rho_j^2 = (M_1^j, M_2^j \cup M_3^j)$. As $T_1 \simeq T_2$, we have $|L_1| = |L_2|$ and $|R_1| = |R_2|$.

In the proof of Lemma 14.4 it was obtained that last two equalities are equivalent to $\{n_1^1 + n_2^1, n_3^1\} = \{n_1^2 + n_2^2, n_3^2\}$ and $\{n_1^1, n_2^1 + n_3^1\} = \{n_1^2, n_2^2 + n_3^2\}$ correspondingly. So, we have four possibilities:

The first one is $n_1^1 + n_2^1 = n_1^2 + n_2^2$, $n_3^1 = n_3^2$, $n_1^1 = n_1^2$ and $n_2^1 + n_3^1 = n_2^2 + n_3^2$. In this case we clearly get $(n_1^1, n_2^1, n_3^1) = (n_1^2, n_2^2, n_3^3)$.

The second one is $n_1^1 + n_2^1 = n_3^2$, $n_3^1 = n_1^2 + n_2^2$, $n_1^1 = n_2^2 + n_3^2$ and $n_2^1 + n_3^1 = n_1^2$. This is impossible, because this system does not have any solution in positive integers.

The third one is $n_1^1 + n_2^1 = n_1^2 + n_2^2$, $n_3^1 = n_3^2$, $n_1^1 = n_2^2 + n_3^2$ and $n_2^1 + n_3^1 = n_1^2$. Put $n_3^1 = l$, $n_2^1 = k$ and $n_2^2 = m$. Then $n_3^2 = l$, $n_1^1 = l + m$ and $n_1^2 = k + l$. If $k = m$, we get $(n_1^1, n_2^1, n_3^1) = (n_1^2, n_2^2, n_3^3)$. If $k \neq m$, we can assume $k < m$ without loss of generality. Let us prove that in this case $|T_1| < |T_2|$ and thus this case is impossible. It is enough to consider $m = k + 1$. By arguments of Lemma 14.1, we can assume $N_3^1 = N_3^2$, $N_2^2 = N_2^1 \cup \{a\}$, $N_1^2 = N_1^1 \setminus \{a\}$. To each $\pi \in T_1$ we associate $p(\pi)$ in the following way: If $a \notin \text{dom}(\pi)$ or $\pi(a) \in N_3^1$, then $p(\pi) = \pi$. If $a \in \text{dom}(\pi)$ and $\pi(a) \in N_2^1$, let $i_1 < i_2 < \cdots < i_s$ be all elements from $N_2^1 \setminus \pi(N_1^2)$, increasingly ordered, and $\pi(a) = i_r$. We also increasingly order all elements of $N_1^2 \setminus \text{dom}(\pi)$: $j_1 < j_2 < \cdots < j_q$. As $n_1^2 = k + l$, $n_2^1 + n_3^1 = k + 1$, $\pi(N_1^2) \subset N_2^1 \cup N_3^1$ and π is a partial bijection, we have $q \geq s$. Now define $p(\pi)$ as follows: for all $x \in \text{dom}(\pi) \setminus \{a\}$, $p(\pi)(x) = \pi(x)$, $p(\pi)(j_r) = i_r$ and in all other points $p(\pi)$ is undefined. It is obvious from the construction that $p : T_1 \to T_2$ is an injection and we have $|T_1| \leq |T_2|$. Now we note that for any $\tau \in p(T_1)$ we have $a \notin \text{dom}(\tau) \cap \text{ran}(\tau)$. Take any $a \in N_1^2$ and $y \in N_3^2$ and consider μ such that $\mu(x) = a$ and $\mu(a) = y$ and μ is not defined in all other pints. Clearly $\mu \in T_2 \setminus p(T_1)$ and thus $|T_1| < |T_2|$.

And the last case $n_1^1 + n_2^1 = n_3^2$, $n_3^1 = n_1^2 + n_2^2$, $n_1^1 = n_1^2$ and $n_2^1 + n_3^1 = n_2^2 + n_3^2$ reduces to the third one if we consider $T_1^{\#}$ and $T_2^{\#}$.

Finally, if $T_1 \simeq T_2$, we get $\text{type}(\rho_1) = \text{type}(\rho_2)$. The statement is proved. □

Theorem 14.1. *Two maximal nilpotent subsemigroups T_1 and T_2 of $\mathcal{IS}_n$ of nilpotency degree $k > 2$ are isomorphic if and only if* $\text{type}(T_1) = \text{type}(T_2)$ *and antiisomorphic if and only if* $\text{type}(T_1) = \text{type}(T_2)^{\#}$.

Proof. Of course, we have only to show that for $k > 3$ the isomorphism of S and T implies $\text{type}(T_1) = \text{type}(T_2)$. We use induction in k with $k = 3$ as the basis (Lemma 14.5) and prove the induction step from k to $k + 1$. Let $(N_1^1, \ldots, N_{k+1}^1)$ and $(N_1^2, \ldots, N_{k+1}^2)$ be the decompositions, corresponding to T_1 and T_2 and $(n_1^1, \ldots, n_{k+1}^1)$ and $(n_1^2, \ldots, n_{k+1}^2)$ be their types. Define

$$L_j = \{x \in T_j | x \circ a_1 \cdots \circ a_{k-1} = 0 \text{ for all } a_1, \dots, a_{k-1} \in T_j\}, j = 1, 2.$$

These sets will clearly be some maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree k, corresponding to decompositions $(N_1^j \cup N_2^j, N_3^j, \dots, N_{k+1}^j)$. As $T_1 \simeq T_2$, we get $L_1 \simeq L_2$ and by inductive assumption $n_1^1 + n_2^1 = n_1^2 + n_2^2$, $n_3^1 = n_3^2$ and so on. Considering

$$R_j = \{x \in T_j | a_1 \cdots \circ a_{k-1} \circ x = 0 \text{ for all } a_1, \dots, a_{k-1} \in T_j\}, j = 1, 2,$$

we get $n_1^1 = n_1^2, \dots, n_{k-1}^1 = n_{k-1}^2$ and $n_k^1 + n_{k+1}^1 = n_k^2 + n_{k+1}^2$. Hence $(n_1^1, \dots, n_{k+1}^1) = (n_1^2, \dots, n_{k+1}^2)$ and the proof is complete. □

Corollary 14.1. *If two maximal nilpotent subsemigroups T_1 and T_2 of $\mathcal{IS}_n$ of nilpotency degree $k > 2$ are isomorphic then there exists $\pi \in \mathbb{S}_n$ such that $T_1 = \pi^{-1} \circ T_2 \circ \pi$.*

Proof. Follows from Theorem 14.1 and Lemma 14.1. □

15 Cardinality of maximal nilpotent subsemigroups of $\mathcal{IS}_n$

The next natural question about maximal nilpotent subsemigroups: can we compute $|T|$ in terms of type(T)? This is done in the present section, but we will need two notation to be able to state the main result. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial with integer coefficients, we set $f(B) = a_n B_n + a_{n-1} B_{n-1} + \dots + a_1 B_1 + a_0$, where B_i's are Bells numbers. We also set $[x]_k = x(x-1)(x-2)\dots(x-k+1)$.

Theorem 15.1. *Let T be the maximal nilpotent subsemigroup in $\mathcal{IS}_n$ of nilpotency degree k, which corresponds to an ordered decomposition, $(N_1, \dots, N_k)$, of $\mathcal{N}_n$. Let $n_i = |N_i|$, $i = 1, \dots, n$, and set $f_{n_1,\dots,n_k}(x) = [x]_{n_1}[x]_{n_2}\dots[x]_{n_k}$. Then $|T| = f_{n_1,\dots,n_k}(B)$.*

We start the proof with considering the case $k = n$ in the following lemma. In this case all $n_i = 1$ and $f_{1,1,\dots,1}(x) = x^n$, hence $f_{1,1,\dots,1}(B) = B_n$.

Lemma 15.1. *If T is a maximal nilpotent subsemigroup in $\mathcal{IS}_n$, then $|T| = B_n$.*

Proof. We construct a bijection between decompositions of $\mathcal{N}_n$ and elements of T. If $\pi \in T$, we associate with π the decomposition of Γ_π into connected components. Since π is nilpotent, this also coincides with the decomposition of $\mathcal{IS}_n$ into the disjoint union of maximal chains with respect to π. Conversely, let $\mathcal{N}_n = N_1 \cup N_2 \cup \dots \cup N_k$. For $1 \le i \le k$ write $N_i = \{j_1 <_T j_2 <_T \dots <_T j_p\}$ and set $\pi_i = [j_p, j_{p-1}, \dots, j_1]$. Then all π_i commute and the element $\pi = \prod_{i=1}^k \pi_i$ clearly belongs to T. It is obvious that these two maps between T and decompositions of $\mathcal{N}_n$ are mutually inverse and our lemma is proved. □

Set $|T| = t(n_1, \dots, n_k)$. This is well-defined as we know that $|T| = |S|$ provided type$(T) =$ type(S). Next we want to show that $t(n_1, \dots, n_k)$ satisfy a recursive relation.

Lemma 15.2.

$$t(n_1,\dots,n_{i-1},1,n_{i+1},\dots,n_k) = \\ = t(n_1,\dots,n_{i-1},n_{i+1}+1,\dots,n_k) + n_{i+1}t(n_1,\dots,n_{i-1},n_{i+1},\dots,n_k).$$

Proof. We decompose the elements of $T = T(N_1,\dots,N_{i-1},\{a\},N_{i+1},\dots,N_k)$ into two groups. The first one, M_1, will contain all elements in which $\{a\}$ is not the image of any element from N_{i+1}; and the second one, M_2, will contain all other elements.

Each element of M_1 can be considered as an element of the semigroup

$$T_1 = T(N_1,\dots,N_{i-1},N_{i+1}\cup\{a\},\dots,N_k)$$

and this correspondence between M_1 and T_1 is bijective. Hence $|M_1| = t(n_1,\dots,n_{i-1},n_{i+1}+1,\dots,n_k$.

Now consider M_2. Let $N_{i+1} = \{b_1,\dots,b_{n_{i+1}}\}$. Decompose M_2 into $M_2^1\cup\dots\cup M_2^{n_{i+1}}$, where $M_2^j = \{\pi\in M_2|\pi^{-1}(a)=b_j\}$. Then for any j the map

$$\varphi_j : M_2^j \to T(N_1,\dots,N_{i-1},N_{i+1},\dots,N_k),$$

defined by: $\varphi_j(\pi)(x)=\pi(x)$, $x\neq b_j$, and $\varphi_j(\pi)(b_j)=\pi(a)$ is a bijection. Hence $|M_2|$ is equal to $n_{i+1}t(n_1,\dots,n_{i-1},n_{i+1},\dots,n_k)$. □

Proof of Theorem 15.1. We use induction in n', which is the sum of all $n_i\neq 1$. If $n'=0$, T is a maximal nilpotent subsemigroup of $\mathcal{IS}_n$ and $t(1,\dots,1) = f_{1,\dots,1}(B)$ be Lemma 15.1. Now assume that we have $n_i\geq 2$ for some i. From $[x]_m = (x-m+1)[x]_{m-1}$ we deduce the equality

$$f_{(n_1,\dots,n_i,n_{i+1},\dots,n_k)}(x) = f_{(n_1,\dots,n_i,1,n_{i+1}-1,\dots,n_k)}(x) - (n_{i+1}-1)f_{(n_1,\dots,n_i,n_{i+1}-1,\dots,n_k)}(x).$$

From the other hand, we have

$$t(n_1,\dots,n_{i-1},1,n_{i+1},\dots,n_k) = \\ = t(n_1,\dots,n_{i-1},n_{i+1}+1,\dots,n_k) + n_{i+1}t(n_1,\dots,n_{i-11},n_{i+1},\dots,n_k).$$

from Lemma 15.2. So, we have two recursive sequences with the same recursion rule and the same initial values. We conclude that these sequences coincide. □

16 Description of automorphisms of maximal nilpotent subsemigroups of $\mathcal{IS}_n$

Let T be the maximal nilpotent subsemigroup in $\mathcal{IS}_n$ of nilpotency degree k, which corresponds to an ordered decomposition, $\rho = (N_1,\dots,N_k)$, of $\mathcal{N}_n$. The aim of this section is to describe the group $\mathrm{Aut}(T)$. We need some preparation to be able to state our results. We start with the description of the obvious part of $\mathrm{Aut}(T)$. Denote by S the direct sum $\oplus_i\mathbb{S}(N_i)$ of the symmetric groups acting on the blocks of ρ. Clearly, $S\subset\mathbb{S}_n\subset\mathcal{IS}_n$. This inclusion defines the action of S on T via conjugation: $S\ni\pi\mapsto\hat{\pi}$, where $\hat{\pi}(x) = \pi^{-1}\circ x\circ\pi$.

Lemma 16.1. *The map $\pi \mapsto \hat{\pi}$ is a homomorphic injection of S into* $\mathrm{Aut}(T)$*.*

Proof. The statement about homomorphism with image in $\mathrm{Aut}(T)$ follows from $\pi^{-1} \circ x \circ y \circ \pi = \pi^{-1} \circ x \circ \pi \circ \pi^{-1}$
$circy \circ \pi$ and $(\pi \circ \tau)^{-1} \circ x \circ (\pi \circ \tau) = \tau^{-1} \circ (\pi^{-1} \circ x \circ \pi) \circ \tau$.

To prove the injectivity of the map, consider some $1 \neq \pi \in S$. Then there are $x, y \in \mathcal{N}_n$ such that $\pi(x) = y$ and $x \neq y$. We consider any element $f = [a, b]\varepsilon_{\{a,b\}} \in T$ such that $x \in \{a, b\}$. Then $\pi^{-1} \circ f \circ \pi \neq f$ and the proof is complete. ☐

An element, $f \in T$, will be called *decomposable* if there exist $a, b \in T$ such that $f = a \circ b$ and *indecomposable* otherwise.

Lemma 16.2. *Each element from T decomposes into a finite product of indecomposable elements (this decomposition is not unique in general).*

Proof. Let $x \in N_1$, $y \in N_2$ and $b = [y, x]\varepsilon_{\{x,y\}}$. Then $\mathbf{0} = b \circ b$ is a desired decomposition of $\mathbf{0}$. Now assume that for some $T \ni a \neq 0$ such decomposition does not exist. Then we decompose $a = a_1 \circ a_2$ and at least one factor is decomposable. Continuing this procedure we get a decomposition $a = c_1 \circ c_2 \circ \dots c_{k+1} = 0$ as T is nilpotent of nilpotency degree k, a contradiction. ☐

For an indecomposable element, $a \in T$, we set

$$\overline{\mathrm{dom}}(a) = \mathrm{dom}(a) \setminus \{x \in \mathrm{dom}(a) | x \in N_k \text{ and } a(x) \in N_1\}.$$

For $a \in T$ we denote by a_* the restriction of a on $\overline{\mathrm{dom}}(a)$ if a is indecomposable and $a_* = a$ otherwise. We define an equivalence relation, $\sim$, on $T \setminus \{\mathbf{0}\}$ as follows: $a \sim b$ if and only if $a_* = b_*$ and extend this to T assuming that $\{\mathbf{0}\}$ is a separate class.

Lemma 16.3. *$\sim$ is a congruence on T.*

Proof. We prove that $\sim$ is right stable. The left stability can be proved analogously. Let $a, b \in T \setminus \{\mathbf{0}\}$, $a \sim b$, and $f \in T$. As $a \circ f$ and $b \circ f$ are decomposable, $(a \circ f)_* = a \circ f$ and $(b \circ f)_* = b \circ f$. Moreover, from the definition of a_* it follows $a \circ f = a_* \circ f$. Thus $b \circ f = b_* \circ f$ as well and from $a_* = b_*$ we derive $(a \circ f)_* = (b \circ f)_*$, which means $a \circ f \sim b \circ f$. ☐

Decompose T into equivalence classes $\cup_i M_i$ with respect to $\sim$. Denote by $G = \oplus_i \mathbb{S}(M_i)$ the direct sum of symmetric groups, acting on these classes.

Theorem 16.1. $\mathrm{Aut}(T) = S \rtimes G$.

The rest of this section will be devoted to the proof of Theorem 16.1. But before turning to it we give two immediate corollaries of this result for the cases of maximal and minimal possible k.

Corollary 16.1. *If T is maximal nilpotent (i.e. $k = n$), then* $\mathrm{Aut}(T)$ *is a finite direct sum of cyclic groups of order* 2*.*

Proof. Clearly, all blocks of ρ are 1-element and all classes of $\sim$ contain not more than 2 elements. $\square$

Corollary 16.2. *Let T be a maximal semigroup with zero multiplication (i.e. $k = 2$). Then* $\mathrm{Aut}(T)$ *equals the stabilizer of* $\{\mathbf{0}\}$ *in* $\mathbb{S}(T)$.

Proof. Clearly, $\mathrm{Aut}(T)$ is a subgroup of this stabilizer. Let σ be an element of the stabilizer. As for any $a, b \in T$ we have $\sigma(a \circ b) = \sigma(\mathbf{0}) = \mathbf{0} = \sigma(a) \circ \sigma(b)$, the element σ belongs to $\mathrm{Aut}(T)$. This completes the proof. $\square$

To prove Theorem 16.1 we will need several lemmas.

Lemma 16.4. *Let $r \in \mathbb{N}$, $r > 1$, and $a_1, \dots, a_r \in T$. Then*

$$a_1 \circ a_2 \circ \cdots \circ a_r = (a_1)_* \circ (a_2)_* \circ \cdots \circ (a_r)_*.$$

Proof. Left to the reader. $\square$

Lemma 16.5. *Let $\pi \in G$ and $a \in T$ then $\pi(a) \sim a$.*

Proof. Obvious. $\square$

For $\pi \in S$ and $a \in T$ we set for simplicity $a^\pi = \pi^{-1} \circ a \circ \pi = \pi(a)$.

Lemma 16.6. *Let $\pi \in S$ and $a \in T$ be indecomposable. Then $(a_*)^\pi = (a^\pi)_*$.*

Proof. If $a_* = 0$ we have $\mathrm{dom}(a) \subset N_k$ and $\mathrm{ran}(a) \subset N_1$. Hence $\mathrm{dom}(a^\pi) \subset N_k$ and $\mathrm{ran}(a^\pi) \subset N_1$ and we get $(a^\pi)_* = 0$. Now assume $a_* \neq 0$. We want to show that $\mathrm{dom}((a_*)^\pi) = \mathrm{dom}((a^\pi)_*)$ and that $(a_*)^\pi(x) = (a^\pi)_*(x)$ for any $x \in \mathrm{dom}((a_*)^\pi)$. We have $\mathrm{dom}(a^\pi) = \pi(\mathrm{dom}(a))$ and hence

$$\mathrm{dom}((a^\pi)_*) = \pi(\mathrm{dom}(a) \setminus \{x \in \mathrm{dom}(a) \cap N_k | a(x) \in N_1\}).$$

Analogously, from the definition of a_* we get

$$\mathrm{dom}((a_*)^\pi) = \pi(\mathrm{dom}(a) \setminus \{x \in \mathrm{dom}(a) \cap N_k | a(x) \in N_1\}).$$

If $y = \pi(x) \in \mathrm{dom}((a_*)^\pi)$, we have $(a_*)^\pi(y) = \pi(a_*(x)) = \pi(a(x))$. From the other hand, $(a^\pi)_*(y) = a^\pi(y) = \pi(a(x))$. $\square$

Lemma 16.7. *Let $a, b \in T$ and $\pi \in S$. Then $a \sim b$ implies $a^\pi \sim b^\pi$.*

Proof. If at least one of a and b is decomposable, $a \sim b$ implies $a = b$ and the statement is obvious. If both a and b are indecomposable, $q \sim b$ means $a_* = b_*$ and hence $(a_*)^\pi = \pi^{-1} \circ a_* \circ pi = \pi^{-1} \circ b_* \circ \pi = (b_*)^\pi$. Now Lemma 16.6 guarantees $(a^\pi)_* = (b^\pi)_*$ and thus $a^\pi \sim b^\pi$. $\square$

If for $a \in T$ we can find $x \in \mathrm{dom}(a) \cap N_i$ such that $a(x) = y \in N_j$, we will say that a has the arrow $x \mapsto y$ from N_i to N_j.

Lemma 16.8. *Assume that $a_1 \circ \dots \circ a_{k-1} \neq 0$. Then for any $1 \leq i \leq k-1$ the element a_i has an arrow from N_{i+1} to N_i.*

Proof. Take $x \in \mathcal{N}_n$ such that $a_1(\dots(a_{k-1}(x))) \neq 0$. As

$$a_1(\dots(a_{k-1}(x))) <_T a_2(\dots(a_{k-1}(x))) <_T \dots <_T x$$

is a chain of length k, we get $a_i(\dots(a_{k-1}(x))) \in N_i$. $\square$

Lemma 16.9. *Let $\sigma \in \mathrm{Aut}(T)$ and $a \in T$ such that $\mathrm{rank}(a_*) = 1$, $\mathrm{dom}(a_*) = \{x\} \subset N_i$ and $\mathrm{ran}(a_*) = \{y\} \subset N_{i-1}$. Then $\mathrm{rank}(\sigma(a)_*) = 1$, $\mathrm{dom}(\sigma(a)_*) \subset N_i$ and $\mathrm{ran}(\sigma(a)_*) \subset N_{i-1}$.*

Proof. Choose some representatives $x_j \in N_j$, $j \neq i, i-1$, set $x_i = x$ and $x_{i-1} = y$, and denote by a_j, $j \neq i$, the elements of T defined as follows: $\mathrm{rank}(a_j) = 1$ and $a_j(x_j) = x_{j-1}$. We have $b = a_{k-1}a_{k-2}\dots a_{i+1}aa_{i-1}\dots a_1 \neq \mathbf{0}$ as $b(x_k) = x_1$. Hence $\sigma(b) \neq \mathbf{0}$ and $\sigma(a)$ has an arrow from N_i to N_{i-1} by Lemma 16.8. Thus the same is true for $\sigma(a)_*$

Consider the left annihilator $Ann_L(T) = \{a \in T | a \circ b = \mathbf{0}$ for any $b \in T\}$ and the set $T(b) = T \circ b \cap Ann_L(T)$ for some $b \in T$. It is easy to see that $T(b)$ consists of all elements from T, whose domain is a subset of $\mathrm{dom}(b_*)$ and whose range is a subset of N_1. In particular, if $m = \min(|\mathrm{dom}(b_*)|, |N_k|)$ then $|T(b)| = \sum_{i=0}^{m} \binom{|\mathrm{dom}(b_*)|}{i}\binom{|N_k|}{i} i!$ and $\mathrm{rank}(b_*) = 1$ if and only if $|T(b)| = |N_k| + 1$. As $|T(b)| = |T(\sigma(b))|$ for any $b \in T$ we get that $\mathrm{rank}(b_*) = 1$ is equivalent to $\mathrm{rank}(\sigma(b_*)) = 1$, which completes the proof. $\square$

Lemma 16.10. *Let $\sigma \in \mathrm{Aut}(T)$, $a, b \in T$ such that $\mathrm{rank}(a_*) = \mathrm{rank}(b_*) = 1$, $\mathrm{dom}(a_*)$, $\mathrm{dom}(b_*) \subset N_i$; $\mathrm{ran}(a_*), \mathrm{ran}(b_*) \subset N_{i-1}$. Then*

1. $\mathrm{dom}(a_*) = \mathrm{dom}(b_*)$ *if and only if* $\mathrm{dom}(\sigma(a)_*) = \mathrm{dom}(\sigma(b)_*)$;
2. $\mathrm{ran}(a_*) = \mathrm{ran}(b_*)$ *if and only if* $\mathrm{ran}(\sigma(a)_*) = \mathrm{ran}(\sigma(b)_*)$.

Proof. As $\mathrm{Aut}(T)$ is a group, it is enough in both cases to prove that the first condition implies the second one. We will consider the first implication and the second one can be done by analogous arguments. Let $\mathrm{dom}(a_*) = \mathrm{dom}(b_*) = \{x\}$.

First we will work with the case $i > 2$ and consider sets $T \circ a$ and $T \circ b$. Each contains $\mathbf{0}$ and all other elements have rank 1, the domain $\{x\}$ and the range being an elements from $N_{i-2} \cup \dots \cup N_1$. Hence, $T \circ a = T \circ b$. As σ is an automorphism we get $T \circ \sigma(a) = T \circ \sigma(b)$ and hence $T \circ \sigma(a)_* = T \circ \sigma(b)_*$, which means $\mathrm{dom}(\sigma(a)_*) = \mathrm{dom}(\sigma(b)_*)$, see Section 10.

Now let $i = 2$ and $y \in N_3$. Consider the rank 1 element $c \in T$ such that $c(y) = x$. We have $a \circ c \neq \mathbf{0}$ and $b \circ c \neq \mathbf{0}$. Hence $\sigma(a) \circ \sigma(c) \neq \mathbf{0}$ and $\sigma(b) \circ \sigma(c) \neq \mathbf{0}$. Now from Lemma 16.4 we get $\sigma(a)_* \circ \sigma(c)_* \neq \mathbf{0}$ and $\sigma(b)_* \circ \sigma(c)_* \neq \mathbf{0}$. Hence Lemma 16.9 implies that the elements $\sigma(a)_*$, $\sigma(b)_*$ and $\sigma(c)_*$ have rank 1, $\sigma(c)_*$ has an arrow from N_3 to N_2 and both $\sigma(a)_*$ and $\sigma(b)_*$ have an arrow from N_2 to N_1. So, the last equalities imply $\mathrm{ran}(\sigma(c)_*) = \mathrm{dom}(\sigma(a)_*)$ and $\mathrm{ran}(\sigma(c)_*) = \mathrm{dom}(\sigma(b)_*)$, which finally means $\mathrm{dom}(\sigma(a)_*) = \mathrm{dom}(\sigma(b)_*)$. $\square$

Lemma 16.11. *Under conditions of Lemma 16.10 we have $a \sim b$ if and only if $\sigma(a) \sim \sigma(b)$.*

Proof. By definition of $\sim$, taking into account that $\mathrm{rank}(a) = \mathrm{rank}(b) = 1$, we have that $a \sim b$ is equivalent to $\mathrm{dom}(a_*) = \mathrm{dom}(b_*)$ and $\mathrm{ran}(a_*) = \mathrm{ran}(b_*)$. This is equivalent to $\mathrm{dom}(\sigma(a)_*) = \mathrm{dom}(\sigma(b)_*)$ and $\mathrm{ran}(\sigma(a)_*) = \mathrm{ran}(\sigma(b)_*)$, which means $\sigma(a) \sim \sigma(b)$. □

Lemma 16.12. *Let $\sigma \in \mathrm{Aut}(T)$ and a be an indecomposable element from T. Then for any i, j, $1 \leq j < i \leq k$, the elements a_* and $\sigma(a)_*$ have the same number of arrows from N_i to N_j.*

Proof. Fix i, j and denote by $n_{i,j}$ (resp. $n'_{i,j}$) the number of arrows from N_i to N_j for the element a_* (resp. $\sigma(a)_*$); and let $x_l \mapsto y_l$, $l = 1, \dots, n_{i,j}$ be the list of these arrows for a_*. As both a and $\sigma(a)$ are indecomposable, a_* and $\sigma(a)_*$ do not have arrows from N_1 to N_k. Now consider three possibilities:

The first one is $i < k$ and $j > 1$. It is enough to prove that $n'_{i,j} \geq n_{i,j}$, which is trivial if $n_{i,j} = 0$. So, we can assume $n_{i,j} > 0$. Fix some $u \in N_{i+1}$ and $v \in N_{j-1}$ and consider the elements a_l, b_l, $l = 1, \dots, n_{i,j}$, of rank 1 in T such that $a_l(u) = x_l$ and $b_l(y_l) = v$. By Lemma 16.9, the rank of $\sigma(a_l)_*$ equals 1 and this element has an arrow from N_{i+1} to N_i. Analogously, $\mathrm{rank}(\sigma(b_l)_*) = 1$ and it has an arrow from N_j to N_{j-1}. Set $x'_l = \mathrm{ran}(\sigma(a_l)_*)$ and $y'_l = \mathrm{dom}(\sigma(b_l)_*)$. From Lemma 16.10 it follows that $x'_l \neq x'_s$ and $y'_l \neq y'_s$ for $l \neq s$. By Lemma 16.4, $b_l \circ a \circ a_l \neq \mathbf{0}$ implies $\sigma(b_l)_* \circ \sigma(a)_* \circ \sigma(a_l)_* \neq \mathbf{0}$ and hence $x'_l \in \mathrm{dom}(\sigma(a)_*)$, $y'_l \in \mathrm{ran}(\sigma(a)_*)$, moreover, $\sigma(a)_*(x'_l) \in N_j$. This means precisely that $n'_{i,j} \geq n_{i,j}$.

The second case is $i = k$ and $j > 1$. We again will prove $n'_{k,j} \geq n_{k,j}$ assuming $n_{k,j} > 0$. Fix some $v \in N_{j-1}$ and consider the elements b_l, $l = 1, \dots, n_{i,j}$ of rank 1 in T such that $b_l(y_l) = v$. As above we get that $\sigma(b_l)_*$ has rank 1 and an arrow from N_j to N_{j-1}. Set $y'_l = \mathrm{dom}(\sigma(b_l)_*) \in N_j$ and again $y'_l \neq y'_s$ and $y'_l \in \mathrm{ran}(\sigma(a)_*)$ by the same arguments as above. Let $(\sigma(a)_*)^{\#}(y'_l) \in N_{i'}$. If $i' \neq k$ then $\sigma(a)_*$ will have an arrow from $N_{i'}$ to N_j, which, by Lemma 16.10, does not coincide with any of arrows constructed in the first case and we get a contradiction with the conclusion of the first case. Hence $i' = k$. This implies $n'_{k,j} \geq n_{k,j}$ and the second case is complete.

The last case $i < k$ and $j = 1$ is proved by arguments analogous to the second case. □

As an immediate corollary we get.

Corollary 16.3. *Let $\sigma \in \mathrm{Aut}(T)$ and a be an indecomposable element from T. Then* $\mathrm{rank}(a_*) = \mathrm{rank}(\sigma(a)_*)$.

We have arrived the key lemma of this section.

Lemma 16.13. *Let $\sigma \in \mathrm{Aut}(T)$. Then there exists $\pi \in S$ such that $\sigma(a) \sim a^{\pi}$ for all $a \in T$.*

Proof. First we construct such π. For each N_i, $1 \leq i \leq k$, we set $N_i = \{x_i^s | s \in I_{N_i}\}$. Choose representatives x_i^1 in blocks N_i for all i.

Fix N_i, $i > 1$, and let $A_i = \{a_i^s | s \in I_{N_i}\}$ be the set of elements of rank 1 such that $a_i^s(x_i^s) = x_{i-1}^1$. From Lemma 16.9 we have that $(\sigma(a_i^s))_*$ has rank 1 and an arrow from N_i to N_{i-1} for any a_i^s. Hence we can defined the map $\pi_i : N_i \to N_i$ by $\pi_i(x_i^s) = \mathrm{dom}(\sigma(a_i^s)_*)$ and Lemma 16.10 guarantees us that π_i is injective, hence bijective.

Denote by $B = \{b^s | s \in I_{N_1}\}$ the set of all elements of rank 1 such that $b^s(x_2^1) = x_1^s$. As above, $\sigma(b^s)_*$ has rank 1 and an arrow from N_2 to N_1. So we can define $\pi_1 : N_1 \to N_1$ by $\pi_1(x_1^s) = \mathrm{ran}(\sigma(b^s)_*)$ and as above get that π_1 is injective and hence bijective. Put $\pi = (\pi_1, \dots, \pi_k)$.

Now we want to check that $\sigma(a) \sim a^\pi$, which is equivalent to $\sigma(a)_* = \pi^{-1} \circ a_* \circ \pi$. The part of the proof is naturally divided into three steps.

Step 1. First let a coincide with one of a_i^s. As $\mathrm{rank}(a) = 1$, it is enough to show that $\mathrm{dom}(\sigma(a)_*) = \pi(\mathrm{dom}(a_*))$ and $\mathrm{ran}(\sigma(a)_*) = \pi(\mathrm{ran}(a_*))$. The first equality follows directly from the construction of π. To prove the second one we first assume that $i > 2$. As $a_{i-1}^1 \circ a_i^s \neq \mathbf{0}$, we have $\sigma(a_{i-1}^1)_* \circ \sigma(a_i^s)_* \neq \mathbf{0}$ and both factors have rank 1. So, we get $\mathrm{ran}(\sigma(a_i^s)_*) = \mathrm{dom}(\sigma(a_{i-1}^1)_*) = \{\pi(x_{i-1}^1)\} = \pi(\mathrm{ran}((a_i^s)_*))$. Now assume that $i = 2$. Then $\mathrm{ran}(a_2^s) = \mathrm{ran}(b^1)$ implies $\mathrm{ran}(\sigma(a_2^s)_*) = \mathrm{ran}(\sigma(b^1)_*) = \{\pi(x_1^1)\} = \pi(\mathrm{ran}((a_2^s)_*))$. So, the case $a = a_i^s$ is complete.

Now let $a = b^s$. As $\mathrm{rank}(a) = 1$ we have to prove the same equalities as above. The second equality follows from the construction of π. For the first one we take $b^s \circ a_3^s \neq \mathbf{0}$ and hence $\sigma(b^s)_* \circ \sigma(a_3^s)_* \neq \mathbf{0}$ and both factors are of rank 1. We get $\mathrm{dom}(\sigma(b^s)_*) = \mathrm{ran}(\sigma(a_3^s)_*) = \{\pi(x_2^1)\} = \pi(\mathrm{dom}(b_*^s))$.

Step 2. Now let a be any indecomposable element in T, which does not equal any of a_i^s and b^s. If $a_* = \mathbf{0}$, then $\sigma(a)_* = \mathbf{0}$ by Lemma 16.12 and, using Lemma 16.6, we get $(a^\pi)_* = (a_*)^\pi = \mathbf{0}$ and hence $\sigma(a)_* = (a^\pi)_* = \mathbf{0}$. So, we can assume $a_* \neq \mathbf{0}$. Take any $x \in \mathrm{dom}(a_*)$. As $\mathrm{rank}(a_*) = \mathrm{rank}(\sigma(a)_*)$ by Corollary 16.3, we need only $\pi(x) \in \mathrm{dom}(\sigma(a)_*)$ and $\sigma(a)_*(\pi(x)) = \pi(a_*(x))$. Let $x = x_i^s$ and $a(x) = x_j^t$. As a_* does not have any arrow from N_k to N_1, we have $i - j < k - 1$. We have to consider two cases.

The first one is $j > 1$. Take $a_j^t \circ a = a_j^1 \circ \cdots \circ a_{i-1}^1 \circ a_i^s$, apply σ and use Lemma 16.4. We get $\sigma(a_j^t)_* \circ \sigma(a) = \sigma(a_j^1) \circ \cdots \circ \sigma(a_{i-1}^1) \circ \sigma(a_i^s)$ and, using the arguments above, obtain $\pi(x_i^s) = \mathrm{dom}(\sigma(a_i^s)_*) \in \mathrm{dom}(\sigma(a)_*)$, $\pi(x_i^t) = \mathrm{dom}(\sigma(a_j^t)_*) \in \mathrm{ran}(\sigma(a)_*)$ and, finally, $\sigma(a)_*(\pi(x_i^s)) = \sigma(a_j^t)_*^\#(\pi(x_{j-1}^1)) = \pi(x_j^t)$.

The second case is $j = 1$ and $i < k$. We consider the auxiliary element b_{i+1}^s of rank 1 such that $\mathrm{dom}(b_{i+1}^s) = \{x_{i+1}^1\}$ and $\mathrm{ran}(b_{i+1}^s) = \{x_i^s\}$. From Lemma 16.10 we derive $\mathrm{dom}(\sigma(b_{i+1}^s)_*) = \{\pi(x_{i+1}^1)\}$. Moreover, since $a_i^s \circ b_{i+1}^s \neq \mathbf{0}$, we get $\sigma(a_i^s)_* \circ \sigma(b_{i+1}^s)_* \neq \mathbf{0}$ and hence $\mathrm{ran}(\sigma(b_{i+1}^s)_*) = \mathrm{dom}(\sigma(a_i^s)_*) = \{\pi(x_i^s)\}$. At the same time from $a \circ b_{i+1}^s = b^t \circ a_{j+2}^1 \circ \cdots \circ a_{i+1}^1$ for some b^t of rank 1 we get $\sigma(a)_* \circ \sigma(b_{i+1}^s)_* = \sigma(b^t)_* \circ \sigma(a_{j+2}^1)_* \circ \cdots \circ \sigma(a_{i+1}^1)_*$ and, combining all the equalities, we derive $\pi(x_i^s) = \mathrm{ran}(\sigma(b_{i+1}^s)_*) \in \mathrm{dom}(\sigma(a)_*)$, $\pi(x_1^t) = \mathrm{ran}(\sigma(b^t)_*) \in \mathrm{ran}(\sigma(a)_*)$ and $\pi(x_1^t) = \sigma(b_{i_1}^s)_* \circ \sigma(a)_*(\pi(x_{i+1}^1))$. Hence $\sigma(a)_*(\pi(x_i^s)) = \pi(x_1^t)$.

Step 3. Finally, we assume that a is decomposable in T. If $a = \mathbf{0}$, then $\sigma(a) = \mathbf{0} = a^\pi$, so we can assume $a \neq \mathbf{0}$. Let $a = a_r \circ \cdots \circ a_1$ be a longest decomposition of a into a product of indecomposable elements. We note that $\mathrm{dom}(a) \subset \mathrm{dom}(a_1)$ and denote by b_1 the restriction of a_1 on $\mathrm{dom}(a)$. Now $a = a_r \circ \cdots \circ a_2 \circ b_1$ should again be a decomposition of a into indecomposable elements, as one we started with was a longest one. But now we have $\mathrm{dom}(a) = \mathrm{dom}(b_1)$. Using Lemma 16.4, we get $a = a_* = (a_r)_* \circ \cdots \circ (a_2)_* \circ (b_1)_*$. Applying σ we get $\sigma(a)_* = \sigma(a_r)_* \circ \cdots \circ \sigma(a_2)_* \circ \sigma(b_1)_*$ and so $\mathrm{dom}(\sigma(a)_*) \subset \mathrm{dom}(\sigma(b_1)_*)$ and $\mathrm{rank}(\sigma(a)_*) \leq \mathrm{rank}(\sigma(b_1)_*)$. But $\mathrm{dom}(a) = \mathrm{dom}(b_1)$ and $\mathrm{dom}((b_1)_*) \subset \mathrm{dom}(b_1)$, hence $\mathrm{dom}((b_1)_*) \subset \mathrm{dom}(a) = \mathrm{dom}(a_*)$. Hence previous inequalities are, in fact, equalities and Corollary 16.3 implies that $\mathrm{rank}((b_1)_*) = \mathrm{rank}(\sigma(b_1))_*$ and we get $\mathrm{rank}(\sigma(a)_*) \geq \mathrm{rank}(a_*)$. Applying σ^{-1} we will obtain the opposite inequality and hence these ranks coincide. So, $\mathrm{rank}(\sigma(a)_*) = \mathrm{rank}(a_*) = \mathrm{rank}((b_1)_*) = \mathrm{rank}(\sigma(b_1)_*)$.

Take any $x \in \mathrm{dom}(a_*) = \mathrm{dom}(b_1)$. Taking into account the indecomposability of b_1 we get $\pi(x) \in \mathrm{dom}(\sigma(b_1)_*)$. From this we easily derive $\pi(x) \in \mathrm{dom}(\sigma(a)_*)$. Hence $\sigma(b_1)_*(\pi(x)) \in \mathrm{dom}(\sigma(a_2)_*)$, ... , $\sigma(a_{r-1})_* \circ \cdots \circ \sigma(b_1)_*(\pi(x)) \in \mathrm{dom}(\sigma(a_r)_*)$ and applying the statement for indecomposable elements we get $\sigma(a)_*(\pi(x)) = \pi(a_*(x))$.

Finally we prove the uniqueness of π. Assume that for some $\tau \in S$ we have $\sigma(a) \sim a^\tau$ for all $a \in T$. Then $a^\pi \sim a^\tau$ for all $a \in T$, in particular, $(a_i^s)^\pi \sim (a_i^s)^\tau$ and $(b^s)^\pi \sim (b^s)^\tau$, which means $(a_i^s)_*^\pi = (a_i^s)_*^\tau$ and $(b^s)_*^\pi = (b^s)_*^\tau$. The last implies $\pi(x_i^s) = \tau(x_i^s)$ and hence $\pi = \tau$. □

Corollary 16.4. *Let $\sigma \in \mathrm{Aut}(T)$ and $\pi \in S$ be such that $\sigma(a) \sim a^\pi$ for all $a \in T$. Then $\pi = \{\mathbf{1}\}$ if and only if $a \sim \sigma(a)$ for all $a \in T$.*

Proof. We have only to prove that $a \sim \sigma(a)$ for all $a \in T$ implies $\pi = \mathbf{1}$. Let A_i, $2 \leq i \leq k$, and B be sets from the proof of Lemma 16.13. Then $(\sigma(a_i^s))_* = (a_i^s)_*$ and $(\sigma(b^s))_* = (b^s)_*$ for all $a_i^s \in A_i$ and $b^s \in B$. Hence the construction of π gives us $\pi(x_i^s) = x_i^s$ for all $x_i^s \in N_i$, $1 \leq i \leq k$. □

Corollary 16.5. *Let $\sigma \in \mathrm{Aut}(T)$ and $a, b \in T$. Then $a \sim b$ implies $\sigma(a) \sim \sigma(b)$.*

Proof. This is an immediate corollary from Lemmas 16.7 and 16.13. □

Corollary 16.6. *G is a normal subgroup of $\mathrm{Aut}(T)$.*

Proof. First we show that G is a subgroup of $\mathrm{Aut}(T)$. Indeed, by Lemma 16.4, for any $\pi \in G$ and $a, b \in T$ we have $\pi(a) \circ \pi(b) = \pi(a)_* \circ \pi(b)_* = a_* \circ b_* = a \circ b = \pi(ab)$. Now we show that G is normal in $\mathrm{Aut}(T)$. Choose any $\sigma \in \mathrm{Aut}(T)$, $\pi \in G$ and $a \in T$. From Lemma 16.5 we get $\sigma^{-1}(\pi(a)) \sim \sigma^{-1}(a)$. This and Corollary 16.5 imply $\sigma^{-1}(\pi(\sigma(a))) \sim \sigma^{-1}(\sigma(a)) = a$. Hence $\sigma^{-1}\pi\sigma \in G$. □

Now we are ready to prove our main result, Theorem 16.1.

Proof of Theorem 16.1. It is enough to show that $\mathrm{Aut}(T)/G \simeq S$. Let $\sigma \in \mathrm{Aut}(T)$. From Lemma 16.13 it follows that there exists unique $\pi = \pi_\sigma \in S$ such that $\sigma(a)_* = \pi_\sigma^{-1} \circ a_* \circ \pi_\sigma$ holds for any $a \in T$. Define the map $\xi : \mathrm{Aut}(T) \to S$ by $\xi(\sigma) = \pi_\sigma$. For any $\sigma_1, \sigma_2 \in \mathrm{Aut}(T)$ and $a \in T$ we have $\sigma_1\sigma_2(a)_* = \pi_{\sigma_1\sigma_2}^{-1} \circ a_* \circ \pi_{\sigma_1\sigma_2}$. From the other hand,

$$(\sigma_1\sigma_2(a))_* = (\sigma_2(\sigma_1(a)))_* = \pi_{\sigma_2}^{-1} \circ \sigma_1(a)_* \circ \pi_{\sigma_2} = \pi_{\sigma_2}^{-1} \circ \pi_{\sigma_1}^{-1} \circ a_* \circ \pi_{\sigma_1} \circ \pi_{\sigma_2}$$

and hence $\pi_{\sigma_1\sigma_2} = \pi_{\sigma_1} \circ \pi_{\sigma_2}$, so ξ is a homomorphism. From Corollary 16.4 we get that $\mathrm{Ker}(\xi) = G$ and our theorem is proved. □

17 The number of automorphisms of maximal nilpotent subsemigroups of $\mathcal{IS}_n$

It is quite surprising, but some extension of arguments from Section 15 can be used to compute $|\operatorname{Aut}(T)|$ for T from the previous section. Our main result is the following:

Theorem 17.1. *Let T be as in Section 16. Then*

$$|Aut(T)| = \prod_{i=1}^{k} n_i! \prod_{p=1}^{n_1-1} \prod_{q=1}^{n_k-1} (t(n_1-p, n_k-q)!)^{\alpha(p,n_2,\dots,n_{k-1},q)}, \tag{4}$$

where $t(x,y)$ are as in Section 15 and

$$\alpha(p, n_2, \dots, n_{k-1}, q) = \sum_{i=0}^{\min(p,q)} \binom{n_1}{p}\binom{n_2}{q} \cdot \\ \cdot \sum_{j=0}^{p+q-i} (-1)^{p+q-i-j} i! \binom{p}{i}\binom{q}{i}\binom{p+q-i}{j} t(n_2, \dots, n_{k-1}, j).$$

To prove this we will need two lemmas and new notation. Consider the following set: $T^* = \{f \in T | N_k \subset \operatorname{dom}(f), N_1 \subset \operatorname{ran}(f), \text{ and } f(x) \notin N_1, x \in N_k\}$ and put $t^*(n_1, \dots, n_k) = |T^*|$. Let $T' = \{f \in T | N_1 \subset \operatorname{dom}(f)\}$ and $t'(n_1, \dots, n_k) = |T'|$.

Lemma 17.1.

$$t(n_1, \dots, n_k) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_k} \binom{n_1}{i}\binom{n_k}{j} t(n_1-i, n_k-j) t^*(i, n_2, \dots, n_{k-1}, j).$$

Proof. For $f \in T$ set $\operatorname{dom}'(f) = \{x \in N_k | f(x) \notin N_1\}$ and $\operatorname{ran}'(f) = \{y \in N_1 | \pi^{\#}(y) \notin N_k\}$. We can assume that f is "glued" from $f_1 \in T(N_1 \setminus \operatorname{ran}'(f), N_k \setminus \operatorname{dom}'(f))$ and $f_2 \in T(\operatorname{ran}'(f), N_2, \dots, N_{k-1}, \operatorname{dom}'(f))$. If $|\operatorname{dom}'(f)| = j$ and $|\operatorname{ran}'(f)| = i$, there are precisely $t(n_1-i, n_k-j)$ choices for f_1 and $t^*(i, n_2, \dots, n_{k-1}, j)$ choices for f_2. Now we just need to decompose T into classes with respect to values of i and j and count the number of elements in each class using elementary combinatorics. □

The importance of Lemma 17.1 becomes clear if we note that $f \sim g$ if and only if $f_2 \sim g_2$, as defined in Lemma 17.1. If $f \in T$ and $y \in \operatorname{ran}(f)$, in the next proof the notation $(f, x) \mapsto y$ will mean that there exists a sequence $x = z_1, z_2, \dots . z_r = y$ of maximal length such that $f(z_j) = z_{j+1}$.

Lemma 17.2.

$$t^*(n_1, \dots, n_k) = \sum_{i=1}^{\min(n_1,n_k)} \binom{n_1}{i}\binom{n_k}{i} i! t'(n_2, \dots, n_{k-1}, n_1+n_k-i).$$

Proof. The elements $x \in N_k$ and $y \in N_1$ will be called *connected* by f if they belong to the same connected component of Γ_f. Decompose elements from T into classes with respect to the number of pairs of connected elements. For the set P of those elements of T^*, whose set of connected pairs is $\{(a_1, b_1), \dots, (a_i, b_i)\}$ we consider the map $\varphi : P \to T'(N_2, \dots, N_{k-1}, N_k \cup (N_1 \setminus \{b_1, \dots, b_i\}))$ defined as follows:

$$\begin{cases} \varphi(f)(x) = f(x), & x \in N_2 \cup \dots \cup N_k, f(x) \notin N_1 \\ \varphi(f)(x) = \varnothing, & f(x) \in B = \{b_1, \dots, b_i\} \\ \varphi(f)(x) = z, & x \in N_1 \setminus B; (f, z) \mapsto x. \end{cases}.$$

From the construction of φ it follows that it is surjective and $\varphi(f) = \varphi(g)$ if and only if $f^{\#}(B) = g^{\#}(B)$ and the restrictions of f and g on $\mathcal{N}_n \setminus f^{\#}(B)$ coincide. Now the statement follows by elementary combinatorial computation. $\square$

Proof of Theorem 17.1. By Theorem 16.1, $|\operatorname{Aut}(T)| = |S||G|$ and by definition, $|S| = \prod_{i=1}^{k} n_i!$. So, we have to compute $|G|$, which reduces to the computation of cardinalities of equivalence classes with respect to $\sim$.

If we decompose the elements of T into classes with respect to the cardinality of $\operatorname{dom}(f) \cap N_k$, we get

$$t(n_1, \dots, n_k) = \sum_{j=0}^{n_k} \binom{n_k}{j} t'(n_1, \dots, n_{k-1}, j).$$

Using the Möbius inversion we obtain

$$t'(n_1, \dots, n_k) = \sum_{j=0}^{n_k} (-1)^{n_k - j} \binom{n_k}{j} t(n_1, \dots, n_{k-1}, j).$$

After Lemma 17.2, each element of T' has exactly $i!$ pre-images under φ and hence for any pair (p, q), $0 \le p \le n_1$, $0 \le q \le n_k$, of parameters the relation $\sim$ on T has $\binom{n_1}{p}\binom{n_k}{q} t^*(p, n_2, \dots, n_{k-1}, q)$ equivalence classes of cardinality $t(n_1 - p, n_k - q)$ each. Distinct pairs of parameters correspond to disjoint groups of equivalence classes and each equivalence class is contained in one of these groups. As $t(0, b) = t(a, 0) = 1$ and one element equivalence classes does not effect on $|\operatorname{Aut}(T)|$, we can consider only $p < n_1$ and $q < n_k$. Now, using Lemma 17.1, we compute that a pair, (p, q) corresponds to

$$\binom{n_1}{p}\binom{n_2}{q} \sum_{i=0}^{\min(p,q)} \binom{p}{i}\binom{q}{i} i! t'(n_2, \dots, n_{k-1}, p + q - i) =$$

$$= \binom{n_1}{p}\binom{n_2}{q} \sum_{i=0}^{\min(p,q)} \binom{p}{i}\binom{q}{i} i! \sum_{j=0}^{p+q-i-j} (-1)^{p+q-i-j} \binom{p + q - i}{j} t(n_2, \dots, n_{k-1}, j)$$

equivalence classes of cardinality $t(n_1 - p, n_k - q)$ each. As l equivalence classes of cardinality r give the factor $(r!)^l$ in $|\operatorname{Aut}(T)|$, our statement follows. $\square$

For example, if $|N_1| = |N_2| = |N_3| = 3$, then $|T| = 2971$ and at the same time $|\operatorname{Aut}(T)| = (3!)^3(2!)^{324}(3!)^{324}(4!)^{36}(7!)^{81}(13!)^{18}34! \approx 2 \cdot 10^{916}$.

18 Automorphisms of $\mathcal{IS}_n$

We have already described all automorphisms of maximal nilpotent subsemigroups of $\mathcal{IS}_n$. At this stage the description of all automorphisms of $\mathcal{IS}_n$ itself will look very easy.

Theorem 18.1. *Any automorphism of $\mathcal{IS}_n$ is an inner automorphism, i.e. has the form $\varphi_g : x \mapsto g^{-1} \circ x \circ g$ for some $g \in \mathbb{S}_n$. In particular,* $\operatorname{Aut}(\mathcal{IS}_n) \simeq \mathbb{S}_n$.

First we recall that the analogous result for the group $\mathbb{S}_n$ looks a little bit different: $\operatorname{Aut}(\mathbb{S}_n) \simeq \mathbb{S}_n$ for $n \neq 2, 6$. The group $\mathbb{S}_2$ is abelian and does not have any non-trivial automorphisms, whereas $\mathbb{S}_6$ has an outer automorphism, which sends any transposition into a product of three commuting transpositions.

Lemma 18.1. *Let $\varphi \in \operatorname{Aut}(\mathcal{IS}_n)$. Then $\varphi(\mathbb{S}_n) = \mathbb{S}_n$, in particular, the restriction $\psi = \varphi|_{\mathbb{S}_n}$ is an automorphism of $\mathbb{S}_n$.*

Proof. Follows from Lemma 5.1, where $\mathbb{S}_n$ is described as the unique subgroup of $\mathcal{IS}_n$ of cardinality $n!$. □

Lemma 18.2. *Let $\varphi \in \operatorname{Aut}(\mathcal{IS}_n)$. Then for any $x \in \mathcal{N}_n$ there exists $y \in \mathcal{N}_n$ such that $\varphi(\varepsilon(x)) = \varepsilon(y)$.*

Proof. If $A \subset \mathcal{N}_n$, we have $\varepsilon_A = \prod_{z \in \mathcal{N}_n \setminus A} \varepsilon(z)$, which means that the only idempotents, which can not be decomposed into a non-trivial product of other idempotents are $\varepsilon(t)$'s. Hence the set of these idempotents should be preserved by φ. □

Lemma 18.3. *Let $n > 2$ and $\varphi \in \operatorname{Aut}(\mathcal{IS}_n)$ such that $\varphi|_{\mathbb{S}_n}$ is trivial. Then φ is trivial.*

Proof. Recall that $\mathcal{IS}_n$ is generated by $\mathbb{S}_n$ and $e = \varepsilon(1)$ (see Section 11). So, we have only to prove that $\varphi(e) = e$. By Lemma 18.2, $\varphi(e) = \varepsilon(x)$ for some $x \in \mathcal{N}_n$. Assume that $x \neq 1$. As $n > 2$, there exists $y \neq 1, x$. We have $(x, y) \circ e \circ (x, y) = e$. But, applying φ we get $(x, y) \circ \varepsilon(x) \circ (x, y) = \varepsilon(y) \neq x$, a contradiction. □

Lemma 18.4. *Let $\varphi \in \operatorname{Aut}(\mathcal{IS}_6)$. Then $\psi = \varphi|_{\mathbb{S}_6}$ is an inner automorphism of $\mathbb{S}_6$.*

Proof. Let $e = \varepsilon(x)$, $x \in \mathcal{N}_6$. If y and z are distinct elements from $\mathcal{N}_6$ such that $x \notin \{y, z\}$, we have $(x, y) \circ e = e \circ (x, y)$. So, e commutes with many transpositions. From the other hand, it is clear that e does not commute with any product of three commuting transpositions. Each $\varphi \in \operatorname{Aut}(\mathcal{IS}_6)$ should send e to some $e(y)$ and, if ψ is outer, all transpositions to products of three commuting transpositions and we get a contradiction. □

Proof of Theorem 18.1. Clearly all φ_g, $g \in \mathbb{S}_n$, are automorphisms of $\mathcal{IS}_n$. Let $\varphi \in \mathrm{Aut}(\mathcal{IS}_n)$ and $\psi = \varphi|_{\mathbb{S}_n}$. From Lemma 18.4 it follows that ψ is an inner automorphism of $\mathbb{S}_n$. Let us assume that φ is the conjugation with $g^{-1} \in \mathbb{S}_n$. Consider $\tau = \varphi_g \circ \varphi$. The restriction of the last one to $\mathbb{S}_n$ is trivial. Hence, if $n > 2$, τ is trivial by Lemma 18.3 and thus $\varphi = \varphi_{g^{-1}}$. Let $n = 2$. Then ψ is always trivial and either $\varphi(e_{\{2\}}) = e_{\{2\}}$, hence φ is trivial, or $\varphi(e_{\{2\}}) = e_{\{1\}}$, hence $\varphi = \varphi_{(1,2)}$. This completes the proof. $\square$

19 Endomorphisms of $\mathcal{IS}_n$

In contrast with description of automorphisms, the description of endomorphisms of $\mathcal{IS}_n$ is not so concise and unified. In particular, cases $n = 2, 4, 6$ happened to differ from all the others. We will start the description with the definition of several endomorphisms, which we will then prove to be all.

Lemma 19.1. *Let $\pi \in \mathrm{Aut}(\mathbb{S}_n)$ and define the map $\varphi_\pi : \mathcal{IS}_n \to \mathcal{IS}_n$ as follows: $\varphi_\pi(f) = \pi(f)$, $f \in \mathbb{S}_n$, and $\varphi_\pi(f) = \mathbf{0}$, $f \in \mathcal{IS}_n \setminus \mathbb{S}_n$. Then $\varphi_\pi \in \mathrm{End}(\mathcal{IS}_n)$.*

Proof. We have to check $\varphi_\pi(x \circ y) = \varphi_\pi(x) \circ \varphi_\pi(y)$. If $x, y \in \mathbb{S}_n$ this reduces to $\pi(x \circ y) = \pi(x) \circ \pi(y)$, which is true as π is an automorphisms of $\mathbb{S}_n$, and if at least one of x, y does not belong to $\mathbb{S}_n$, both sides of the equality equal $\mathbf{0}$. $\square$

We note that for $n \neq 6$ the elements π above are conjugations with respect to an element from $\mathbb{S}_n$, i.e. has the form $x \mapsto g^{-1} \circ x \circ g$ for some $g \in \mathbb{S}_n$, whereas for $n = 6$ the non-inner automorphisms are also involved. To formulate the next lemma we need a new notation. For $i, j \in \mathcal{N}_n$ by $f_{i,j}$ we will denote the element of rank 1 such that $f_{i,j}(i) = j$.

Lemma 19.2. *Let $g \in \mathbb{S}_n$ and define the map $\psi_g : \mathcal{IS}_n \to \mathcal{IS}_n$ as follows: $\psi_g(f) = g^{-1} \circ f \circ g$, $f \in \mathbb{S}_n$; $\psi_g(f) = f_{g(i),g(j)}$, $\mathrm{rank}(f) = n-1$, $i \notin \mathrm{dom}(f)$, $j \notin \mathrm{ran}(f)$, and $\psi_g(f) = \mathbf{0}$, $\mathrm{rank}(f) < n-1$. Then $\psi_g \in \mathrm{End}(\mathcal{IS}_n)$.*

Proof. Again we have to check $\psi_g(x \circ y) = \psi_g(x) \circ \varphi_g(y)$. If $x, y \in \mathbb{S}_n$ this reduces to $\pi(x \circ y) = \pi(x) \circ \pi(y)$, which is true as π is an automorphisms of $\mathbb{S}_n$. If at least one of x, y belongs to I_{n-2}, both sides of the equality equal $\mathbf{0}$. The same happens if $x, y \in I_{n-1} \setminus I_{n-2}$ $\mathrm{ran}(y) \neq \mathrm{dom}(x)$. Finally, if $i \notin \mathrm{dom}(y)$, $j \notin \mathrm{ran}(y) = \mathrm{dom}(x)$ and $k \notin \mathrm{ran}(x)$ then $i \notin \mathrm{ran}(x \circ y)$, $k \notin \mathrm{ran}(x \circ y)$ and the equality reduces to $f_{i,k} = f_{j,k} \circ f_{i,j}$, which is true. This completes the proof. $\square$

Lemma 19.3. *Choose $a, b \in \mathcal{IS}_n$ such that $a^3 = a$ and $b^2 = a \circ b = b \circ a = b$ and define $\xi_{a,b}(f) = b$, $f \in \mathbb{S}_n \setminus \mathbb{A}_n$; $\xi_{a,b}(f) = b^2$, $f \in \mathbb{A}_n$, and $\xi_{a,b}(f) = a$, $a \in \mathcal{IS}_n \setminus \mathbb{S}_n$. Then $\xi_{a,b}$ is an endomorphisms of $\mathcal{IS}_n$.*

Proof. By the choose of a and b, the set $\{a, a^2, b\}$ is a semigroup with identity a^2 and zero b. So, $\xi_{a,b}|_{\mathbb{S}_n}$ is the projection of $\mathbb{S}_n$ to $\mathbb{S}_n/\mathbb{A}_n$, hence a homomorphisms, and if at least one of x, y does not belong to $\mathbb{S}_n$, the both sides of $\xi_{a,b}(x \circ y) = \xi_{a,b}(x) \circ \xi_{a,b}(y)$ equal b. This completes the proof. $\square$

We note that if $a^2 = a = b$ then the image of $\xi_{a,b}$ has one element which is an idempotent. If $a^2 = a \neq b$ the image of $\xi_{a,b}$ has two elements, both being idempotents (automatically commuting). In all other cases the image is $\{a, a^2, b\}$, which is a three-element semigroup with unit a^2 and zero b. The set $\{a, a^2\}$ is a two-element subgroup. We are almost done and have to consider only some special endomorphisms in the case $n = 4$. In this case we denote by K the normal subgroup $\{\mathbf{1}; (1,2)(3,4); (1,3)(2,4); (1,4)(2,3)\}$ of $\mathbb{S}_4$. Each of six cosets Kg, $g \in \mathbb{S}_4$, contains exactly one permutation $\sigma_1(g)$, which fixes 4. Let $\sigma_0(g)$ denote the permutation of $\{1,2,3\}$ induced by $\sigma_1(g)$, so we have two maps $\sigma_1 : \mathbb{S}_4 \to \mathbb{S}_4$ and $\sigma_0 : \mathbb{S}_4 \to \mathbb{S}_3$.

Lemma 19.4. *Let $n = 4$. Define $\sigma_{i,j} : \mathcal{IS}_4 \to \mathcal{IS}_4$, $i, j \in \{0, 1\}$, as follows:*

$$\sigma_{i,0}(f) = \begin{cases} \sigma_i(f), & f \in \mathbb{S}_4 \\ \mathbf{0}, & f \in \mathcal{IS}_4 \setminus \mathbb{S}_4, \end{cases} \qquad \sigma_{i,1}(f) = \begin{cases} \sigma_i(f), & f \in \mathbb{S}_4 \\ \varepsilon_{\{1,2,3\}}, & f \in \mathcal{IS}_4 \setminus \mathbb{S}_4, \end{cases}$$

and for $g \in \mathbb{S}_4$ set $\sigma_{i,j}^g(f) = g^{-1} \circ \sigma_{i,j}(f) \circ g$. Then all $\sigma_{i,j}^g$ are endomorphisms of $\mathcal{IS}_4$.

Proof. Checking $\sigma_{i,j}^g(x \circ y) = \sigma_{i,j}^g(x) \circ \sigma_{i,j}^g(y)$ as in the previous lemma the case $x, y \in \mathbb{S}_4$ reduces to an endomorphism of $\mathbb{S}_4$ and in all other cases both sides are equal $\mathbf{0}$ (resp. $\varepsilon_{\{1,2,3\}}$). □

Now we are ready to state our classification result.

Theorem 19.1. 1. *If $n \neq 4$ then each endomorphism of $\mathcal{IS}_n$ is either an automorphism or coincides with one of φ_π, ψ_g or $\xi_{a,b}$.*

2. *Each endomorphism of $\mathcal{IS}_4$ is either an automorphism or coincides with one of φ_π, ψ_g, $\xi_{a,b}$ or $\sigma_{i,j}^g$.*

We note that for $n = 2$ the endomorphism ψ_g, $g \in \mathbb{S}_2$, are in fact automorphisms and $\varphi_{\mathbf{1}} = \varphi_{(1,2)} = \xi_{(1,2),\mathbf{0}}$. For $n = 1$ the list reduces to the trivial automorphism and two endomorphisms $\xi_{\mathbf{1},\mathbf{1}}$ and $\xi_{\mathbf{0},\mathbf{0}}$.

Proof. We left the trivial case $n = 1$ to the reader and assume $n > 1$.

Step 1. Let $\varphi \in \mathrm{End}(\mathcal{IS}_n)$ and $\psi = \varphi_{\mathbb{S}_n}$. First we assume that the kernel of ψ is trivial. This implies that ψ is an automorphism of $\mathbb{S}_n$. Let $n \neq 6$. Then ψ is an inner automorphism and hence there exist an inner automorphism, ψ', of $\mathcal{IS}_n$, whose restriction to $\mathbb{S}_n$ coincides with ψ. Set $\varphi' = (\psi')^{-1} \circ \varphi$ and we get $\varphi'(x) = x$, $x \in \mathbb{S}_n$. As $\mathcal{IS}_n$ is generated by $\mathbb{S}_n$ and $e = \varepsilon_{\mathcal{N}_n \setminus \{1\}}$ it is enough to determine $f = \varphi'(e)$, which is an idempotent, say $f = \varepsilon_A$. Now, if $1 \neq yx, \in \mathcal{N}_n$, then $e \circ (x, y) = (x, y) \circ e$ and hence $f \circ (x, y) = (x, y) \circ f$, which implies that either $\{x, y\} \subset A$ or $\{x, y\} \cap A = \varnothing$. Hence either $\mathcal{N}_n \setminus \{1\} \subset A$ or $\mathcal{N}_n \setminus \{1\} \cap A = \varnothing$. This reduces the possibilities for A to $\mathcal{N}_n$, $\varnothing$, $\{1\}$ and $\mathcal{N}_n \setminus \{1\}$. If $A = \{1\}$, then φ' is the identity and hence φ is an automorphism. If $A = \mathcal{N}_n \setminus \{1\}$, then $\varphi' = \psi_{\mathbf{1}}$ and hence $\varphi = \psi_g$ for some $g \in \mathbb{S}_n$. If $A = \varnothing$, then $\varphi' = \varphi_{id}$, where id is the identity automorphism of $\mathbb{S}_n$. Let us prove that $A = \mathcal{N}_n$ is impossible.

Indeed, in this case $\varphi'(e) = \mathbf{1}$ and hence $\varphi'(\varepsilon_{\mathcal{N}_n\setminus\{x\}}) = \varphi'((1,x)) \circ \varphi'(e) \circ \varphi'((1,x)) = \mathbf{1}$ for any $x \in \mathcal{N}_n$. Decomposing all other non-invertible idempotents into a product of $\varepsilon_{\mathcal{N}_n\setminus\{x\}}$, we get $\varphi'(g) = \mathbf{1}$ for any $g^2 = g \in \mathbb{S}_n$. In particular, $\varphi'(\mathbf{0}) = \mathbf{1}$ and hence $\mathbf{1} = \varphi'(\mathbf{0}) = \varphi'(h \circ \mathbf{0}) = \varphi'(h) \circ \varphi'(\mathbf{0}) = \varphi'(h)$ for any $h \in \mathcal{IS}_n$, which contradicts the fact that $\varphi'|_{\mathbb{S}_n}$ is identical and $n > 1$.

Now assume that $n = 6$ in the situation above. For this case the only difference with arguments above is that ψ can be non-inner. Let us prove that this is possible only if $\varphi(\mathcal{IS}_n \setminus \mathbb{S}_n) = \mathbf{0}$. Indeed, let $f = f^2 = \varphi(e)$. As e commutes with all transpositions (x,y), $x,y \neq 1$, we get that f commutes with all their images, which are products of three transpositions and direct calculation gives us $f = \mathbf{1}$ or $f = \mathbf{0}$. But the first case is impossible by the same arguments as in the previous paragraph, so $f = \mathbf{0}$ and hence $\varphi(\mathcal{IS}_n \setminus \mathbb{S}_n) = \mathbf{0}$.

Step 2. Now assume that $\mathrm{Ker}(\varphi)$ is not trivial. Then we have three possibilities: $\mathrm{Ker}(\varphi) = \mathbb{S}_n$, $\mathrm{Ker}(\varphi) = \mathbb{A}_n$ and $n = 4$ and $\mathrm{Ker}(\varphi) = K$. Consider now the first two cases. Set $b = \varphi(e)$ and $a = \varphi((1,2))$. Then $a^2 = \varphi(\mathbf{1})$ and we obviously get $b \circ a^2 = b$. If $n = 2$ then $\mathrm{Ker}(\varphi)$ non-trivial implies $a = a^2$ commutes with b. Let $n > 2$. Then $a = \varphi((2,3))$ commutes with b as $(2,3)$ commutes with e. Hence for all n from $e_{\mathcal{N}_n\setminus\{x\}} = (1,x) \circ e \circ (1,x)$ we also get $\varphi(e_{\mathcal{N}_n\setminus\{x\}}) = a \circ b \circ a = b$ and hence $\varphi(g) = b$ for any non-invertible idempotent g, in particular, for $h = \varepsilon_{\mathcal{N}_n\setminus\{1,2\}}$. But $(1,2) \circ h = h \circ (1,2) = h$ and hence $a \circ b = b \circ a = b$. Now any element from $\mathcal{IS}_n \setminus \mathbb{S}_n$ can be expressed as product of elements from $\mathbb{S}_n$ and e, moreover, at least one e should be present. This implies $\varphi(\mathcal{IS}_n \setminus \mathbb{S}_n) = b$ and $\varphi = \xi_{a,b}$.

Step 3. To complete the proof we are left to consider the case $n = 4$ and $\mathrm{Ker}(\psi) = K$. Multiplying with an inner automorphism of $\mathcal{IS}_n$, we can assume that $\varphi(\mathbb{S}_4)$ either stabilizes 4 or is not defined in this point. Using the arguments of Step 1, we also get that $\varphi(\varepsilon_{\{1,2,3\}})$ is either $\mathbf{0}$ or $\varepsilon_{\{4\}}$ and using the arguments of Step 2 we get that $\varphi(\mathcal{IS}_4 \setminus \mathbb{S}_4) = \varphi(\varepsilon_{\{1,2,3\}})$. Hence φ coincides with one of $\sigma_{i,j}$. This completes the proof. $\square$

20 Transitive representations of $\mathcal{IS}_n$ by partial bijections

The aim of this section is to study representations of $\mathcal{IS}_n$ by partial bijections, that is to study homomorphism from $\mathcal{IS}_n$ into $\mathcal{IS}(X)$ subject to some additional conditions. To formulate our main result here we will need to recall some standard notions concerning representations. Let S be an inverse semigroup. A *representation of S by partial bijections* is a homomorphism, $\varphi : S \to \mathcal{IS}(X)$, for some X. This representation is called *effective* provided $X = \cup_{f\in S} \mathrm{dom}(\varphi(f))$, and *transitive* provided for and $x, y \in X$ there exists $f \in S$ such that $\varphi(f)(x) = y$. Effective transitive representations of S are closely connected with the notion of closed subsemigroups in S, in fact they can be realized as representations on the cosets with respect to such subsemigroups. These are defined in the following way. Let $H \subset S$. The *closure* $H\omega$ of H is the set $\{f \in S | g\omega f \text{ for some } g \in S\}$. H is called *closed* if $H = H\omega$. We first describe all closed subsemigroups in $\mathcal{IS}_n$. It worth to note that for

$f, g \in \mathcal{IS}_n$ the relation $f\omega g$ holds if and only if $\mathrm{dom}(f) \subset \mathrm{dom}(g)$ and $f|_{\mathrm{dom}(f)} = g|_{\mathrm{dom}(f)}$. Our main result will be the following.

Theorem 20.1. *Let $M \subset \mathcal{N}_n$ and G is a subgroup of $\mathbb{S}(M)$. Then $G \oplus \mathcal{IS}(\mathcal{N}_n \setminus M)$ is a closed inverse subsemigroup of $\mathcal{IS}_n$, moreover, each closed inverse subsemigroup of $\mathcal{IS}_n$ is of this form.*

Proof. Obviously, each $H = G \oplus \mathcal{IS}(\mathcal{N}_n \setminus M)$ from formulation is an inverse subsemigroup of $\mathcal{IS}_n$. Let $h \in H$ and $h\omega a$. Then $M \subset \mathrm{dom}(h) \subset \mathrm{dom}(a)$ and $h(x) = a(x)$ for all $x \in M$. Hence M is invariant under a and $a|_M \in G$. Hence $a = a|_M \oplus a|_{\mathcal{N}_n \setminus M}$ and $a \in H$. Therefore H is closed.

Now let H be a closed inverse subsemigroup of $\mathcal{IS}_n$. Then $E(H)$ is a lower semi-lattice with respect to the divisability order and hence contains the minimal idempotent, say e. Set $M = \mathrm{dom}(e)$. First we prove that $f(M) = M$ for any $f \in H$. First of all $M \subset \mathrm{dom}(f)$ for any $f \in H$. Indeed, otherwise there should exist $g \in H$ and $x \in M$ such that $x \notin \mathrm{dom}(g)$. Consider the idempotent $f = e \circ g \circ g^{\#} \in H$. We have $\mathrm{dom}(f) = \mathrm{dom}(g \circ g^{\#}) \cap \mathrm{dom}(e) \neq M$ and hence $f \neq e$, which contradicts the minimality of e. Second, take $f \in H$ and let $f(M) = M'$. Consider $f \circ e \in H$. Then $\mathrm{dom}(f \circ e) = \mathrm{dom}(e) = M$ and $\mathrm{dom}((f \circ e)^{\#}) = M'$. But $g = (f^{\#} \circ e)^{\#} \circ (f^{\#} \circ e) \in E(H)$ hence $\mathrm{dom}(g) = M' \cap M$ should contain M as the idempotent e is minimal. This implies $M' = M$.

As the next step we prove that $e \oplus \mathcal{IS}(\mathcal{N}_n \setminus M) \subset H$. Indeed, for any $s = e \oplus h$, $h \in \mathcal{IS}(\mathcal{N}_n \setminus M)$, we have $se = e$, hence $e\omega s$ and $s \in H$ as H is closed.

We proceed with the fact that $H \circ e$ is a subgroup of H. We have $\mathrm{dom}(f) = M$ for any $f \in H \circ e$ and $f(M) = M$ from above. This implies that $f^k = e$ for some k and hence e is a unit in $H \circ e$ and all elements of $H \circ e$ are invertible. Clearly, $H \circ e \subset \mathbb{S}(M)$.

And finally we claim that $H = H \circ e \oplus \mathcal{IS}(\mathcal{N}_n \setminus M)$. From the previous paragraph we have $H \subset H \circ e \oplus \mathcal{IS}(\mathcal{N}_n \setminus M)$, but it is easy to see that for any $h \in H$ we have $h \circ e \oplus \mathcal{IS}(\mathcal{N}_n \setminus M) \subset H$ and hence both semigroups should coincide. $\square$

Two representations $\varphi : S \to \mathcal{IS}(X)$ and $\psi : S \to \mathcal{IS}(Y)$ are called *equivalent* if there exists a bijection, $\theta : X \to Y$, such that $\theta(\varphi(s)(x)) = \psi(s)(\theta(x))$ for all $x \in X$ and $s \in S$. The last essentially says that $\psi(s)(x) = y$ is equivalent to $\psi(\theta(x)) = \theta(z)$ for all $x, z \in X$ and $s \in S$.

The Schein's construction of representation involves the following data: an inverse semigroup, S, and a closed inverse subsemigroup, H, of S. Set $\pi_H = \{(s,t) \in S \times S | t^{\#}s \in H\}$. It is a partial left congruence on S with the domain $D_H = \{s \in S | s^{\#}s \in H\}$. The equivalence classes modulo π_H are $(sH)\omega$, $s \in D_H$. In particular, $s \in (sH)\omega$ and H is an equivalence class. Denote by X the set of these equivalence classes, i.e. the set of all $(sH)\omega$, $s^{\#}s \in H$. Define the map $\varphi_H : S \to \mathcal{IS}(X)$ as follows: $(\varphi_H(s))((tH)\omega) = (stH)\omega$ if $(stH)\omega \in X$ otherwise $(\varphi_H(s))((tH)\omega)$ is not defined. This is a representation of S in $\mathcal{IS}(X)$ which is called the *representation of S on left ω-classes with respect to H.*

Lemma 20.1. *Let S be an inverse semigroup and H be a closed inverse subsemigroup of S. Then the representation of S on left ω-classes with respect to H is effective and* *transitive. Conversely, each effective and transitive representation of S is equivalent to the representation on left ω-classes with respect to some closed inverse subsemigroup.*

Proof. The equalities $s(H\omega) = (sH)\omega$ and $s^{\#}(sH)\omega = ((s^{\#}s)H)\omega = H\omega$ imply that the representation of S on left ω-classes with respect to H is effective and transitive.

Conversely, let φ be an effective and transitive representation of S in $\mathcal{IS}(x)$. Fix $x \in X$ and denote by $H = H_x$ the semigroup of all $s \in S$, which stabilize x, in particular, they are defined on x. The equality $ss^{\#}s = s$ guarantees that H is an inverse semigroup. Let $h \in H$ and $s \in S$ such that $h\omega s$. Then $x = hh^{\#}(x) = sh^{\#}(x) = s(x)$ and hence $s \in H$. This means that H is closed. If $y \in X$ then the transitivity of φ implies that there exists $s_y \in S$ such that $s_y(x) = y$. Then the left ω-class $(s_yH)\omega$ belongs to the stabilizer H_y of y. But, conversely, $s_y^{\#}(H_y) \subset H_x$, from which it follows that the map $y \mapsto (s_yH)\omega$ provided an equivalence between φ and the representation of S on left ω-classes with respect to H. This completes the proof. $\square$

The following Lemma cane be found in [KP, Theorem 7.27] and we refer the reader to this book for the proof.

Lemma 20.2. *Let S be an inverse semigroup and H_1, H_2 be two closed inverse subsemigroups of S. Then φ_{H_1} and φ_{H_2} are equivalent if and only if there exists $a \in S$ such that $aa^{\#} \in H_1$, $a^{\#}a \in H_2$, $a^{\#}H_1a \subset H_2$ and $aH_2a^{\#} \subset H_1$.*

Now we will use Lemma 20.1 to describe all *faithful* (i.e. monomorphic) effective transitive representations of $\mathcal{IS}_n$ in the following statement.

Theorem 20.2. *Each faithful effective transitive representation of $\mathcal{IS}_n$ is equivalent to standard representation $id : \mathcal{IS}_n \to \mathcal{IS}_n$.*

Lemma 20.3. *Let $M \subset \mathcal{N}_n$, G a subgroup of $\mathbb{S}(M)$ and $H = G \oplus \mathcal{IS}(\mathcal{N}_n \setminus M)$. Then the number of left ω-classes of $\mathcal{IS}_n$ with respect to H equals $n!/(|G| \cdot |\mathcal{N}_n \setminus M|!)$, each class has the form $g \circ H$ for some $g \in \mathbb{S}_n$ and all classes have the same cardinality $|H|$.*

Proof. If $g \in \mathbb{S}_n$, $g^{\#} \circ g = \mathbf{1} \in H$ and hence each such g is contained in $(g \circ H)\omega$. Hence $(g_1 \circ H)\omega = (g_2 \circ H)\omega$ if and only if $g_2^{-1} \circ g_1 \in H \cap S_n = G \oplus \mathbb{S}(\mathcal{N}_n \setminus M)$. Hence $(g_1 \circ H)\omega = (g_2 \circ H)\omega$ if and only if g_1 and g_2 belong to the same left class of $\mathbb{S}_n$ modulo $G \oplus \mathbb{S}(\mathcal{N}_n \setminus M)$. In particular, the number of left ω-classes of such form is precisely $n!/(|G| \cdot |\mathcal{N}_n \setminus M|!)$. As for $g \in \mathbb{S}_n$ $x \mapsto g \circ x$ is a bijective map on $\mathcal{IS}_n$, all the above classes have the same cardinality $|H|$.

It is left to prove that each left ω-class has the above form. Consider some class, say $(a \circ H)\omega$. The set of maximal elements with respect to ω on $\mathcal{IS}_n$ is $\mathbb{S}_n$ hence there exists $g \in S_n$ such that $a\omega g$ and thus $g \in (a \circ H)\omega$ as the last is closed. We get $(a \circ H)\omega = (g \circ H)\omega$ and the proof is completed. $\square$

Lemma 20.4. *The representation of $\mathcal{IS}_n$ on left ω-classes with respect to $H_1 = G_1 \oplus \mathcal{IS}(\mathcal{N}_n \setminus M_1)$ and $H_2 = G_2 \oplus \mathcal{IS}(\mathcal{N}_n \setminus M_2)$ are equivalent if and only if the actions (G_1, M_1) and (G_1, M_2) are equivalent.*

Proof. First we prove the "only if" part. We use Lemma 20.2 and find $a \in \mathcal{IS}_n$ such that $a^{\#} \circ a \in H_1$, $a \circ a^{\#} \in H_2$, $aH_1a^{\#} \subset H_2$ and $a^{\#}H_2a \subset H_1$. Then $\mathrm{dom}(a^{\#}a) = \mathrm{dom}(a) \supset M_1$ and $\mathrm{dom}(aa^{\#}) = \mathrm{dom}(a^{\#}) \supset M_2$. Set $A_1 = \mathrm{dom}(a)$, $A_2 = \mathrm{dom}(a^{\#})$. We want to show that $a^{\#}(M_2) = M_1$. If not, there exists $x \in M_2$ such that $a^{\#}(x) \in A_1 \setminus M_1$. Take $\pi \in H_1$ such that $\pi(a^{\#}(x)) = \varnothing$, which obviously exists, and we have $a \circ \pi \circ a^{\#}(M_2) \neq M_2$, which means $aH_1a^{\#} \not\subset H_2$. Obtained contradiction implies $a^{\#}(M_2) = M_1$ and analogously $a(M_1) = M_2$.

Set $X_i = \{(g \circ H_i)\omega, g \in \mathbb{S}_n\}$, $i = 1, 2$, and let θ denote the bijection from X_1 to X_2, which provides an equivalence of φ_{H_1} and φ_{H_2}. From Lemma 20.3 it follows that each left ω-class with respect to H_i contains a left coset of $\mathbb{S}_n$ modulo $B_i = G_i \oplus \mathbb{S}(\mathcal{N}_n \setminus M_i)$, $i = 1, 2$. Hence θ induces a bijection, say $\hat{\theta}$, between left cosets of $\mathbb{S}_n$ modulo B_i. It is easy to see that this bijection is in fact an equivalence of actions $(\mathbb{S}_n, \mathbb{S}_n/B_1)$ and $(\mathbb{S}_n, \mathbb{S}_n/B_2)$. In particular, $|\mathbb{S}_n/B_1| = |\mathbb{S}_n/B_2|$ and hence $|G_1| = |G_2|$. Choose $s \in \mathbb{S}_n$ such that $\hat{\theta(B_2)} = s \circ B_1$. Consider any $g \in G_2$ and set $\varphi_{G_i} = \varphi_{H_i}|_{G_i}$, $i = 1, 2$. We have $B_2^{\varphi_{G_2}(g)} = B_2$ and hence $g \circ s \circ B_1 = s \circ B_2$, which is equivalent to $g \in s \circ A_1 \circ s^{-1}$. Set $s_* = s|_{M_1}$ and $s_*^{\#} = s^{\#}|_{M_2}$. Then $g \in s_*^{\#} \circ G_1 \circ s_*$ and therefore $G_2 = s_*^{\#} \circ G_1 \circ s_*$, which means that G_1 and G_2 are conjugated in $\mathbb{S}_n$.

Define $\psi : G_1 \to G_2$ by $\psi(g) = s_*^{\#} \circ g \circ s_*$, $g \in G_1$. At the same time s_* defines a bijection, $\tau : M_1 \to M_2$, by $\tau(x) = s_*(x)$, $x \in M_1$. Moreover, $\psi(g)(\tau(x)) = s_*^{\#} \circ g \circ s_*(s_*(x)) = \tau(g(x))$ and hence the actions (G_1, M_1) and (G_2, M_2) are equivalent.

We proceed with the "if" part. We have that the actions (G_1, M_1) and (G_2, M_2) are equivalent. Let $\tau : M_1 \to M_2$ and $\psi : G_1 \to G_2$ be some bijection and isomorphism, which give us this equivalence. Then $|G_1| = |G_2|$ and $|M_1 = |M_2|$. In particular, $|\mathbb{S}_n/B_1| = |\mathbb{S}_n/B_2|$ and hence $|X_1| = |X_2|$. Choose $s \in \mathbb{S}_n$ such that $s(x) = \tau(x)$ for all $x \in M_1$. For any $g \in G_1$ and $y \in M_2$ we have $\psi(g)(y) = \tau(g(\tau^{-1}(y)))$, which gives $\psi(g) = s \circ g \circ s^{\#}$. Hence $G_2 = s \circ G \circ s^{\#}$ and G_1 is conjugated with G_2 in $\mathbb{S}_n$.

Take $f \in \mathcal{IS}_n$ such that $f|_{M_1} = s|_{M_1}$ and $f(\mathcal{N}_n \setminus M_1) = \varnothing$. Then $f^{\#}|_{M_2} = s^{\#}|_{M_2}$ and $f^{\#}(\mathcal{N}_n \setminus M_2) = \varnothing$. Moreover, $f^{\#} \circ f \in H_1$ and $f \circ f^{\#} \in H_2$. Let $a = f \circ h \circ f^{\#}$, $h \in H_1$. Clearly $a(M_2) = M_2$ and $a(\mathcal{N}_n \setminus M_2) = \varnothing$. Set $g = h|_{M_1}$ and take any $y \in M_2$. We have $a(y) = f \circ h \circ f^{\#}(y) = f \circ h \circ s^{\#}(y) = f \circ h(\tau^{-1}(y)) = f \circ g(\tau^{-1}(y)) = s(g(\tau^{-1}(y))) = \tau(g(\tau^{-1}(y))) = \psi(g)(\tau(\tau^{-1}(y))) = \psi(g)(y)$. But $\psi(g) \in G_2$ and hence $a \in H_2$, which means $f \circ H_1 \circ f^{\#} \subset H_2$. The same arguments imply $f^{\#} \circ H_2 \circ f \subset H_1$, which in turn means that φ_{H_1} and φ_{H_2} are equivalent by Lemma 20.2. $\square$

Proof of Theorem 20.2. Let $H = G \oplus \mathcal{IS}(\mathcal{N}_n \setminus M)$ be a closed inverse subsemigroup of $\mathcal{IS}_n$ and φ_H be the corresponding representation of $\mathcal{IS}_n$ on left ω-classes with respect to H, which we assume to be faithful. If $M = \varnothing$ then $H = \mathcal{IS}_n$ and φ_H is obviously non-faithful, so $M \neq \varnothing$. Consider an idempotent, $e \in \mathcal{IS}_n$, such that $|\,\mathrm{dom}(e)| < |M|$. If $\varphi_H(e)$ is defined on some left ω-class, $(s \circ H)\omega$, then $e \circ s \in ((s \circ H)\omega)^{\varphi_H(e)}$ and $|\,\mathrm{dom}(e \circ s)| \leq |\,\mathrm{dom}(e)| < |M|$. But, by Lemma 20.3, any left class is $g \circ H$ for some $g \in \mathbb{S}_n$, hence $|\,\mathrm{dom}(x)| \geq |M|$ for any $x \in g \circ H$. This contradiction means that $\varphi_H(e)$ is not defined on all classes and thus $e = \mathbf{0}$ by faithfulness of φ_H. Hence, $|M| = 1$. Thus G is trivial and from Lemma 20.4 it now follows that all there exists exactly one, up to equivalence, faithful effective and transitive representation of $\mathcal{IS}_n$. But the standard representation

$id : \mathcal{IS}_n \to \mathcal{IS}_n$ is faithful, effective and transitive. This completes the proof. $\square$

21 Appendix I: Theorem of Preston and Wagner

Here we present one of the most famous results about $\mathcal{IS}_n$, the so-called Theorem of Preston and Wagner, which illustrates the universal property of $\mathcal{IS}_n$ to contain all inverse semigroups. This is analogous to the Kelly's Theorem for $\mathbb{S}_n$.

Theorem 21.1. *Let $(T, *)$ be an inverse semigroup with identity. Then $(T, *)$ is an inverse subsemigroup of $\mathcal{IS}(T)$.*

Proof. With each $a \in T$ we associate a partial transformation, $\varphi_a : T * a^\# \to T * a$, defined by $\varphi_a(x) = x * a$, $x \in T * a^{-1}$. We have to check three things. First, we want φ_a to be a partial bijection. Second, we want the map $a \mapsto \varphi_a$ to be injective. And third, we want the last map to be a homomorphism of inverse semigroups.

First we prove that φ_a is a partial bijection, i.e. that for $x, y \in T * a^\#$ the equality $x * a = y * a$ implies $x = y$. As $x, y \in T * a^\#$, there exist $x_1, y_1 \in T$ such that $x = x_1 * a^\#$ and $y = y_1 * a^\#$. From $x * a = y * a$ we get $x_1 * a^\# * a = y_1 * a^\# * a$. Multiplying with $a^\#$ from the right, we get $x_1 * a^\# * a * a^\# = y_1 * a^\# * a * a^\#$. But $x_1 * a^\# * a * a^\# = x_1 * a^\#$ and $y_1 * a^\# * a * a^\# = y_1 * a^\#$ and hence $x = x_1 * a^\# = x_1 * a^\# * a * a^\# = y_1 * a^\# * a * a^\# = y_1 * a^\# * a * a^\# = y_1 * a^\# = y$.

Now let us show that $a \mapsto \varphi_a$ is an injection. Assume that $\varphi_a = \varphi_b$ and consider the idempotents $a * a^\#$ and $b * b^\#$. If we succeed to prove that $a * a^\# = b * b^\#$, from $\varphi_a = \varphi_b$ we will get

$$a = a * a^\# * a = \varphi_a(a * a^\#) = \varphi_b(a * a^\#) = \varphi_b(b * b^\#) = b * b^\# * b = b.$$

As T is inverse, all idempotents of it commute and we have $a*a^\#*b*b^\# = b*b^\#*a*a^\#$. Further $T * a * a^\# \subset T * a^\# = T * a^\# * a * a^\# \subset T * a * a^\#$ and we have $T * a^\# = T * a * a^\#$. From $\varphi_a = \varphi_b$ we now have $T * a * a^\# = T * a^\# = T * b^\# = T * b * b^\#$. Hence, there exist $x, y \in T$ such that $a * a^{-1} = x * b * b^\#$ and $b * b^\# = y * a * a^\#$. It is left only to calculate:

$$\begin{aligned} a * a^\# = x * b * b^\# = x * b * b^\# * b * b^\# = a * a^\# * b * b^\# = \\ = b * b^\# * a * a^\# = y * a * a^\# * a * a^\# = y * a * a^\# = b * b^\# \end{aligned}$$

and the second step is completed.

So, the only thing we are left to check is that $a \mapsto \varphi_a$ is a homomorphism. We know already that $\varphi_a : T * a^\# \to T * a$ and $\varphi_{a^\#} : T * a \to T * a^\#$. We have

$$x * a^\# \xmapsto{\varphi_a} x * a^\# * a \xmapsto{\varphi_{a^\#}} x * a^\# * a * a^\# = x * a^\#$$

and hence φ_a and $\varphi_{a^\#}$ are inverse to each other bijections between $T * a^\#$ and $T * a$. It is obvious that $\varphi_b(\varphi_a(x)) = x * a * b = \varphi_{a*b}(x)$, if $x \in T * a^\#$ and $x * a \in T * b^\#$. So, to complete the proof we have to show that $\mathrm{dom}(\varphi_b \circ \varphi_a)$ coincides with $\mathrm{dom}(\varphi_{a*b})$. We have $\mathrm{dom}(\varphi_b \circ \varphi_a) = (\varphi_a)^{-1}(T * a \cap T * b^{\#}) = \varphi_{a^{\#}}(T * a \cap T * b^{\#}) = (T * a \cap T * b^{\#}) * a^{\#}$ and we have to check that this coincides with $T * (a * b)^{\#} = T * b^{\#} * a^{\#}$. The inclusion $(T * a \cap T * b^{\#}) * a^{\#} \subset T * b^{\#} * a^{\#}$ is obvious. Let $x \in T * b^{\#} * a^{\#}$. Then $x = y * b^{\#} * a^{\#} = y * b^{\#} * a^{\#} * a * a^{\#} \in T * a * a^{-1}$ and the proof of our theorem is completed by applying the canonical anti-automorphism of $\mathcal{IS}(T)$. $\square$

22 Appendix II: Theorem of Nemirovskaya

The classical Theorem of Frucht, [F], states that each finite group is an automorphism group of a finite non-oriented graph. The aim of this appendix is to prove an analogous result for inverse semigroups. As a first step for this it is natural to switch from automorphisms to partial automorphisms, which we define as follows. Let Γ be a graph, a *partial automorphism* of Γ is a partial bijection, φ, on the set $V(\Gamma)$ of all vertices of Γ satisfying the following: for any vertices x, y from the domain of φ the existence of the arrow (x, y) is equivalent to the existence of the arrow $(\varphi(x), \varphi(y))$, i.e. that "partiality" of the map essentially relates to the vertices of Γ. It is natural to ask if any finite inverse semigroup is a semigroup of partial automorphisms of a finite graph. The answer is no and one of the main reasons is that the inverse semigroup $\mathrm{PAut}(\Gamma)$ of all partial automorphisms of Γ has too many ideals, in particular, in will never be a group. We improve the situation by considering *weight* graphs, i.e. a graph, Γ, together with a surjection $\omega : V(\Gamma) \to P$, where P is a lower semi-lattice. A set, $W \subset V(\Gamma)$, will be called *principal ideal* provided $W = \{x \in V(\Gamma) | \omega(x) \leq \alpha\}$ for some $\alpha \in P$. A partial automorphism, φ, of Γ will be called *weight automorphism* if both $\mathrm{dom}(\varphi)$ and $\mathrm{ran}(\varphi)$ are principal ideals and φ preserves the order on weights, i.e. for any $a, b \in V(\Gamma)$ the equality $\omega(a) = \omega(b)$ implies $\omega(\varphi(a)) = \omega(\varphi(b))$ and the inequality $\omega(a) < \omega(b)$ implies $\omega(\varphi(a)) < \omega(\varphi(b))$.

Lemma 22.1. *The set* $\mathrm{PAut}_{\omega}(\Gamma)$ *of all weight automorphisms of a weight graph,* Γ*, is an inverse subsemigroup of* $\mathrm{PAut}(\Gamma)$.

Proof. Clearly, if $\varphi \in \mathrm{PAut}_{\omega}(\Gamma)$ then $\varphi^{\#} \in \mathrm{PAut}_{\omega}(\Gamma)$ and that the composition of two elements, $\varphi, \psi \in \mathrm{PAut}_{\omega}(\Gamma)$ preserves the order of weight. So, it is left to show that both $\mathrm{dom}(\varphi \circ \psi)$ and $\mathrm{ran}(\varphi \circ \psi)$ are principal ideals. Let $\mathrm{ran}(\psi)$ and $\mathrm{dom}(\varphi)$ be principal ideals defined by α and β in P respectively. Then $\mathrm{ran}(\psi) \cap \mathrm{dom}(\varphi) = \{x | x \leq \inf(\alpha, \beta)\}$ as P is a lower semi-lattice. Now if we choose one vertex c of weight $\inf(\alpha, \beta)$ in $\mathrm{ran}(\psi) \cap \mathrm{dom}(\varphi)$ we will get $\mathrm{dom}(\varphi \circ \psi) = \{y | \omega(y) \leq \omega(\psi^{\#}(c))\}$ and $\mathrm{ran}(\varphi \circ \psi) = \{z | \omega(z) \leq \omega(\varphi(c))\}$. This completes the proof. $\square$

Now we can formulate the following result.

Theorem 22.1. *If S is a finite inverse semigroup, then there exists a weight graph,* Γ*, such that* $S \simeq \mathrm{PAut}_{\omega}(\Gamma)$.

We note that $|P| = 1$ implies $\mathrm{PAut}_{\omega}(\Gamma) = \mathrm{Aut}(\Gamma)$ and hence the statement is true if S is a group according to the Frucht's theorem mentioned above. To prove this statement we will need certain constructions and several lemmas. First we associate with S a colored oriented graph, $\tilde{\Gamma}$, whose vertices are S, there is an arrow from a to b if an only if $b \in Sa$ and this arrow has color $ba^{\#}$. Clearly, it is enough to consider the semigroups up to an anti-isomorphism, as the semigroup, anti-isomorphic to an inverse semigroup, is itself inverse. So, we consider the Preston-Wagner (anti-) representation $a \mapsto \varphi_a : Sa^{\#} \to Sa$, $\varphi_a(x) = xa$, of S by partial bijections on S, given by Theorem 21.1.

Lemma 22.2. *For any $a \in S$ the map φ_a is a partial automorphism of $\tilde{\Gamma}$, which preserves colors of arrows.*

Proof. Let $x, y \in Sa^{\#}$ and there is an arrow from x to y of color $yx^{\#}$. By definition of φ_a there exist $s, t \in S$ such that $y = sa^{\#}$ and $y = tx$. Then $ya = txa \in Sa$ and $(ya)(xa)^{\#} = yaa^{\#}x^{\#} = sa^{\$}aa^{\#}x^{\#} = sa^{\#}x^{\#} = yx^{-1}$ that is there is an arrow from xa to ya of color $yx^{\#}$.

Let now $x = ua^{\#}, y = va^{\#} \in Sa^{\#}$ and $y \notin Sx$. Assume that there is an arrow from xa to ya. We use $\varphi_{a^{\#}}$ and by the conclusion above obtain an arrow from $xaa^{\#} = ua^{\$}aa^{\#} = ua^{\#} = x$ to $yaa^{\#} = va^{\$}aa^{\#} = va^{\#} = y$, a contradiction. □

Recall that $E(S)$ is a lower semi-lattice with respect to the natural partial order. To the vertex a of $\tilde{\Gamma}$ we assign the weight $\omega(a) = a^{\#}a = \omega(a^{\#}a)$.

Lemma 22.3. *There hold the following equivalences:*

1. *$b \in Sa$ if and only if $\omega(b) \le \omega(a)$;*
2. *$Sb \subset Sa$ if and only if $\omega(b) \le \omega(a)$;*
3. *$Sb = Sa$ if and only if $\omega(b) = \omega(a)$.*

Proof. Clearly, it is enough to prove the first statement. As $Sa = Sa^{\#}a$, from $b = xa^{\#}a$ it follows $b^{\#}b = a^{\#}ax^{\#}xa^{\#}a$ and thus $(b^{\#}b)(a^{\#}a) = b^{\#}b$, that is $\omega(b) \le \omega(a)$. Conversely, from $\omega(b) \le \omega(a)$ we get $b^{\#}b = b^{\#}ba^{\#}a$ and $b = ba^{\#}a \in Sa$. □

Lemma 22.4. *$\varphi_{a^{\#}a}$ is the identity map on Sa.*

Proof. Follows from $S(a^{\#}a)^{\#} = Sa$ and $xaa\#a = xa$. □

Lemma 22.5. *Let $a \in S$ and φ be a partial automorphism of $\tilde{\Gamma}$, which preserves colors of edges, with domain $\{x | \omega(x) \le \omega(a)\}$. If $\varphi_t(a^{\#}a)$ is an idempotent of S, then $\varphi = \varphi_{a^{\#}a}$.*

Proof. From Lemma 22.3 it follows that $\mathrm{dom}(\varphi) = Sa = Sa^{\#}a$ and hence for any $ya \in Sa$ there is an arrow from $a^{\#}a$ to ya of color ya. But then there is an arrow of the same color from $\varphi_t(a^{\#}a)$ to $\varphi_t(ya)$ and $\varphi_t(ya) = s\varphi_t(a^{\#}a)$ for some $s \in S$. From the other hand, as $\varphi_t(a^{\#}a)$ is an idempotent, the arrow from $\varphi_t(a^{\#}a)$ to $\varphi_t(ya)$ should have color

$$\varphi_t(ya)(\varphi_t(a^{\#}a))^{\#} = s\varphi_t(a^{\#}a)(\varphi_t(a^{\#}a))^{\#} = s\varphi_t(a^{\#}a) = \varphi_t(ya).$$

hence $\varphi(ya) = ya$ for all $ya \in Sa$ and the statement follows from Lemma 22.4. □

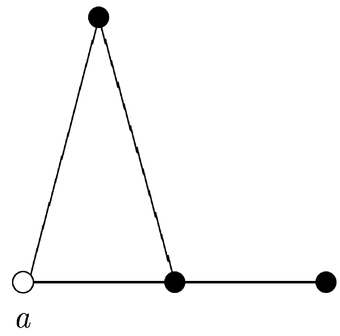

Figure 1: The graph Γ_a.

Lemma 22.6. *Let $a \in S$ and φ be a partial automorphism of $\tilde{\Gamma}$, which preserves colors of edges, with domain $\{x|\omega(x) \le \omega(a)\}$. Then there exists $b \in S$ such that $\varphi = \varphi_b$.*

Proof. Assume that such element does not exist. Then Lemma 22.5 says that $c = \varphi(a^{\#}a)$ is not an idempotent. But $\varphi(Sa) \subset Sc$. Consider the partial automorphism $\psi = \varphi \circ \varphi_{c^{\#}}$. If $\psi = \varphi_y$ for some $y \in S$, we would have $\varphi = \varphi \circ \varphi_{c^{\#}c} = \varphi \circ \varphi_{c^{\#}} \circ \varphi_c = \psi \circ \varphi_c = \varphi_y \circ \varphi_c = \varphi_{yc}$, which contradicts our assumption. But ψ preserves the colors of the edges and $\mathrm{dom}(\psi) = \mathrm{dom}(\varphi)$. Therefore, by Lemma 22.5, $\psi(a^{\#}a)$ is not an idempotent either. But $\psi(a^{\#}a) = (\varphi \circ \varphi_{c^{\#}})(a^{\#}a) = \varphi_{c^{\#}}(c) = cc^{\#}$. This contradiction completes the proof. □

Lemma 22.7. *Each partial automorphism φ of $\tilde{\Gamma}$ preserves the order of weights of vertices.*

Proof. Let $a, b \in \mathrm{dom}(\varphi)$ and $\omega(a) = \omega(b)$, i.e. by Lemma 22.3, $a \in Sb$ and $b \in Sa$. Then, by construction of $\tilde{\Gamma}$, there is an arrow from a to b and an arrow from b to a. Hence there are arrows in two directions between $\varphi(a)$ and $\varphi(b)$ as well. So, we get $\omega(\varphi(a)) = \omega(\varphi(b))$ by construction of $\tilde{\Gamma}$ and Lemma 22.3. That $\omega(a) < \omega(b)$ implies $\omega(\varphi(a)) < \omega(\varphi(b))$ is proved analogously. □

So, for any $a \in S$ the partial automorphism φ_a of $\tilde{\Gamma}$ is weight and preserves the colors of edges. Hence, by Lemma 22.6, we get that all weight partial automorphisms of $\tilde{\Gamma}$, which preserves the colors of vertices from a semigroup, (anti-) isomorphic to S. Now we are ready to prove Theorem 22.1.

Proof of Theorem 22.1. Denote by Δ the set of colors of $\tilde{\Gamma}$. Take any injections: $\Delta \to \{n \in \mathbb{N}|n \ge 2\}$, $x \mapsto n_x$, and $E(S) \to \{m \in \mathbb{N}|m > M\}$, $e \mapsto m_e$, where $M = \max_{x\in\Delta} n_x$. Now we can transform $\tilde{\Gamma}$ into a weight graph Γ. From $a = aa^{\#}a \in Sa$ it follows that $\tilde{\Gamma}$ contains a loop of color $aa^{\#}$ in the vertex a. If $a \neq 0$, we replace this loop by subgraph Γ_a from Picture 1. Otherwise (if $a = 0$) we consider a as a vertex without loop. Each arrow from a to b, $a \neq b$ we replace by subgraph $\Gamma_{a,b}$ from Picture 2 and the weight of all new vertices of these subgraphs is $\omega(a) = aa^{\#}$ by definition.

We note that for any $b \in Sa^{\#}$ from $ba = 0$ it follows $b = 0$. Hence, the construction of Γ implies that for any $0 \neq b \in Sa^{\#}$ there exists the unique isomorphism $\psi_b : \Gamma_b \to \Gamma_{\varphi_a(b)}$,

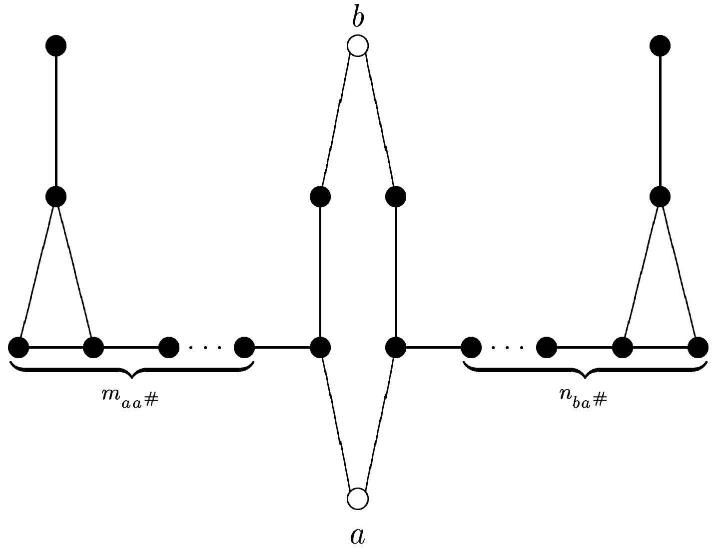

Figure 2: The graph $\Gamma_{a,b}$.

satisfying $\psi_b(b) = \psi_a(b)$. Moreover, for any $b \in Sa^{\#}$ and $c \in Sb$ the construction of Γ implies the existence of the unique isomorphism $\psi_{b,c} : \Gamma_{b,c} \to \Gamma_{\varphi_a(b),\varphi_a(c)}$. Now for any $a \in S$ we define a partial map, φ_a^*, on the vertices of Γ in the following way:

$$\varphi_a^*(x) = \begin{cases} \varphi_a(x), & x \in Sa^{\#}; \\ \psi_b(x), & x \text{ is a vertex of } \Gamma_b \text{ for some } b \in Sa^{\#}; \\ \psi_{b,c}(x), & x \text{ is a vertex of } \Gamma_{b,c} \text{ for some } b, c \in Sa^{\#}; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Since $\varphi_a(x) = \psi_b(x) = \psi_{b,c}(x)$ for any $b \in Sa^{\#}$, the map φ_a^* is well-defined.

It is obvious from the construction that φ_a^* is a partial automorphism of Γ and that it preserves the order of weights. Let us prove that $\mathrm{dom}(\varphi_a^*) = \{x | \omega(x) \leq \omega(a^{\#})\}$. Indeed, if $x \in \mathrm{dom}(\varphi_a^*)$, then $\omega(x)$ coincides with the weight of some $b \in Sa^{\#}$. Hence $\omega(x) \leq \omega(a^{\#})$ by Lemma 22.3. Conversely, if $\omega(x) \leq \omega(a^{\#})$, then there exists $b \in Sa^{\#}$ such that $\omega(x) = \omega(b)$ and x is a vertex of either Γ_b or $\Gamma_{b,c}$. As $c \in Sb$, we have $b \in Sa^{\#}$ implies $c \in Sa^{\#}$ and we have $x \in \mathrm{dom}(\varphi_a^*)$ in both cases.

From $\mathrm{ran}(\varphi_a^*) = \mathrm{dom}(\varphi_{a^{\#}}^*)$ it follows that $\mathrm{ran}(\varphi_a^*)$ is a principal ideal as well. Hence $\varphi_a^* \in \mathrm{PAut}_\omega(\Gamma)$. It is obvious that $a \mapsto \varphi_a^*$ is a homomorphisms from S into $\mathrm{PAut}_\omega(\Gamma)$. As φ_a^* coincides with φ_a on vertices, this homomorphism is injective and it is left to prove that each partial weight automorphism of Γ is of the form φ_a^*.

Let $\varphi \in \mathrm{PAut}_\omega(\Gamma)$. Then $\mathrm{dom}(\varphi) = \{x | \omega(x) \leq \omega(e)\}$, where $e \in E(S)$. If $e = 0$, then $\mathrm{dom}(\varphi) = \{0\}$ and $\varphi = \varphi_0^*$ as $\{\}$ is the unique principal ideal containing exactly one element.

Now let $e \neq 0$. As for any $0 \neq b \in Se$ all vertices of Γ_b and $\Gamma_{b,c}$ have weights $\leq \omega(b)$, we get that φ is defined on all such vertices. But $|Sb| > 1$ as $b \neq 0$. Hence $\tilde{\Gamma}$ contains a loop in b and an arrow from b to c, $b \neq c \in Sb$. Hence any vertex b, $0 \neq b \in Se$, has degree ≥ 4 in the subgraph of Γ, generated by $\mathrm{dom}(\varphi)$, and any vertex of Γ, which was not vertex in $\tilde{\Gamma}$, has the degree ≤ 3. Thus φ maps vertices of $\tilde{\Gamma}$ into vertices of $\tilde{\Gamma}$ and new vertices of Γ into new vertices of Γ respectively. In particular, we can restrict φ onto $\tilde{\Gamma}$ and denote the result by $\tilde{\varphi}$.

It is obvious that $\tilde{\varphi}$ preserves the order of vertices and that both $\mathrm{dom}(\tilde{\varphi})$ and $\mathrm{ran}(\tilde{\varphi})$ are principal ideal. Moreover, if there is an arrow from a to b for $a, b \in \mathrm{dom}(\tilde{\varphi})$, then φ should map $\Gamma_{a,b}$ into $\Gamma_{\varphi(a),\varphi(b)}$. But $\Gamma_{\varphi(a),\varphi(b)}$ is isomorphic to some subgraph of $\Gamma_{c,d}$ if and only if $\Gamma_{a,b} \simeq \Gamma_{c,d}$. This implies that the arrows from a to b and from $\varphi(a)$ to $\varphi(b)$ has the same color. Therefore $\tilde{\varphi}$ is a weight partial automorphism of $\tilde{\Gamma}$, which preserves the color of edges. That is $\tilde{\varphi} = \varphi_a$ for some $a \in S$. But $\tilde{\varphi}_a^* = \varphi_a$ and $\tilde{\varphi}_1 = \tilde{\varphi}_2$ implies $\varphi_1 = \varphi_2$. And we finally get $\varphi = \varphi_a^*$. $\square$

23 Some facts without comments

1. $\mathrm{dom}(f \circ g) \subset \mathrm{dom}(g)$ and $\mathrm{ran}(f \circ g) \subset \mathrm{ran}(f)$ for any $f, g \in \mathcal{IS}_n$.
2. $|\mathcal{IS}_n| = \sum_{i=0}^{n} \binom{n}{i}^2 i!$.
3. The number of elements in $\mathcal{IS}_n$, the chain decomposition of which contains l_i cycles and m_i chains of length i, $i = 1, \dots, n$ equals $n! \left(\prod_{i=1}^{n} (i^{l_i} l_i! m_i!)\right)^{-1}$.
4. Say that two elements $f, g \in \mathcal{IS}_n$ have the same *type* if for any $i = 1, \dots, n$ the number of cycles (resp. chains) of length i in the chain decomposition of f and g coincide. The number of different types among element in $\mathcal{IS}_n$ equals $\sum_{k=0}^{n} s(k)s(n-k)$, where $s(i)$, $i \in \mathbb{N}$ denotes the number of different cyclic types of elements in $\mathbb{S}_i$ and $s(0) = 1$.
5. $|\langle f \rangle_{inv}| = \frac{\mathrm{i}(f)(\mathrm{i}(f)+1)(2\mathrm{i}(f)+1)}{6} + \mathfrak{p}(f) - 1$.
6. The number of maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree k equals $\sum_{m=k}^{n} \sum_{i=1}^{k} (-1)^i \binom{n}{m} \binom{k}{i} (k-i)^m$.
7. The number of non-isomorphic maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree $k \geq 3$ equals $\begin{pmatrix} n+k-1 \\ k-1 \end{pmatrix}$.
8. The number of non-isomorphic maximal nilpotent subsemigroups of $\mathcal{IS}_n$ of nilpotency degree 2 equals $[\frac{n+1}{2}]$.
9. Each maximal nilpotent subsemigroup of $\mathcal{IS}_n$ of nilpotency degree k and of type $(n_1, \dots, n_k)$ is contained in precisely $n_1! n_2! \dots n_k!$ maximal nilpotent subsemigroups of $\mathcal{IS}_n$.

10. Let $k = 2$ and $f(n, m)$ be as in Section 15, then

$$f(n, m) = \sum_{i=0}^{\min(m,n)} \binom{n}{i}\binom{m}{i} i!.$$

11. For $n > 2$, $n \neq 4$ the semigroup $\mathcal{IS}_n$ has proper endomorphisms only of rank 1, 2, 3, $n! + 1$, $n! + n^2 + 1$ and $\mathcal{IS}_4$ has additionally endomorphisms of rank 7.

12. For $n > 2$ the semigroup $\mathcal{IS}_n$ has 2^n endomorphisms of rank 1, $3^n - 2^n$ endomorphisms of rank 2,

$$n! \sum_{m=0}^{n} \sum_{k=1}^{[m/2]} \frac{2^{m-3k}}{(n-m)!(m-2k)!k!}$$

endomorphisms of rank 3, $n!$ endomorphisms of rank $n! + 1$ and $n!$ endomorphisms of rank $n! + n^2 + 1$ and that $\mathcal{IS}_4$ also has 96 endomorphisms of rank 7. Compute then the formula for $|\operatorname{End}(\mathcal{IS}_n)|$.

24 Historical comments

Definition of $\mathcal{IS}_n$ and its elementary properties can be found in [LI] or in [KP].

Centralizers in $\mathcal{IS}_n$ were described by Lipskomb, [Li1], this approach can be found in [LI] in very detailed exposition. We follow closely an independent paper [GK1]. The results of Section 8 and 9 are also taken from [GK1]. An alternative approach of description of G-conjugated elements can be found in [LI]. The presentation, discussed in Section 11 is due to Lipskomb, [LI], but is a refinement of the result of Popova, [Po]. Isolated and completely isolated subsemigroups of $\mathcal{IS}_n$ are described by Ganyushkin and Kormysheva in [GK2], where the general philosophy, how to study nilpotent subsemigroups of $\mathcal{IS}_n$, was also formulated. The description of the maximal nilpotent subsemigroups of a given nilpotency degree in $\mathcal{IS}_n$, their inclusions and the isomorphism problem was obtained in [GK3, GK4]. The isomorphism problem here was technically difficult and the proofs in [GK3] and [GK4] are different. We follows the shorter version from [GK4]. On this place it is natural to say that the philosophy of study of maximal nilpotent subsemigroups of transformation semigroups in terms of partial orders was quite successful also for study of the semigroup of all partial linear transformation of a finite-dimensional vector-space over a finite field. This was done by A.Shafranova (maiden name - Kudryavtseva) in [Sh1, Sh2].

The results about the cardinality of maximal nilpotent semigroups is a recent result of Ganyushkin and Pavlov, [GP], and the paper is still quoted as “to appear”. The description of the automorphisms of the maximal nilpotent subsemigroups in $\mathcal{IS}_n$ is taken from [GTS]. In fact, in [GTS] the authors consider the case of $\mathcal{IS}(M)$, where M is an arbitrary set and study maximal nilpotent semigroups of finite nilpotency degree. Their result is analogous to one in the finite case but several arguments in the proof require more care because of consideration of maps between infinite sets. The cardinality of these automorphisms groups for $\mathcal{IS}_n$ is also due to Ganyushkin and Pavlov, [GP].

The automorphisms of $\mathcal{IS}_n$ were described by Liber, [Lib], and independently by Lyapin. The endomorphisms are recently described by Schein and Teclezghi in [ST1]. The results of Section 20 are due to Voloshyna, [Vo]. The result of Appendix I is the famous Theorem of Preston and Wagner, see [KP]. The result of Appendix II is due to Nemirovskaya, [Ne].

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