$\mathcal{L}$– and $\mathcal{R}$–cross-sections in the Brauer semigroup

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Abstract

We classify all cross-sections of Green's relations $\mathcal{L}$ and $\mathcal{R}$ in the Brauer semigroup. The regular behavior of such cross-sections starts from $n = 7$. We show that in the regular case there are essentially two different cross-sections and all others are $\mathcal{S}_n$-conjugated to one of these two. We also classify all cross-sections up to isomorphism.

1 Introduction

Let S be a semigroup, and ρ be an equivalence relation on S. A subsemigroup, $T \subset S$, is called a *cross-section* with respect to ρ provided that T contains exactly one element from every equivalence class. Certainly the most natural equivalence relations on a semigroup are congruences and Green's relations and so finding descriptions of the cross-sections on these relations is a natural problem to consider.

In what follows we will call the cross-sections with respect to Green's relations $\mathcal{L}$ ($\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$, $\mathcal{J}$) the $\mathcal{L}$– ($\mathcal{R}$–, $\mathcal{H}$–, $\mathcal{D}$–, $\mathcal{J}$–) *cross-sections*, respectively. During the last decade the cross-sections of Green's relations for some classical semigroups were studied by different authors. In particular, for the symmetric inverse semigroup $\mathcal{IS}_n$ all $\mathcal{H}$–cross-sections were classified in [CR] and all $\mathcal{L}$– and $\mathcal{R}$–cross-sections were classified in [GM]. For the infinite symmetric inverse semigroup $\mathcal{IS}_X$ all $\mathcal{H}$–, $\mathcal{L}$– and $\mathcal{R}$–cross-sections were classified in [Pe2], and for the full transformation semigroup $\mathcal{T}_n$ all $\mathcal{H}$– and $\mathcal{R}$–cross-sections were classified in [Pe1]. In [Ml1] it was shown that the description of $\mathcal{L}$– and $\mathcal{R}$–cross-sections for the partial Brauer semigroup $\mathcal{P}\mathfrak{B}_n$ reduces to the corresponding description for $\mathcal{IS}_n$, and a classification of $\mathcal{L}$– and $\mathcal{R}$–cross-sections for the composition semigroup $\mathfrak{C}_n$ (defined in [Mr] and studied in [Ma2]) was obtained.

The semigroups $\mathcal{P}\mathfrak{B}_n$ and $\mathfrak{C}_n$ are generalizations of the Brauer semigroup $\mathfrak{B}_n$, introduced in [Br]. In the present paper we give a classification of all $\mathcal{L}$– and $\mathcal{R}$–cross-section in $\mathfrak{B}_n$. Surprisingly enough, for $\mathfrak{B}_n$ the problem happens to be much more difficult than for $\mathcal{P}\mathfrak{B}_n$ and $\mathfrak{C}_n$. Because of the existence of an anti-involution on $\mathfrak{B}_n$ it is enough to classify only one kind of cross-section. We do this for $\mathcal{R}$–cross-sections. It happens that $\mathcal{R}$–cross-sections exhibit a regular behavior starting with $n = 7$. The symmetric group $\mathcal{S}_n \subset \mathfrak{B}_n$ acts on the set of all $\mathcal{R}$–cross-sections of $\mathfrak{B}_n$ in a natural way. For $n \geq 7$ we show that this action has exactly 2 orbits, each containing $n!/2$ elements. We also describe the canonical representatives in these orbits, which we call the regular and the alternating $\mathcal{R}$–cross-sections, respectively. We show that these two $\mathcal{R}$–cross-sections are not isomorphic as monoids. The cases $n \leq 6$ are considered separately as the descriptions in these cases do not fit into the "regular" picture.

The paper is organized as follows. In Section 2 we give all the necessary background about the Brauer semigroups required for the sequel. In Section 3 we study and completely determine a class of $\mathcal{R}$–cross-sections, which we call canonical. We use these results in Section 4 to give a classification of all $\mathcal{R}$–cross-sections of $\mathfrak{B}_n$. We also classify all $\mathcal{R}$–cross-sections of $\mathfrak{B}_n$ up to isomorphism. We finish the paper with a discussion of the problems to classify $\mathcal{D}$– and $\mathcal{H}$–cross-sections of $\mathfrak{B}_n$ in Section 5.

2 Preliminaries about the Brauer semigroup

Let $n \in \mathbb{N}$, $M = \{1, 2, \dots, n\}$ and $M' = \{1', 2', \dots, n'\}$. We consider the map $' : M \to M'$ as a fixed bijection and will denote the inverse bijection by the same symbol, that is $(a')' = a$.

Denote by $\mathfrak{B}_n = \mathfrak{B}(M)$ the set of all possible partitions of $M \cup M'$ into two-element subsets. It is a simple exercise to verify that $|\mathfrak{B}_n| = (2n-1)!!$, see for example [Ke]. For $\alpha \in \mathfrak{B}_n$ and $a \neq b \in M \cup M'$ we set $a \equiv_\alpha b$ provided that $\{a, b\} \in \alpha$. That is $\equiv_\alpha$ is the equivalence relation corresponding to the partition α. Let $\alpha = X_1 \cup \dots \cup X_n$ and $\beta = Y_1 \cup \dots \cup Y_n$ be two elements from $\mathfrak{B}_n$. Define a new equivalence relation, $\equiv$, on $M \cup M'$ as follows:

- for $a, b \in M$ we have $a \equiv b$ if and only if $a \equiv_\alpha b$ or there is a sequence, $c_1, \cdots, c_{2s}$, $s \geq 1$, of elements in M, such that $a \equiv_\alpha c_1'$, $c_1 \equiv_\beta c_2$, $c_2' \equiv_\alpha c_3'$, ..., $c_{2s-1} \equiv_\beta c_{2s}$, and $c_{2s}' \equiv_\alpha b$;
- for $a, b \in M$ we have $a' \equiv b'$ if and only if $a' \equiv_\beta b'$ or there is a sequence, $c_1, \cdots, c_{2s}$, $s \geq 1$, of elements in M, such that $a' \equiv_\beta c_1$, $c_1' \equiv_\alpha c_2'$, $c_2 \equiv_\beta c_3$, ..., $c_{2s-1}' \equiv_\alpha c_{2s}'$, and $c_{2s} \equiv_\beta b'$;

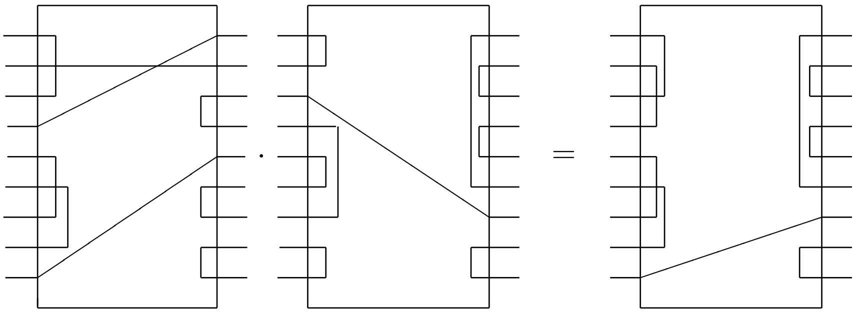

Figure 1: Chips and their multiplication.

- for $a, b \in M$ we have $a \equiv b'$ if and only if $b' \equiv a$ if and only if there is a sequence, $c_1, \cdots, c_{2s-1}$, $s \geq 1$, of elements in M, such that $a \equiv_\alpha c_1'$, $c_1 \equiv_\beta c_2$, $c_2' \equiv_\alpha c_3', \dots, c_{2s-2}' \equiv_\alpha c_{2s-1}'$, and $c_{2s-1} \equiv_\beta b'$.

It is easy to see that $\equiv$ determines a partition of $M \cup M'$ into two-element subsets and so belongs to $\mathfrak{B}_n$.

One can think about the elements from $\mathfrak{B}_n$ as certain "microchips" with n pins on the left hand side (corresponding to M) and n pins on the right hand side (corresponding to M'). Having $\alpha \in \mathfrak{B}_n$ we connect two pins in the corresponding chip if and only if they belong to the same set of the partition α. The operation described above can then be viewed as a "composition" of such chips: having $\alpha, \beta \in \mathfrak{B}_n$ we identify (connect) the right pins of α with the corresponding left pins of β, which uniquely defines a connection of the remaining pins (which are the left pins of α and the right pins of β). An example of multiplication of two chips from $\mathfrak{B}_n$ is given on Figure 1. Note that performing the operation we can obtain some "dead circles" formed by some identified pins from α and β, see for example the two lowest identified pins on Figure 1. These circles should be disregarded. From such an interpretation it is not hard to see that the composition of elements from $\mathfrak{B}_n$ defined above is associative (see [Br]).

The obtained semigroup is called the *Brauer* semigroup (monoid). The (deformed) semigroup algebra of $\mathfrak{B}_n$ is the famous Brauer algebra, which was introduced in [Br], and which plays an important role in the study of certain representations of orthogonal groups. The Brauer algebra has been extensively studied in the literature (see for example [KX] and the references therein). The Brauer semigroup was studied in, for example, [Ke, Ma1, Ma2, Ml2].

There is a natural monomorphism $\mathcal{S}_n \hookrightarrow \mathfrak{B}_n$ defined as follows:

$$\sigma \mapsto \{1, \sigma(1)'\} \cup \{2, \sigma(2)'\} \cup \dots \cup \{n, \sigma(n)'\}, \quad \text{where} \quad \sigma \in \mathcal{S}_n,$$

and we will identify $\sigma \in \mathcal{S}_n$ with its image in $\mathfrak{B}_n$ via this embedding. The image of $\mathcal{S}_n$ under the above embedding coincides with the set of all invertible elements in $\mathfrak{B}_n$. For $n \geq 2k$ there is also a natural monomorphism from the symmetric inverse semigroup $\mathcal{IS}_k$ on $\{1, \dots, k\}$ into $\mathfrak{B}_n$, which is constructed in the following way: an element, $\sigma \in \mathcal{IS}_k$, is sent to the element $\gamma \in \mathfrak{B}_n$, uniquely defined by the following conditions:

- $i \equiv_\gamma i'$ for all $i > 2k$;
- $2i \equiv_\gamma (2\sigma(i))'$ and $2i - 1 \equiv_\gamma (2\sigma(i) - 1)'$ for all $i \in \{1, \dots, k\}$, which belong to the domain of σ;
- $2i \equiv_\gamma 2i - 1$ for all $i \in \{1, \dots, k\}$, which do not belong to the domain of σ;
- $(2i)' \equiv_\gamma (2i-1)'$ for all $i \in \{1, \dots, k\}$, which do not belong to the range of σ.

For $\alpha \in \mathfrak{B}_n$ a partition set, $X \in \alpha$, having the form $X = \{a, b'\}$ for $a, b \in M$, will be called a *line* in α. The number of different lines in α is called the *rank* of α and is denoted by $\mathrm{rank}(\alpha)$. The number $n - \mathrm{rank}(\alpha)$ is called the *corank* of α and is denoted by $\mathrm{corank}(\alpha)$. For example, $\sigma \in \mathcal{S}_n$ if and only if $\mathrm{rank}(\sigma) = n$ if and only if $\mathrm{corank}(\sigma) = 0$. Note that $\mathrm{corank}(\alpha)$ is even for every $\alpha \in \mathfrak{B}_n$ and that $\mathrm{corank}(\alpha\beta) \leq \mathrm{corank}(\alpha) + \mathrm{corank}(\beta)$ for all $\alpha, \beta \in \mathfrak{B}_n$. Since $\mathfrak{B}_n$ is finite, for every $\alpha \in \mathfrak{B}_n$ there exists a unique idempotent, $\pi \in \mathfrak{B}_n$, such that $\pi = \alpha^i$ for some $i \in \mathbb{N}$. The rank of π is thus an invariant for α and will be called the *stable rank* $\mathrm{strank}(\alpha)$ of α. Note that $\mathrm{strank}(\alpha) \leq \mathrm{rank}(\alpha)$.

Call $\alpha, \beta \in \mathfrak{B}_n$ *left neighbors* or *right neighbors* provided that $\{a, b\} \in \alpha$ if and only if $\{a, b\} \in \beta$ for all $a, b \in M$ and for all $a, b \in M'$, respectively. Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{H}$, $\mathcal{J}$, in $\mathfrak{B}_n$ can be described as follows:

Theorem 1. *[Ma1, Theorem 7] Let $\alpha, \beta \in \mathfrak{B}_n$. Then*

(i) $\alpha \mathcal{L} \beta$ if and only if α and β are right neighbors if and only if there exists $\sigma \in \mathcal{S}_n$ such that $\sigma\alpha = \beta$;

(ii) $\alpha \mathcal{R} \beta$ if and only if α and β are left neighbors if and only if there exists $\sigma \in \mathcal{S}_n$ such that $\alpha\sigma = \beta$;

(iii) $\alpha \mathcal{H} \beta$ if and only if α and β are both left and right neighbors;

(iv) $\alpha\mathcal{D}\beta$ *if and only if* $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta)$ *if and only if there exist* $\sigma, \tau \in \mathcal{S}_n$ *such that* $\sigma\alpha\tau = \beta$*;*

(v) $\mathcal{D} = \mathcal{J}$*.*

A subset, $X \subset M$, will be called *α-invariant* for some $\alpha \in \mathfrak{B}_n$ provided that for any $a \in X \cup X'$ and any $b \in M \cup M'$ the condition $a \equiv_\alpha b$ implies $b \in X \cup X'$. If X is invariant with respect to α, then we define the element $\alpha|_X \in \mathfrak{B}(X)$ in the following way:

- for all $a, b \in X \cup X'$ we have $a \equiv_{\alpha|_X} b$ if and only if $a \equiv_\alpha b$.

The element $\alpha|_X$ is called the *restriction* of α to X. Note that if X is α-invariant then $M \setminus X$ is α-invariant as well.

The involution $' : M \cup M' \to M \cup M'$ extends in a natural way to the anti-involution $^* : \mathfrak{B}_n \to \mathfrak{B}_n$, which, in the language of chips, acts on a chip by taking the mirror image of it. It is obvious that $\alpha\alpha^*\alpha = \alpha$, in particular, $\mathfrak{B}_n$ is a regular semigroup. Since * is an anti-involution, it interchanges $\mathcal{L}$– and $\mathcal{R}$–classes and the corresponding cross-sections. Hence, it is enough to classify one type of cross-sections, say the $\mathcal{R}$–cross-sections. The classification of the $\mathcal{L}$–cross-sections is then obtained by applying *.

3 Canonical $\mathcal{R}$–cross-sections in $\mathfrak{B}_n$

It does not follow from the definition that $\mathcal{R}$–cross-sections in $\mathfrak{B}_n$ exist. In this section we construct and investigate a special (rather big) family of $\mathcal{R}$–cross-sections in $\mathfrak{B}_n$, in particular, showing that they exist. We call an $\mathcal{R}$–cross-section, Λ, of $\mathfrak{B}_n$ *canonical* provided that for every $\alpha \in \Lambda$ there exists $k \in \{0, 1, \dots, \lfloor \frac{n}{2} \rfloor\}$ such that $\mathrm{corank}(\alpha) = 2k$ and

$$(n-2k+1)' \equiv_\alpha (n-2k+2)', (n-2k+3)' \equiv_\alpha (n-2k+4)', \dots, (n-1)' \equiv_\alpha n'.$$

Our aim in this section is to show that canonical $\mathcal{R}$–cross-sections of $\mathfrak{B}_n$ exist and classify all such cross-sections. However, even the existence is not obvious and will be established only in Proposition 9 and Proposition 12. To be able to prove these results in the early part of the section we describe what must happen within a canonical $\mathcal{R}$–cross-section of $\mathfrak{B}_n$, should one exist.

Let Λ be a canonical $\mathcal{R}$–cross-section in $\mathfrak{B}_n$. For $1 \leq i < j \leq n$ denote by $\alpha_{i,j}$ the (unique) element of Λ such that $\mathrm{corank}(\alpha_{i,j}) = 2$ and $i \equiv_{\alpha_{i,j}} j$. For $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$ set

$$\Lambda_k = \{\alpha \in \Lambda \, : \, \mathrm{corank}(\alpha) = 2k\}.$$

Note that $\Lambda_0 = \{\mathrm{id}\}$ and $\Lambda_1 = \{\alpha_{i,j} : 1 \leq i < j \leq n\}$. Hence for $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$ we can define

$$\Lambda_1^{(k)} = \{\alpha_{i,j} : j \leq n - 2(k-1)\}.$$

In particular, $\Lambda_1^{(1)} = \Lambda_1$. Later on we will show that the elements $\alpha_{i,j}$ completely determine (generate) Λ. Our main idea is to collect enough information (mainly technical) about $\alpha_{i,j}$ to be able to explicitly describe Λ.

For every $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$ define the element β_i of corank $2i$ as follows: $j \equiv_{\beta_i} j'$ for all $j = 1, \dots, n-2i$; $j \equiv_{\beta_i} j+1$ and $j' \equiv_{\beta_i} (j+1)'$ for all $j = n-2i+1, n-2i+3, \dots, n-1$ (see example in Figure 2). We start with the following observation:

Lemma 2. *(i) Let $\alpha \in \Lambda$ and suppose that there exists $\pi \in \mathfrak{B}_n$ such that $\alpha \mathcal{H} \pi$ and $\pi^2 = \pi$. Then $\alpha = \pi$.*

(ii) For every $i = 0, \dots, \lfloor \frac{n}{2} \rfloor$, the element β_i is an idempotent, and so belongs to Λ.

(iii) For $j = n-1, n$ and for all $i = 1, \dots, j-1$ the element $\alpha_{i,j}$ of Λ is an idempotent and satisfies $s \equiv_{\alpha_{i,j}} s'$ for all $s \neq i, n-1, n$. Moreover, in the case $(i,j) \neq (n-1,n)$ we also have $i' \equiv_{\alpha_{i,j}} \overline{j}$, where $\{j, \overline{j}\} = \{n-1, n\}$ (see example on Figure 2). In the case $(i,j) = (n-1,n)$ we have $\alpha_{i,j} = \beta_1$.

(iv) The element $\alpha_{i,j}$ is an idempotent if and only if $j = n-1$ or $j = n$.

(v) If two elements of Λ are $\mathcal{D}$-related then they are $\mathcal{L}$-related.

Proof. Since π is an idempotent, its $\mathcal{H}$-class is a (finite) maximal subgroup of $\mathfrak{B}_n$. In particular, $\alpha^i = \pi$ for some $i \in \mathbb{N}$. This implies $\pi \in \Lambda$. Furthermore, $\alpha \mathcal{H} \pi$ implies $\alpha \mathcal{R} \pi$ and hence $\pi = \alpha$ as Λ is an $\mathcal{R}$–cross-section. This proves (i).

For any i, β_i is easily seen to be an idempotent by direct calculation. If $\alpha \in \Lambda$ and $\alpha \mathcal{R} \beta_i$ then we have $\alpha \mathcal{H} \beta_i$ by Theorem 1(iii), and so $\alpha = \beta_i$ by part (i). This proves (ii). (iii) and (iv) are proved by direct calculation, and (v) follows from the definition of a canonical $\mathcal{R}$–cross-section and Theorem 1(iv). □

Corollary 3. *Let $1 \leq i < j \leq n-2$.*

(i) Assume that j and n have the same parity. Then $s \equiv_{\alpha_{i,j}} s'$ for all $s \in \{1, \dots, j-2\} \setminus \{i\}$, and, if $i \neq j-1$, we also have $(j-1) \equiv_{\alpha_{i,j}} i'$.

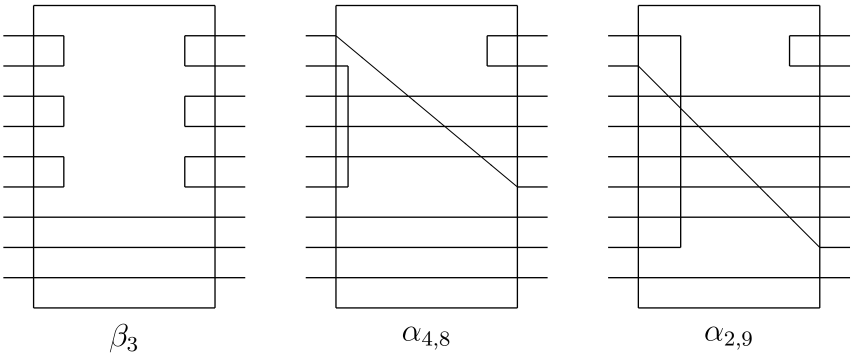

Figure 2: Some idempotents in Λ for $\mathfrak{B}_9$.

(ii) Assume that j and n have different parities. Then $s \equiv_{\alpha_{i,j}} s'$ for all $s \in \{1, \dots, j-1\} \setminus \{i\}$, and $(j+1) \equiv_{\alpha_{i,j}} i'$.

(iii) For every $k = 1, \dots, \lfloor \frac{n-j}{2} \rfloor$ and for every $s \in \{n-2k+2, n-2k+1\}$ there exists $t \in \{n-2k, n-2k-1\}$ such that $s \equiv_{\alpha_{i,j}} t'$.

Proof. Assume that j and n have the same parity, set $k = \frac{n-j+2}{2}$, and consider the product $\gamma = \beta_k \alpha_{i,j}$. We claim that for any $s \in \{1, \dots, j-2\}$ there exists $t \in M'$ such that $s \equiv_\gamma t'$ (see illustration on Figure 3). Indeed, for $s \neq i$ this follows directly from the definitions of β_k and $\alpha_{i,j}$. For $s = i$ we have $i \equiv_{\beta_k} i'$, $i \equiv_{\alpha_{i,j}} j$, $j' \equiv_{\beta_k} (j-1)'$ and $(j-1) \equiv_{\alpha_{i,j}} t'$ for some $t \in M$, which is exactly what we wanted to prove. This implies that $\gamma \mathcal{R} \beta_k$ and hence $\gamma = \beta_k$ since Λ is an $\mathcal{R}$–cross-section. Now all equalities in (i) follow from $\gamma = \beta_k$ and the definitions of β_k and $\alpha_{i,j}$. One also proves (ii) by analogous arguments.

To prove (iii) we use induction on k. For $k = 1$ we consider $\beta_1 \alpha_{i,j}$. We obviously have $\mathrm{corank}(\beta_1 \alpha_{i,j}) = 4$ and as Λ is canonical, we get that either $n \equiv_{\alpha_{i,j}} (n-2)'$ and $(n-1) \equiv_{\alpha_{i,j}} (n-3)'$, or else $n \equiv_{\alpha_{i,j}} (n-3)'$ and $(n-1) \equiv_{\alpha_{i,j}} (n-2)'$. This implies our statement for $k = 1$. Using analogous arguments, the cases $k = 2, \dots, \lfloor \frac{n-j}{2} \rfloor$ are shown by considering $\beta_k \alpha_{i,j}$ and proceeding by induction. This completes the proof. □

Proposition 4. *Let $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$.*

(i) Let $\alpha \in \Lambda_k$ and $\eta \in \Lambda_1$. Then $\alpha\eta = \alpha$ if $\eta \notin \Lambda_1^{(k+1)}$ and $\alpha\eta \in \Lambda_{k+1}$ otherwise.

(ii) Let $\eta_1, \dots, \eta_k \in \Lambda_1$. Then $\eta_1 \cdot \dots \cdot \eta_k \in \Lambda_k$ if and only if $\eta_i \in \Lambda_1^{(i)}$ for all $i \in \{1, \dots, k\}$.

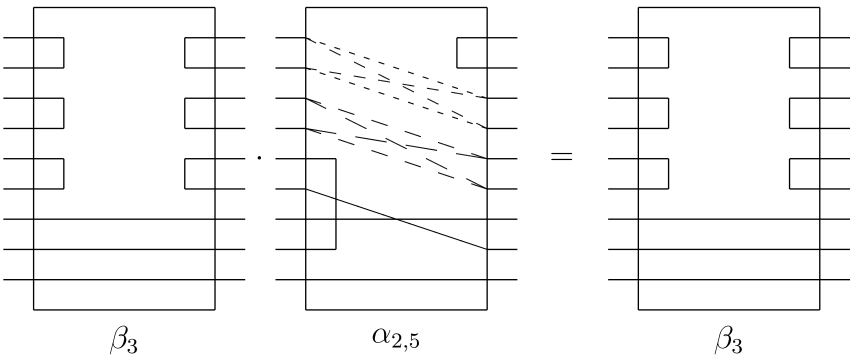

Figure 3: Illustration of the proof of Corollary 3.

(iii) If $k < \lfloor \frac{n}{2} \rfloor$ then $\Lambda_k \Lambda_1^{(k+1)} = \Lambda_{k+1}$. In particular, Λ is generated by Λ_1 as a monoid.

Proof. We start with (i). From the definition of a canonical $\mathcal{R}$–cross-section we have

$$n' \equiv_\alpha (n-1)', (n-2)' \equiv_\alpha (n-3)', \dots, (n-2(k-1))' \equiv_\alpha (n-2(k-1)-1)',$$

and for every $j < n-2(k-1)-1$ there exists $x \in M$ such that $x \equiv_\alpha j'$. If $\gamma \notin \Lambda_1^{(k+1)}$ then $\mathrm{rank}(\alpha\gamma) = \mathrm{rank}(\alpha)$. Thus $\alpha \mathcal{R} \alpha\gamma$ and hence $\alpha = \alpha\gamma$. In the case when $\gamma \in \Lambda_1^{(k+1)}$ one gets $\mathrm{rank}(\alpha\gamma) = \mathrm{rank}(\alpha) - 2$, which proves (i). (ii) follows immediately from (i) by induction on k.

Let $k < \lfloor \frac{n}{2} \rfloor$ and $\alpha \in \Lambda_{k+1}$. Let further $X_1, \dots, X_{k+1}$ be disjoint two-element subsets of M contained in α. Let β be the unique element of Λ_k, containing $X_1, \dots, X_k$. Let $X_{k+1} = \{a, b\}$ and $u, v \in M$ be such that $a \equiv_\beta u'$ and $b \equiv_\beta v'$. Let $i < j$ be such that $\{i, j\} = \{u, v\}$. Then $\alpha_{i,j} \in \Lambda_1^{(k+1)}$ and we have $\beta\alpha_{i,j} \mathcal{R} \alpha$ by construction, implying $\beta\alpha_{i,j} = \alpha$, since Λ is an $\mathcal{R}$–cross-section. This, together with part (i) establishes (iii). □

Lemma 5. *For all $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$ we have $|\Lambda_k| = \binom{n}{2k}(2k-1)!!$.*

Proof. Let $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. Then $|\Lambda_k|$ coincides with the number of $\mathcal{R}$–classes of corank $2k$. To define such an $\mathcal{R}$–class we have to choose k subsets of M, each containing 2 elements, when the order of subsets is not important. This can be done in $\binom{n}{2k}|\mathfrak{B}_k| = \binom{n}{2k}(2k-1)!!$ different ways, completing the proof. □

The following statement is the key observation in our attempt to understand the structure of Λ.

Proposition 6. *Let $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ and $\alpha \in \Lambda_k$. Then the pre-image of α under the multiplication map*

$$\begin{array}{rccc} \mathrm{mult}: & \Lambda_1^{(1)} \times \Lambda_1^{(2)} \times \cdots \times \Lambda_1^{(k)} & \to & \Lambda_k \\ & (\eta_1, \eta_2, \dots, \eta_k) & \mapsto & \eta_1 \eta_2 \cdot \dots \cdot \eta_k \end{array}$$

consists of exactly $k!$ elements.

Proof. The map mult is well-defined by Proposition 4(ii) and is surjective by Proposition 4(iii). Let $X_1, \dots, X_k$ be disjoint two-element subsets of M, contained in α, and let $\sigma \in \mathcal{S}_k$. For $i \in \{1, \dots, k\}$ denote by $\gamma_i^{(\sigma)}$ the unique element of Λ_i containing $X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(i)}$. Set for convenience $\gamma_0^{(\sigma)} = \mathrm{id}$. Then the same arguments as in the proof of Proposition 4(iii) give a unique element, $\eta_i^{(\sigma)} \in \Lambda_1^{(i)}$, such that $\gamma_{i-1}^{(\sigma)} \eta_i^{(\sigma)} = \gamma_i^{(\sigma)}$. By construction we have $\eta_1^{(\sigma)} \eta_2^{(\sigma)} \cdot \dots \cdot \eta_k^{(\sigma)} \mathcal{R} \alpha$ and hence $\mathrm{mult}(\eta_1^{(\sigma)}, \eta_2^{(\sigma)}, \dots, \eta_k^{(\sigma)}) = \alpha$. Moreover, obviously,

$$(\eta_1^{(\sigma)}, \eta_2^{(\sigma)}, \dots, \eta_k^{(\sigma)}) \neq (\eta_1^{(\sigma')}, \eta_2^{(\sigma')}, \dots, \eta_k^{(\sigma')})$$

if $\sigma \neq \sigma'$. This means that the pre-image of α under mult consists of at least $k!$ elements. From the surjectivity of mult it follows that

$$|\Lambda_1^{(1)} \times \Lambda_1^{(2)} \times \cdots \times \Lambda_1^{(k)}| \geq |\Lambda_k| \cdot k!. \tag{3.1}$$

However, we have

$$|\Lambda_1^{(1)} \times \Lambda_1^{(2)} \times \cdots \times \Lambda_1^{(k)}| = \binom{n}{2} \binom{n-2}{2} \cdots \binom{n-2(k-1)}{2} \tag{3.2}$$

and, comparing (3.1), (3.2) and Lemma 5, we obtain that (3.1) must be an equality. The statement follows. □

We also have the following recursion for the canonical $\mathcal{R}$–cross-section.

Proposition 7. *Let $n > 2$ and Λ be a canonical $\mathcal{R}$–cross-section of $\mathfrak{B}_n$.*

(i) For every $\beta \in \Lambda$ the element $(\alpha_{n-1,n}\beta)|_{\{1,\dots,n-2\}}$ is a well-defined element of $\mathfrak{B}_{n-2}$ and the set $\Phi = \{(\alpha_{n-1,n}\beta)|_{\{1,\dots,n-2\}} : \beta \in \Lambda\}$ is a canonical $\mathcal{R}$–cross-section of $\mathfrak{B}_{n-2}$.

(ii) The map

$$\begin{array}{rccc} \varphi: & \Lambda & \to & \Phi \\ & \beta & \mapsto & (\alpha_{n-1,n}\beta)|_{\{1,\dots,n-2\}} \end{array}$$

is a homomorphism, which sends idempotent elements of corank 2 to the identity and is injective on the set of all nonidempotent elements of corank 2.

Proof. Consider the map $\overline{\varphi} : \Lambda \to \Lambda$, defined via $\overline{\varphi}(\beta) = \alpha_{n-1,n}\beta\alpha_{n-1,n}$ for all $\beta \in \Lambda$. A direct calculation shows that $\overline{\varphi}(\beta) = \alpha_{n-1,n}\beta$, which implies that $\overline{\varphi}$ is a homomorphism. It is easy to see that the image of $\overline{\varphi}$ coincides with $N = \{\beta \in \Lambda : \{n-1, n\}, \{(n-1)', n'\} \in \beta\}$. In particular, $\{1, \dots, n-2\}$ is β-invariant for every $\beta \in N$ and hence for all such β the element $\beta|_{\{1,\dots,n-2\}}$ is a well-defined element of $\mathfrak{B}_{n-2}$. Hence φ is well-defined. Moreover, φ is a homomorphism since both $\overline{\varphi} : \Lambda \to N$ and $\_|_{\{1,\dots,n-2\}} : N \to \mathfrak{B}_{n-2}$ are. This proves the first parts of (i) and (ii). The rest of (ii) is proved by a direct calculation.

So, we are left to show that $\varphi(\Lambda)$ is a canonical $\mathcal{R}$–cross-section of $\mathfrak{B}_{n-2}$. Forgetting $\{n-1, n\}$ identifies the elements in the set of all collections of two-element subsets of $\{1, \dots, n\}$, containing $\{n-1, n\}$, and the set of all collections of two-element subsets of $\{1, \dots, n-2\}$. Since Λ was an $\mathcal{R}$–cross-section of $\mathfrak{B}_n$, it follows that the map $\_|_{\{1,\dots,n-2\}}$ gives rise to a bijection between the elements of N and all collections of two-element subsets of $\{1, \dots, n-2\}$. This means that $\varphi(\Lambda) = N|_{\{1,\dots,n-2\}}$ is an $\mathcal{R}$–cross-section of $\mathfrak{B}_{n-2}$. It is easy to see that this cross-section is canonical. This completes the proof. □

By Proposition 4, Λ is completely determined (generated) by Λ_1. Recall that we are still working with a given Λ (and we still do not know if it exists). Λ_1 consists of $\alpha_{i,j}$ and hence to describe Λ_1 we have to determine all $\alpha_{i,j}$ explicitly. If $j \in \{n-1, n\}$ then $\alpha_{i,j}$ is an idempotent by Lemma 2(iv) and it is explicitly described by Lemma 2(iii). In all other cases, Corollary 3 gives only a precise definition of one part of $\alpha_{i,j}$, since the statement of Corollary 3(iii) describes certain parts of $\alpha_{i,j}$ (the partition sets containing $a \in M$, for $a > j$ or $a > j+1$, depending on some parities), only up to a bijection between two two-element sets (that is, roughly speaking, up to an element of $\mathcal{S}_2$). Our idea now is to write these undetermined parts as "parameters", identifying each of the parameters with an element of $\mathcal{S}_2$ (since in some sense they behave well under multiplication, see Lemma 8 below), and to investigate the relations between these parameters. Let $1 \le i < j < n-1$. For $l = 1, \dots, \lfloor \frac{n-j}{2} \rfloor$ we define $\alpha_{i,j}^{(l)} \in \mathcal{S}_2$ in the following way:

$$\alpha_{i,j}^{(l)} = \begin{cases} \text{id}, & (n-2l+2) \equiv_{\alpha_{i,j}} (n-2l)' \\ (1,2), & (n-2l+2) \equiv_{\alpha_{i,j}} (n-2l-1)' \end{cases}. \tag{3.3}$$

This definition is motivated by the following easy observation.

Lemma 8. *Let $1 \le i < j < n-1$, $l \in \{1, \dots, \lfloor \frac{n-j}{2} \rfloor\}$, and $1 \le s < t < n-1$ be such that $\lfloor \frac{n-t}{2} \rfloor \ge l+1$. Let $\beta = \alpha_{i,j}\alpha_{s,t}$.*

(i) *For every* $u \in \{n-2l+1, n-2l+2\}$ *there exists* $v \in \{n-2l-3, n-2l-2\}$ *such that* $u \equiv_\beta v$.

(ii) *Define* $\beta^{(l)} \in \mathcal{S}_2$ *as follows:*

$$\beta^{(l)} = \begin{cases} \mathrm{id}, & (n-2l+1) \equiv_\beta (n-2l-3)' \\ (1,2), & (n-2l+1) \equiv_\beta (n-2l-2)' \end{cases}. \tag{3.4}$$

Then $\beta^{(l)} = \alpha_{i,j}^{(l)} \alpha_{s,t}^{(l+1)}$.

Proof. (i) follows from Corollary 3(iii), and (ii) follows from the definitions (3.3) and (3.4). □

Now we would like to describe the canonical $\mathcal{R}$–cross-sections for small values of n.

Proposition 9. *(i) For* $n = 1$ *we have* 1 *trivial* $\mathcal{R}$*–cross-section.*

(ii) *For* $n = 2$ *we have only* 1 $\mathcal{R}$*–cross-section, moreover, it is canonical and consists of* $\alpha_{1,2}$ *and* id.

(iii) *For* $n = 3$ *we have* 1 *canonical* $\mathcal{R}$*–cross-section, consisting of idempotents* $\alpha_{1,2}$, $\alpha_{2,3}$, $\alpha_{1,3}$ *and* id.

(iv) *For* $n = 4$ *we have* 2 *canonical* $\mathcal{R}$*–cross-section, for one of them we have* $\alpha_{1,2}^{(1)} = \mathrm{id}$, *for another one we have* $\alpha_{1,2}^{(1)} = (1,2)$.

(v) *For* $n = 5$ *we have* 8 *canonical* $\mathcal{R}$*–cross-section, which correspond to independent choices of the parameters* $\alpha_{1,2}^{(1)}, \alpha_{2,3}^{(1)}, \alpha_{1,3}^{(1)} \in \mathcal{S}_2$.

(vi) *For* $n = 6$ *we have* 16 *canonical* $\mathcal{R}$*–cross-section, which correspond to independent choices of the parameters* $\alpha_{1,2}^{(1)}, \alpha_{1,2}^{(2)}, \alpha_{2,3}^{(1)}, \alpha_{1,3}^{(1)} \in \mathcal{S}_2$.

Proof. The statements (i), (ii), and (iii) are obvious. For $n = 4$ we observe that there is only one parameter, namely $\alpha_{1,2}^{(1)} \in \mathcal{S}_2$. A direct calculation shows that both values of the parameter indeed lead to cross-sections. This proves (iv). For $n = 5$ we observe that there are exactly 3 parameters, namely $\alpha_{1,2}^{(1)} \in \mathcal{S}_2$, $\alpha_{2,3}^{(1)} \in \mathcal{S}_2$, and $\alpha_{1,3}^{(1)} \in \mathcal{S}_2$. A direct calculation again shows that all values of these parameters indeed lead to cross-sections. This proves (v).

Let us now consider the case $n = 6$. In this case we have 7 parameters, namely $\alpha_{1,2}^{(1)} \in \mathcal{S}_2$, $\alpha_{1,2}^{(2)} \in \mathcal{S}_2$, $\alpha_{1,3}^{(1)} \in \mathcal{S}_2$, $\alpha_{1,4}^{(1)} \in \mathcal{S}_2$, $\alpha_{2,3}^{(1)} \in \mathcal{S}_2$, $\alpha_{2,4}^{(1)} \in \mathcal{S}_2$, and

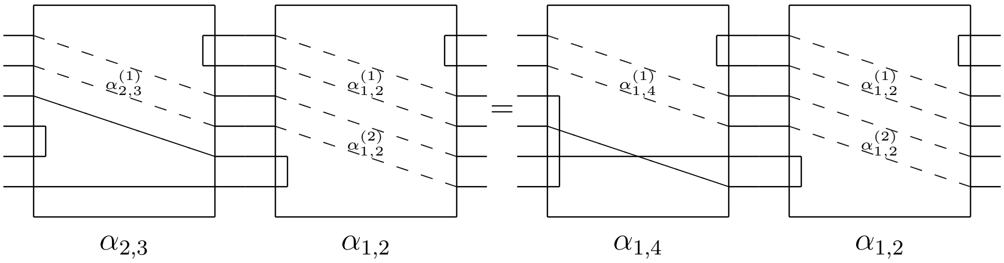

Figure 4: Illustration of the equality (3.6) implying $\alpha^{(1)}_{2,3} = \alpha^{(1)}_{1,4}$.

$\alpha^{(1)}_{3,4} \in \mathcal{S}_2$. Using the fact that Λ is an $\mathcal{R}$–cross-section, by a direct calculation we obtain that the following relations should be satisfied (see Figure 4):

$$\alpha_{1,3}\alpha_{1,2} = \alpha_{2,4}\alpha_{1,2}, \tag{3.5}$$

$$\alpha_{2,3}\alpha_{1,2} = \alpha_{1,4}\alpha_{1,2}, \tag{3.6}$$

$$\alpha_{1,2}\alpha_{1,2} = \alpha_{3,4}\alpha_{1,2}. \tag{3.7}$$

Using now Lemma 8, from (3.5) we obtain $\alpha^{(1)}_{1,3}\alpha^{(2)}_{1,2} = \alpha^{(1)}_{2,4}\alpha^{(2)}_{1,2}$ implying $\alpha^{(1)}_{1,3} = \alpha^{(1)}_{2,4}$. Analogously from (3.6) we obtain $\alpha^{(1)}_{2,3} = \alpha^{(1)}_{1,4}$, from (3.7) we obtain $\alpha^{(1)}_{1,2} = \alpha^{(1)}_{3,4}$. This implies that all parameters can be expressed in terms of $\alpha^{(1)}_{1,2}$, $\alpha^{(2)}_{1,2}$, $\alpha^{(1)}_{2,3}$, and $\alpha^{(1)}_{1,3}$. A direct (but quite long) calculation shows that all values of these parameters indeed lead to distinct cross-sections. This proves (vi). □

Later on we will also need the following relation between the (canonical) $\mathcal{R}$–cross-sections for different n.

Proposition 10. *Let* $\Gamma = \{\alpha \in \Lambda : 1 \equiv_\alpha 1'\}$. *Then the set* $\overline{\Gamma} = \{\alpha|_{\{2,\dots,n\}} : \alpha \in \Gamma\}$ *is an* $\mathcal{R}$*–cross-section of* $\mathfrak{B}(\{2,\dots,n\})$. *Identifying* $\{2,\dots,n\}$ *with* $\{1,\dots,n-1\}$ *via* $x \mapsto x-1$, $\overline{\Gamma}$ *becomes a canonical* $\mathcal{R}$*–cross-section of* $\mathfrak{B}_{n-1}$.

Proof. Left to the reader. □

Now we are ready to go to the general case. Denote by $\mathfrak{A}_n$ the set of all pairs $(n-3, n-2)$, $(n-5, n-4)$, ...

Proposition 11. *(i) Let $n \geq 8$ be even, then we have the following equalities:*

$$\alpha_{1,2}^{(l)} = \alpha_{i,j}^{(l)} \quad \textit{for all} \quad (i,j) \in \mathfrak{A}_n \textit{ and for all } l = 1, \ldots, \frac{n-j}{2}, \tag{3.8}$$

$$\alpha_{2,3}^{(l)} = \alpha_{i,j}^{(l)} \quad \textit{for all} \quad (i,j) \notin \mathfrak{A}_n \textit{ and for all } l = 1, \ldots, \lfloor \frac{n-j}{2} \rfloor, \tag{3.9}$$

$$\alpha_{1,2}^{(l)} \alpha_{2,3}^{(l+1)} = \alpha_{2,3}^{(l)} \alpha_{1,2}^{(l+1)} \textit{ for all } l = 1, \ldots, \frac{n}{2} - 3. \tag{3.10}$$

(ii) Let $n \geq 7$ be odd, then we have the following equalities:

$$\alpha_{2,3}^{(l)} = \alpha_{i,j}^{(l)} \quad \textit{for all} \quad (i,j) \in \mathfrak{A}_n \textit{ and for all } l = 1, \ldots, \frac{n-j}{2}, \tag{3.11}$$

$$\alpha_{1,2}^{(l)} = \alpha_{i,j}^{(l)} \quad \textit{for all} \quad (i,j) \notin \mathfrak{A}_n \textit{ and for all } l = 1, \ldots, \lfloor \frac{n-j}{2} \rfloor, \tag{3.12}$$

$$\alpha_{1,2}^{(l)} \alpha_{2,3}^{(l+1)} = \alpha_{2,3}^{(l)} \alpha_{1,2}^{(l+1)} \textit{ for all } l = 1, \ldots, \frac{n+1}{2} - 3. \tag{3.13}$$

Proof. We prove both statements using induction on n. We start with $n = 7$. In this case we use the fact that Λ is an $\mathcal{R}$–cross-section to obtain the following equalities:

$$\alpha_{2,3}\alpha_{2,3} = \alpha_{4,5}\alpha_{2,3}, \tag{3.14}$$

$$\alpha_{2,4}\alpha_{1,2} = \alpha_{1,5}\alpha_{1,2}, \tag{3.15}$$

$$\alpha_{1,4}\alpha_{1,2} = \alpha_{2,5}\alpha_{1,2}, \tag{3.16}$$

$$\alpha_{3,5}\alpha_{2,3} = \alpha_{2,4}\alpha_{2,3}, \tag{3.17}$$

$$\alpha_{3,4}\alpha_{2,3} = \alpha_{2,5}\alpha_{2,3}, \tag{3.18}$$

$$\alpha_{1,5}\alpha_{1,3} = \alpha_{3,4}\alpha_{1,3}, \tag{3.19}$$

$$\alpha_{1,3}\alpha_{2,3} = \alpha_{4,5}\alpha_{1,3}. \tag{3.20}$$

The arguments, analogous to those in the proof of Proposition 9 give the following: (3.14) implies $\alpha_{2,3}^{(1)} = \alpha_{4,5}^{(1)}$. Further, (3.15) implies $\alpha_{2,4}^{(1)} = \alpha_{1,5}^{(1)}$, (3.16) implies $\alpha_{1,4}^{(1)} = \alpha_{2,5}^{(1)}$, (3.17) implies $\alpha_{3,5}^{(1)} = \alpha_{2,4}^{(1)}$, (3.18) implies $\alpha_{3,4}^{(1)} = \alpha_{2,5}^{(1)}$, (3.19) implies $\alpha_{1,5}^{(1)} = \alpha_{3,4}^{(1)}$. This implies

$$\alpha_{1,5}^{(1)} = \alpha_{1,4}^{(1)} = \alpha_{2,4}^{(1)} = \alpha_{2,5}^{(1)} = \alpha_{3,4}^{(1)} = \alpha_{3,5}^{(1)}.$$

Moreover, (3.20) implies

$$\alpha_{1,3}^{(1)}\alpha_{2,3}^{(2)} = \alpha_{4,5}^{(1)}\alpha_{1,3}^{(2)}. \tag{3.21}$$

Now we have to go to the case-by-case analysis. We consider two cases, $\alpha_{1,3}^{(2)} = \mathrm{id}$ and $\alpha_{1,3}^{(2)} = (1,2)$.

For $\alpha_{1,3}^{(2)} = \mathrm{id}$ we have the identities $\alpha_{1,3}\alpha_{1,2} = \alpha_{2,4}\alpha_{1,3}$ and $\alpha_{1,3}\alpha_{1,3} = \alpha_{2,5}\alpha_{1,3}$ giving

$$\alpha_{1,3}^{(1)}\alpha_{1,2}^{(2)} = \alpha_{2,4}^{(1)}\alpha_{1,3}^{(2)} \text{ and } \alpha_{1,3}^{(1)}\alpha_{1,3}^{(2)} = \alpha_{2,5}^{(1)}\alpha_{1,3}^{(2)}, \tag{3.22}$$

respectively.

For $\alpha_{1,3}^{(2)} = (1,2)$ we have the identities $\alpha_{1,3}\alpha_{1,2} = \alpha_{2,5}\alpha_{1,3}$ and $\alpha_{1,3}\alpha_{1,3} = \alpha_{2,4}\alpha_{1,3}$ giving

$$\alpha_{1,3}^{(1)}\alpha_{1,2}^{(2)} = \alpha_{2,5}^{(1)}\alpha_{1,3}^{(2)} \text{ and } \alpha_{1,3}^{(1)}\alpha_{1,3}^{(2)} = \alpha_{2,4}^{(1)}\alpha_{1,3}^{(2)}, \tag{3.23}$$

respectively. Since we already know that $\alpha_{2,5}^{(1)} = \alpha_{2,4}^{(1)}$, we have that both (3.22) and (3.23) in fact do not depend on the values of $\alpha_{1,3}^{(2)}$.

Combining (3.21) and (3.23) we consequently obtain $\alpha_{1,3}^{(1)} = \alpha_{2,5}^{(1)}$, $\alpha_{1,3}^{(2)} = \alpha_{1,2}^{(2)}$ and $\alpha_{1,3}^{(1)}\alpha_{2,3}^{(2)} = \alpha_{4,5}^{(1)}\alpha_{1,2}^{(2)}$. To complete the proof we are now left to show that $\alpha_{1,3}^{(1)} = \alpha_{1,2}^{(1)}$

Consider two cases: $\alpha_{1,2}^{(2)} = \mathrm{id}$ and $\alpha_{1,2}^{(2)} = (1,2)$. In the first case we obtain $\alpha_{1,2}\alpha_{1,2} = \alpha_{3,4}\alpha_{1,2}$ implying

$$\alpha_{1,2}^{(1)}\alpha_{1,2}^{(2)} = \alpha_{3,4}^{(1)}\alpha_{1,2}^{(2)}. \tag{3.24}$$

In the second case case we obtain $\alpha_{1,2}\alpha_{1,2} = \alpha_{3,5}\alpha_{1,2}$ implying $\alpha_{1,2}^{(1)}\alpha_{1,2}^{(2)} = \alpha_{3,5}^{(1)}\alpha_{1,2}^{(2)}$, which is the same as (3.24) as we already know that $\alpha_{3,4}^{(1)} = \alpha_{3,5}^{(1)}$. This implies that (3.24) holds in all cases, which gives $\alpha_{1,2}^{(1)} = \alpha_{3,4}^{(1)} = \alpha_{2,5}^{(1)} = \alpha_{1,3}^{(1)}$. So, the case $n = 7$ is complete.

Now we prove the induction step and, because of the inductive assumption and Proposition 10, it is enough to prove either the statement (3.12) or the statements (3.8), (3.9), respectively, for the elements $\alpha_{1,j}$, $j = 2, \dots, n-2$, and either the statement (3.10) or (3.13), respectively, depending on the parity of n.

Assume that n is odd. Then $\alpha_{1,j} \notin \mathfrak{A}_n$ for all $j = 2, \dots, n-2$ and hence we have to check (3.12). For even $j \geq 4$ we have $\alpha_{1,j}\alpha_{1,2} = \alpha_{2,j+1}\alpha_{1,2}$ and for odd $j \geq 5$ we have $\alpha_{1,j}\alpha_{1,2} = \alpha_{2,j-1}\alpha_{1,2}$ and in both cases we obtain $\alpha_{1,j}^{(l)} = \alpha_{3,4}^{(l)}$ for all possible l by inductive assumptions.

Let $l = \frac{n-3}{2}$. Then for $\alpha_{1,2}^{(l)} = \mathrm{id}$ we have $\alpha_{1,2}\alpha_{1,2} = \alpha_{3,4}\alpha_{1,2}$ and for $\alpha_{1,2}^{(l)} = (1,2)$ we have $\alpha_{1,2}\alpha_{1,2} = \alpha_{3,5}\alpha_{1,2}$. But since $\alpha_{3,4}^{(s)} = \alpha_{3,5}^{(s)}$ for all possible s by induction, we obtain $\alpha_{1,2}^{(s)} = \alpha_{3,4}^{(s)}$ for all $s < l$.

Let $l=\frac{n-3}{2}$. Then for $\alpha_{1,3}^{(l)}=\mathrm{id}$ we have $\alpha_{1,3}\alpha_{1,3}=\alpha_{2,4}\alpha_{1,3}$ and for $\alpha_{1,3}^{(l)}=(1,2)$ we have $\alpha_{1,3}\alpha_{1,3}=\alpha_{2,5}\alpha_{1,3}$. But, since $\alpha_{2,4}^{(s)}=\alpha_{2,5}^{(s)}$ for all possible s by induction, we obtain $\alpha_{1,3}^{(s)}=\alpha_{2,4}^{(s)}$ for all $s<l$. That $\alpha_{1,3}^{(l)}=\alpha_{1,2}^{(l)}$ is proved using the same arguments as in the paragraph containing the formula (3.24). So, the proof of (3.12) for the elements $\alpha_{1,j}$, $j=2,\dots,n-2$ is complete.

Finally, we have $\alpha_{1,2}\alpha_{2,3}=\alpha_{4,5}\alpha_{1,2}$ which implies (3.13).

Assume now that n is even. Then we have $\alpha_{1,2}\alpha_{1,2}=\alpha_{3,4}\alpha_{1,2}$ which implies (3.8) for $\alpha_{1,2}$. For the elements $\alpha_{1,j}$, $j\geq 4$ the arguments are the same as in the case of odd n. Further, we also have $\alpha_{1,3}\alpha_{1,2}=\alpha_{2,4}\alpha_{1,2}$ implying equalities (3.9) for $\alpha_{1,3}$. And finally, (3.10) follows from the inductive assumption and (3.8). This completes the proof. □

Now we are ready to construct a canonical $\mathcal{R}$–cross-section for $\mathfrak{B}_n$ in the general case.

Proposition 12. *Let $n\in\mathbb{N}$, $l=\lfloor\frac{n-2}{2}\rfloor$, $m=\lfloor\frac{n-3}{2}\rfloor$, and choose $x_1,\dots,x_l$, $y_1,\dots,y_m\in\mathcal{S}_2$ such that $x_iy_{i+1}=y_ix_{i+1}$ for all possible i. For $1\leq i<j\leq n$ let $\alpha_{i,j}$ be the element which satisfies the corresponding equalities of Lemma 2*(iii) *and Corollary 3, and, additionally, the following conditions:*

(a) $\alpha_{i,j}^{(s)}=x_s$ for all $(i,j)\in\mathfrak{A}_n$ and for all possible s;

(b) $\alpha_{i,j}^{(s)}=y_s$ for all $(i,j)\notin\mathfrak{A}_n$ and for all possible s.

Then the elements id *and $\alpha_{i,j}$, $1\leq i<j\leq n$, generate a canonical $\mathcal{R}$–cross-section of $\mathfrak{B}_n$.*

Proof. Set $\Gamma=\{\alpha_{i,j}\,:\,1\leq i<j\leq n\}$ and let $\Phi=\langle\Gamma\rangle$ be the monoid, generated by Γ. For $k=1,2,\dots,\lfloor\frac{n}{2}\rfloor$ set $\Gamma_k=\{\alpha_{i,j}\,:\,j\leq n-2(k-1)\}$. Define $\Phi_0=\{\mathrm{id}\}$ and for $i=1,2,\dots,\lfloor\frac{n}{2}\rfloor$ set $\Phi_i=\Gamma_1\Gamma_2\cdot\dots\cdot\Gamma_i$. Finally, set $\overline{\Phi}=\cup_{i=0}^{\lfloor\frac{n}{2}\rfloor}\Phi_i$. We are going to show that $\overline{\Phi}$ contains exactly one element of each $\mathcal{R}$–class of $\mathfrak{B}_n$, and then that $\overline{\Phi}=\Phi$.

By Theorem 1(ii), every $\mathcal{R}$–class of $\mathfrak{B}_n$ of corank $2k$ is uniquely determined by an unordered collection of k disjoint two-element subsets of M. Using the same arguments as in Proposition 6 one shows that every $\mathcal{R}$–class of $\mathfrak{B}_n$ of corank $2k$ contains exactly $k!$ products of the form $\eta_1\cdot\dots\cdot\eta_k$, where all $\eta_i\in\Gamma_i$.

Step 1. Let us first show that, under the assumptions of our statement, Φ_2 contains exactly one element of each $\mathcal{R}$–class of corank 4. The arguments above show that every such class contains at most two elements of Φ_2. Consider $\alpha_{i,j}$ and $\alpha_{s,t}$ such that $t<n-1$. Then there exist the unique pair

$u < v \in M$ such that $u \equiv_{\alpha_{i,j}\alpha_{s,t}} v$ and $(u,v) \neq (i,j)$, moreover, there exists a unique pair, $p < q \in M$, such that

$$\alpha_{i,j}\alpha_{s,t}\mathcal{R}\alpha_{u,v}\alpha_{p,q}. \tag{3.25}$$

It is enough to show that

$$\alpha_{i,j}\alpha_{s,t} = \alpha_{u,v}\alpha_{p,q}. \tag{3.26}$$

Lemma 13. (3.26) *is equivalent to the collection of the following conditions:*

$$\alpha_{n-1,n}\alpha_{i,j}\alpha_{s,t} = \alpha_{n-1,n}\alpha_{u,v}\alpha_{p,q}, \tag{3.27}$$

$$(n-1) \equiv_{\alpha_{i,j}\alpha_{s,t}} f \Rightarrow (n-1) \equiv_{\alpha_{u,v}\alpha_{p,q}} f \quad \textit{for all} \quad f \in M \cup M', \tag{3.28}$$

$$n \equiv_{\alpha_{i,j}\alpha_{s,t}} f \Rightarrow n \equiv_{\alpha_{u,v}\alpha_{p,q}} f \quad \textit{for all} \quad f \in M \cup M'. \tag{3.29}$$

Proof. That (3.26) implies (3.27), (3.28), and (3.29) is obvious. Hence we assume that (3.27), (3.28), and (3.29) are satisfied and we have to prove (3.26). Set $\alpha = \alpha_{i,j}\alpha_{s,t}$ and $\beta = \alpha_{u,v}\alpha_{p,q}$. We have to show that for every $x, y \in M \cup M'$ the condition $x \equiv_\alpha y$ implies the condition $x \equiv_\beta y$. We know that $\mathrm{corank}(\alpha) = \mathrm{corank}(\beta) = 4$. By the definition of Γ we have that both, $\{(n-1)', n'\}$ and $\{(n-2)', (n-3)'\}$, belong to both α and β. Hence, without loss of generality we can assume that $x \in M$. If $\{x,y\} \cap \{n-1, n\} \neq \varnothing$, then the necessary statement follows from (3.28) and(3.29). If $\{x,y\} \cap \{n-1,n\} = \varnothing$ then it follows from (3.27) since $a \equiv_{\alpha_{n-1,n}} a'$ for all $a < n-1$. □

Now we claim that (3.27) follows by induction on n with the cases $n = 1, 2, 3$ being trivial. The idea of the induction is based on the statement of Proposition 7. From the definition of $\alpha_{x,y}$ we have $\alpha_{x,y}\alpha_{n-1,n} = \alpha_{x,y}$ for all appropriate x, y. Hence (3.27) is equivalent to

$$\alpha_{n-1,n}\alpha_{i,j}\alpha_{n-1,n}\alpha_{s,t} = \alpha_{n-1,n}\alpha_{u,v}\alpha_{n-1,n}\alpha_{p,q}. \tag{3.30}$$

Consider the map

$$\begin{array}{rccc} \psi: & \Phi & \to & \mathfrak{B}_{n-2} \\ & \beta & \mapsto & (\alpha_{n-1,n}\beta)|_{\{1,\dots,n-2\}}. \end{array}$$

As in Proposition 7 one obtains that ψ is a homomorphism from Φ to $\mathfrak{B}_{n-2}$, which maps $\alpha_{x,y}$ to the identity element if $y \in \{n-1, n\}$ and to an element of corank 2 otherwise. Furthermore, it is easy to see that $\{\psi(\alpha_{x,y}) : y < n-1\}$ satisfy all the assumptions of our statement and hence by induction we obtain

$$\psi(\alpha_{i,j}\alpha_{s,t}) = \psi(\alpha_{i,j})\psi(\alpha_{s,t}) = \psi(\alpha_{u,v})\psi(\alpha_{p,q}) = \psi(\alpha_{u,v}\alpha_{p,q})$$

if $j, v < n-1$. If $j \in \{n-1, n\}$ or $v \in \{n-1, n\}$ then (3.27) is straightforward since our "parameters" $\alpha^{(s)}_{x,y}$ do not affect any part of (3.27) at all. This implies (3.27).

Thus we are left to prove (3.28) and (3.29). If $\{i, j\} = \{n-1, n\}$, then both (3.28) and (3.29) follows from (3.25). If there exist $f, g \in M \setminus \{n-1, n\}$ such that $(n-1) \equiv_{\alpha_{i,j}\alpha_{s,t}} f$ and $n \equiv_{\alpha_{i,j}\alpha_{s,t}} g$, then we have either $(n-1) \equiv_{\alpha_{i,j}\alpha_{s,t}} i$ and $n \equiv_{\alpha_{i,j}\alpha_{s,t}} u$ or $(n-1) \equiv_{\alpha_{i,j}\alpha_{s,t}} u$ and $n \equiv_{\alpha_{i,j}\alpha_{s,t}} i$. Combining this with (3.25) we obtain either $(n-1) \equiv_{\alpha_{u,v}\alpha_{p,q}} i$ and $n \equiv_{\alpha_{u,v}\alpha_{p,q}} u$ or $(n-1) \equiv_{\alpha_{u,v}\alpha_{p,q}} u$ and $n \equiv_{\alpha_{u,v}\alpha_{p,q}} i$, respectively.

Assume that there exist $f, g \in M$ such that $(n-1) \equiv_{\alpha_{i,j}\alpha_{s,t}} f'$ and $n \equiv_{\alpha_{i,j}\alpha_{s,t}} g'$. Then we in fact have to prove that

$$\alpha^{(1)}_{i,j}\alpha^{(2)}_{s,t} = \alpha^{(1)}_{u,v}\alpha^{(2)}_{p,q}. \tag{3.31}$$

If the pairs (i, j), (s, t), (u, v), and (p, q) either all belong or all do not belong to $\mathfrak{A}_n$, then our equality reduces to the obvious identities $x_1x_2 = x_1x_2$ and $y_1y_2 = y_1y_2$, respectively.

Assume now that $(i, j) \in \mathfrak{A}_n$ and $(s, t) \notin \mathfrak{A}_n$. If $t < i$, then one obtains $(u, v) = (s, t) \notin \mathfrak{A}_n$ and $(p, q) = (i-2, j-2) \in \mathfrak{A}_n$. If $s \geq i$, then $(s, t) \notin \mathfrak{A}_n$ means that s and t belong to different sets from the $\{n, n-1\}$, $\{n-2, n-3\}$,.... From Corollary 3(iii) it follows that in this case u and v belong to different sets from the $\{n, n-1\}$, $\{n-2, n-3\}$,... as well. This implies that $(u, v) \notin \mathfrak{A}_n$. It also follows that $(p, q) = (i, j) \in \mathfrak{A}_n$. Finally, assume that $s < i \leq t$, then $v - u > 2$ and hence $(u, v) \notin \mathfrak{A}_n$. Moreover, in the last case we also have $(p, q) = (i, j) \in \mathfrak{A}_n$. The case $(i, j) \notin \mathfrak{A}_n$ and $(s, t) \in \mathfrak{A}_n$ is analogous and we obtain that exactly one pair on the right hand side of (3.26) belongs to $\mathfrak{A}_n$. Then (3.31) reduces to $x_1y_2 = y_1x_2$, which is again the case.

Finally, let us assume that there exist $f, g \in M$ such that $(n-1) \equiv_{\alpha_{i,j}\alpha_{s,t}} f$ and $n \equiv_{\alpha_{i,j}\alpha_{s,t}} g'$. In this case we have either $i = f$, $j = n-1$ and $\alpha_{i,j}$ is an idempotent or $u = f$, $v = n-1$, and $\alpha_{u,v}$ is an idempotent. Without loss of generality we assume that $\alpha_{i,j}$ is an idempotent. Now, using (3.25), we obtain $(n-1) \equiv_{\alpha_{u,v}\alpha_{p,q}} f$. Moreover, we obviously have $\alpha_{u,v} = \alpha_{s,t}$ and $f \equiv_{\alpha_{s,t}} g'$. Observe that $p = g$ and $q \in \{n-2, n-3\}$. If $q = n-3$, we have $(n-2) \equiv_{\alpha_{p,q}} g'$. If $q = n-2$, we have $(n-3) \equiv_{\alpha_{p,q}} g'$. Hence in both cases we obtain $n \equiv_{\alpha_{u,v}\alpha_{p,q}} g'$. For the case $(n-1) \equiv_{\alpha_{i,j}\alpha_{s,t}} g'$ and $n \equiv_{\alpha_{i,j}\alpha_{s,t}} f$ the arguments are similar. This completes the proof of Step 1.

Step 2. Now we go to elements of arbitrary coranks. Let $\eta_1 \cdot \dots \cdot \eta_k \in \Phi_k$. If $\zeta \in \mathfrak{B}_n$ has corank 2 then for every $\alpha \in \mathfrak{B}_n$ we have that corank$(\alpha\zeta)$ is either corank(α) or corank$(\alpha) + 2$. The same holds for corank$(\zeta\alpha)$. This implies that for every $i = 1, \dots, k-1$ we have corank$(\eta_i\eta_{i+1}) = 4$. In particular, by Step 1 there exists a unique pair, $(\eta'_i, \eta'_{i+1}) \in \Gamma_1 \times \Gamma_2$ such that $(\eta'_i, \eta'_{i+1}) \neq (\eta_i, \eta_{i+1})$ and $\eta_i\eta_{i+1} = \eta'_i\eta'_{i+1}$, which, in particular, implies that $\eta'_i \in \Gamma_i$ and $\eta'_{i+1} \in \Gamma_{i+1}$. This allows us to define the involution $\mathfrak{i}_i^{(k)}$ on the set $\Gamma(k) = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_k$ via

$$\mathfrak{i}_i^{(k)}(\eta_1, \dots, \eta_k) = (\eta_1, \dots, \eta_{i-1}, \eta'_i, \eta'_{i+1}, \eta_{i+2}, \dots, \eta_k).$$

Note that for $(\eta_1, \dots, \eta_k), (\eta'_1, \dots, \eta'_k) \in \Gamma(k)$ such that

$$(\eta_1, \dots, \eta_k) = \mathfrak{i}_i^{(k)}(\eta'_1, \dots, \eta'_k)$$

from Step 1 we have

$$\eta_1 \cdot \cdots \cdot \eta_k = \eta'_1 \cdot \cdots \cdot \eta'_k. \tag{3.32}$$

Fix $(\eta_1, \dots, \eta_k) \in \Gamma(k)$ and let $\alpha = \eta_1 \cdot \cdots \cdot \eta_k$. Let $X_1, \dots, X_k$ be the ordered collection of disjoint two-element subsets of M such that for every $i = 1, \dots, k$ the element $\eta_1 \cdot \cdots \cdot \eta_i$ contains $X_1, \dots, X_i$. Set

$$\Gamma(k, \alpha) = \{(\eta'_1, \dots, \eta'_k) \in \Gamma(k) \, : \, \eta'_1 \cdot \cdots \cdot \eta'_k \mathcal{R} \alpha\}.$$

We already know that $|\Gamma(k, \alpha)| = k!$, moreover, the arguments used to prove this allow us to define a bijective map, $\varphi : \Gamma(k, \alpha) \to \mathcal{S}_k$, in the following way: to an element, $(\eta'_1, \dots, \eta'_k) \in \Gamma(k, \alpha)$, we associate $\sigma \in \mathcal{S}_k$ such that for every $i = 1, \dots, k$ the element $\eta'_1 \cdot \cdots \cdot \eta'_i$ contains $X_{\sigma(1)}, \dots, X_{\sigma(i)}$. Let $\mathfrak{s}_i$ denote the simple transposition $(i, i+1) \in \mathcal{S}_k$. From the definitions of φ and $\mathfrak{i}_i^{(k)}$ for arbitrary $\beta \in \Gamma(k, \alpha)$ we have

$$\varphi \circ \mathfrak{i}_i^{(k)}(\beta) = \varphi(\beta)\mathfrak{s}_i. \tag{3.33}$$

For every $\sigma \in \mathcal{S}_k$ fix a reduced decomposition, $\mathfrak{s}_{a^\sigma_{l_\sigma}} \dots \mathfrak{s}_{a^\sigma_2}\mathfrak{s}_{a^\sigma_1}$, of σ and define

$$\alpha(\sigma) = \mathfrak{i}^{(k)}_{a^\sigma_1} \circ \mathfrak{i}^{(k)}_{a^\sigma_2} \circ \cdots \circ \mathfrak{i}^{(k)}_{a^\sigma_{l_\sigma}}(\eta_1, \dots, \eta_k).$$

From (3.33) we obtain $\varphi(\alpha(\sigma)) = \sigma$, implying $|\{\alpha(\sigma) \, : \, \sigma \in \mathcal{S}_k\}| = k!$. This yields $\{\alpha(\sigma) \, : \, \sigma \in \mathcal{S}_k\} = \Gamma(k, \alpha)$ and hence (3.32) implies that all elements of $\Gamma(k, \alpha)$ produce the same element, namely α, via multiplication. This means that $\overline{\Phi}$ contains precisely one element in each $\mathcal{R}$–class and is exactly what we wanted to prove.

Step 3. Obviously, $\overline{\Phi} \subset \Phi$, and we are left to show that $\overline{\Phi} = \Phi$. From the definition of $\overline{\Phi}$ it follows that it is enough to show that $\alpha\eta \in \Phi_k$ for every $\alpha \in \Phi_k$ and $\eta \in \Gamma \setminus \Gamma_k$. However, $\alpha \in \Phi_k$ implies that

$$n' \equiv_\alpha (n-1)', (n-2)' \equiv_\alpha (n-3)', \dots, (n-2(k-1))' \equiv_\alpha (n-2(k-1)-1)',$$

by the definition of Φ_k. At the same time $\eta \in \Gamma \setminus \Gamma_k$ implies that $\eta = \alpha_{i,j}$ and $j > n - 2k$. Now Corollary 3 and a direct calculation imply $\alpha\eta = \alpha$, which completes the proof. □

Theorem 14. *Let Λ be a canonical $\mathcal{R}$–cross-section of $\mathfrak{B}_n$. Then the generators $\alpha_{i,j}$, $1 \leq i < j \leq n$, and the following relations:*

(a) $\alpha_{s,t}\alpha_{i,j} = \alpha_{s,t}$ for all possible s, t, i and for all $j = n-1, n$,

(b) $\alpha_{s,t}\alpha_{i,j} = \alpha_{u,v}\alpha_{x,y}(= \beta)$ for all $\beta \in \Lambda$, $\mathrm{corank}(\beta) = 4$, with appropriate s, t, i, j, u, v, x, y given by Step 1 of the proof of Proposition 12,

form a copresentation of the monoid Λ.

Proof. Let Ψ denote the monoid, generated by $\gamma_{i,j}$, $1 \leq i < j \leq n$, satisfying

(c) $\gamma_{s,t}\gamma_{i,j} = \gamma_{s,t}$ for all possible s, t, i and for all $j = n-1, n$,

(d) $\gamma_{s,t}\gamma_{i,j} = \gamma_{u,v}\gamma_{x,y}$ for all s, t, i, j, u, v, x, y such that $\alpha_{s,t}\alpha_{i,j} = \alpha_{u,v}\alpha_{x,y}$ is an element of corank 4.

Let $\mathfrak{f} : \{\gamma_{i,j} : 1 \leq i < j \leq n\} \to \{\alpha_{i,j} : 1 \leq i < j \leq n\}$ be the bijection given by $\mathfrak{f}(\gamma_{i,j}) = \alpha_{i,j}$. Then $\mathfrak{f}$ obviously extends to an epimorphism, $\overline{\mathfrak{f}} : \Psi \to \Lambda$. To complete the proof it is enough to show that $|\Psi| = |\Lambda|$. For $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ let $\Psi^{(k)}$ denote the set $\{\gamma_{i,j} : j \leq n - 2(k-1)\}$. Define $\Psi_0 = \{\mathrm{id}\}$ and for $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ let Ψ_k denote the image of the multiplication map

$$\mathrm{mult} : \Psi^{(1)} \times \cdots \times \Psi^{(k)} \to \Psi.$$

Define $\overline{\Psi} = \cup_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Psi_k$. Using the relations from (d) and the arguments analogous to those of Proposition 6 and Step 2 of Proposition 12 one shows that $|\Psi_k| = |\Lambda_k|$ for every $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. Hence to complete the proof we have just to show that $\Psi = \overline{\Psi}$. To prove this it is enough to show that for any $\gamma \in \Psi_k$ and $\eta \notin \Psi^{(k+1)}$ we have $\gamma\eta \in \Psi_k$. Assume that $\eta \in \Psi^{(l)} \setminus \Psi^{(l+1)}$ for some $l \leq k$, that is $\eta = \gamma_{u,v}$ and $v \in \{n - 2(l-1), n - 2(l-1) - 1\}$. Let $\gamma = \eta_1 \cdot \dots \cdot \eta_k$ and $\eta_i \in \Psi^{(i)}$ for all i. Consider $\eta_1 \cdot \dots \cdot \eta_k \eta$. Using (d) we have $\eta_k \eta = \eta^{(1)} \eta'_k$, moreover, $\mathfrak{f}(\eta_k)\mathfrak{f}(\eta) = \mathfrak{f}(\eta^{(1)})\mathfrak{f}(\eta'_k)$. Hence, using Corollary 3(iii), we obtain $\mathfrak{f}(\eta^{(1)}) \in \Lambda_1^{(l-1)} \setminus \Lambda_1^{(l)}$ and $\mathfrak{f}(\eta'_k) \in \Lambda_1^{(k)}$, which implies $\eta^{(1)} \in \Psi^{(l-1)} \setminus \Psi^{(l)}$ and $\eta'_k \in \Psi^{(k)}$. We proceed inductively and after $l-1$ steps we obtain

$$\eta_1 \cdot \dots \cdot \eta_k \eta = \eta_1 \cdot \dots \cdot \eta_{k-l+1} \eta^{(l-1)} \eta'_{k-l+2} \cdot \dots \cdot \eta'_k,$$

where $\eta^{(l-1)} \in \Psi^{(1)} \setminus \Psi^{(2)}$ and $\eta'_i \in \Psi^{(i)}$ for all $i > k - l + 1$. Now (c) implies $\eta_{k-l+1}\eta^{(l-1)} = \eta_{k-l+1}$ and hence $\eta_1 \cdot \dots \cdot \eta_k \eta \in \Psi_k$. This completes the proof. □

4 The main results

Lemma 15. *Let Θ be an arbitrary $\mathcal{R}$–cross-section in $\mathfrak{B}_n$. Then Θ is generated (as a monoid) by elements of corank 2.*

Proof. Let Θ_1 be the set of elements of corank 2 in Θ. We have to show that $\alpha \in \langle \Theta_1 \rangle$ for every $\alpha \in \Theta$. We use the induction on k such that $2k = \mathrm{corank}(\alpha)$, with the case $k = 1$ being trivial. Let $\alpha \in \Theta$ be such that $\mathrm{corank}(\alpha) = 2k$, and let $i_1, \dots, i_k, j_1, \dots, j_k \in M$ be pairwise different and such that $i_s \equiv_\alpha j_s$ for all $s = 1, \dots, k$. Let β be the element of Θ of corank $2k-2$ such that $i_s \equiv_\beta j_s$ for all $s = 1, \dots, k-1$. By inductive assumption, β can be decomposed into a product of elements of corank 2 from Θ. Further, since $\mathrm{corank}(\beta) = 2k - 2$, we have that there exist $u, v \in M$ such that $i_k \equiv_\beta u'$ and $j_k \equiv_\beta v'$. Let further γ be the element of corank 2 in Θ such that $u \equiv_\gamma v$. Then $\beta\gamma \in \Theta$, $\beta\gamma\mathcal{R}\alpha$ and thus $\beta\gamma = \alpha$ as Θ is an $\mathcal{R}$–cross-section. This proves the induction step and completes the proof. □

The group $\mathcal{S}_n$ acts by isomorphisms on $\mathfrak{B}_n$ via conjugation. This action induces an action of $\mathcal{S}_n$ on the set of all $\mathcal{R}$–cross-sections of $\mathfrak{B}_n$. Our next step towards the classification of all $\mathcal{R}$–cross-sections of $\mathfrak{B}_n$ is the following statement:

Proposition 16. *Every $\mathcal{R}$–cross-section of $\mathfrak{B}_n$ is $\mathcal{S}_n$-conjugated to a canonical $\mathcal{R}$–cross-section.*

Proof. Let Θ be an $\mathcal{R}$–cross-section of $\mathfrak{B}_n$. To prove our statement it is certainly enough to show that there exists a sequence, $X_1, \dots, X_{\lfloor \frac{n}{2} \rfloor}$, of disjoint two-element subsets of M such that for every $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ each element $\alpha \in \Theta$ of corank $2k$ contains $X_1', X_2', \dots, X_k'$.

To prove our statement we will need two lemmas.

Lemma 17. *All elements of Θ of corank 2 belong to the same $\mathcal{L}$-class.*

Proof. Let $\alpha \in \Theta$ be an element of corank 2 and $i' \equiv_\alpha j'$ for some $i, j \in M$. Consider the element $\beta \in \Theta$ of corank 2 such that $i \equiv_\beta j$. Then $\alpha\beta$ has corank 2, moreover, $\alpha\beta\mathcal{R}\alpha$. Thus $\alpha\beta = \alpha$, which implies that β must coincide with the (unique) idempotent $\pi_{\{i,j\}}$ of corank 2, which satisfies $s \equiv_{\pi_{\{i,j\}}} s'$ for all $s \notin \{i, j\}$.

Now assume that there are $\alpha, \beta \in \Theta$ such that $\mathrm{corank}(\alpha) = \mathrm{corank}(\beta) = 2$ and the elements α and β belong to different $\mathcal{L}$–classes. We have to consider two cases. First we assume that there exist different $i, j, t \in M$ such that $i' \equiv_\alpha j'$ and $j' \equiv_\beta t'$. In this case the above observation implies $\pi_{\{i,j\}}, \pi_{\{j,t\}} \in \Theta$. But then $\pi_{\{i,j\}}\pi_{\{j,t\}}\mathcal{R}\pi_{\{i,j\}}$ and $\pi_{\{i,j\}}\pi_{\{j,t\}} \neq \pi_{\{i,j\}}$ (see Figure 5), which

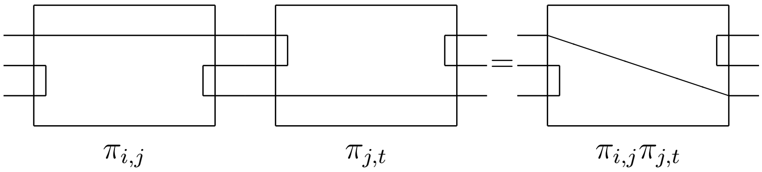

Figure 5: $\pi_{\{i,j\}}\pi_{\{j,t\}}\mathcal{R}\pi_{\{i,j\}}$ and $\pi_{\{i,j\}}\pi_{\{j,t\}} \neq \pi_{\{i,j\}}$.

contradicts the fact that Θ is an $\mathcal{R}$–cross-section. In the second case we assume that there exist pairwise different $i, j, s, t \in M$ such that $i' \equiv_\alpha j'$ and $s' \equiv_\beta t'$. In this case the above observation implies $\pi_{\{i,j\}}, \pi_{\{s,t\}} \in \Theta$. Let $\gamma \in \Theta$ be such that $j \equiv_\gamma s$. Then it is easy to see that $\pi_{\{i,j\}}\gamma\mathcal{R}\pi_{\{i,j\}}$ and thus $i' \equiv_\gamma j'$ since Θ is an $\mathcal{R}$–cross-section. Analogously we obtain $s' \equiv_\gamma t'$, which contradicts the fact that $\mathrm{corank}(\gamma) = 2$. This completes the proof. □

Lemma 17 and Theorem 1(i) say that there exists a two-element subset, X_1, of M such that X_1' belongs to every element of Θ of corank 2. Since the elements of corank 2 generate Θ by Lemma 15 we even have that X_1' belongs to every element of Θ of corank at least 2. Let π denote the unique element of Θ of corank 2, containing X_1. It is obviously an idempotent and $i \equiv_\pi i'$ for all $i \notin X_1$. Define $\overline{\varphi} : \Theta \to \Theta$ via $\overline{\varphi}(\alpha) = \pi\alpha\pi$, $\alpha \in \Theta$, and set $N = \overline{\varphi}(\Theta)$.

Lemma 18. *(i) $\overline{\varphi}(\alpha) = \pi\alpha$ for all $\alpha \in \Theta$.*

(ii) The set $M \setminus X_1$ is $\overline{\varphi}(\alpha)$-invariant for all $\alpha \in \Theta$, in particular, the element $\varphi(\alpha) = (\overline{\varphi}(\alpha))|_{M\setminus X_1}$ is a well-defined element of $\mathfrak{B}(M \setminus X_1)$.

(iii) The map $\varphi : \Theta \to \mathfrak{B}(M \setminus X_1)$ is a homomorphism and its image is an $\mathcal{R}$–cross-section of $\mathfrak{B}(M \setminus X_1)$.

Proof. Analogous to that of Proposition 7 and hence is left to the reader. □

After Lemma 18 we can use induction on n. First we observe that for $n = 1, 2$ our statement is trivial. Let $n > 2$. Since $\varphi(\Theta)$ is an $\mathcal{R}$–cross-section of $\mathfrak{B}(M \setminus X_1)$ by Lemma 18(iii), by induction there exists a sequence, $X_2, \dots, X_{\lfloor \frac{n}{2} \rfloor}$, or disjoint two-element subsets of $M \setminus X_1$ such that for every $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$ each element $\beta \in \varphi(\Theta)$ of corank $2k$ contains $X_2', \dots, X_{k+1}'$.

Now let $k \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ and $\alpha \in \Theta$ be an element of corank $2k$. Let $X_1 = \{x, y\}$. Assume first that either x or y (or both) does not belong to a line in α. Then it is easy to see that the element $\overline{\varphi}(\alpha)$ also has corank $2k$

and thus $\varphi(\alpha)$ has corank $2(k-1)$. Thus $\varphi(\alpha)$ contains $X'_2, \dots, X'_k$, which implies that α contains $X'_1, \dots, X'_k$.

If both, x and y, belong to some lines in α, then the element $\overline{\varphi}(\alpha)$ has corank $2k+2$ and thus $\varphi(\alpha)$ has corank $2k$. Therefore, by the induction hypothesis, $\varphi(\alpha)$ contains $X'_2, \dots, X'_{k+1}$ and thus $\overline{\varphi}(\alpha)$ contains $X'_1, \dots, X'_{k+1}$. Using Lemma 15 we can write $\alpha = \eta\beta$, where η has corank 2 and β has corank $2k-2$. Using induction on k (the case $k=1$ is given by Lemma 17), we can assume that β contains $X'_1, \dots, X'_{k-1}$ and hence so does α. Now we have to find out if α contains X'_k or X'_{k+1} (it must contain exactly one of them since $\mathrm{corank}(\alpha) = 2k$). Assume that α contains X'_{k+1}. Note that if β is any element of corank $2k$ that contains $\{x, y\}$, then β must contain $X'_1, \dots, X'_k$, by the arguments of the previous paragraph. Let $X_k = \{a, b\}$ and $X_{k+1} = \{u, v\}$ and note that $\{a, b\} \cap \{u, v\} = \varnothing$ by our assumptions. Consider the unique element γ of Θ of corank $2k$, which contains $X_1, \dots, X_{k-1}$ and $\{a, u\}$. Then γ has to contain $X'_1, \dots, X'_k$. It is easily checked that $\alpha\gamma\mathcal{R}\alpha$ and hence $\alpha\gamma = \alpha$, a contradiction. Hence α must contain X'_k and the proof is complete. □

Recall that for $i \neq j \in M$ we denote by $\mathfrak{s}_i$ the transposition $(i, i+1) \in \mathcal{S}_n \subset \mathfrak{B}_n$. A canonical $\mathcal{R}$–cross-section, Λ will be called *regular* provided that $\alpha_{i,j}^{(l)} = \mathrm{id}$ for all possible i, j, l. A canonical $\mathcal{R}$–cross-section, Λ will be called *alternating* provided that $\alpha_{i,j}^{(l)} = (1,2)$ for all possible $(i,j) \notin \mathfrak{A}_n$ and all possible l, and $\alpha_{i,i+1}^{(l)} = \mathrm{id}$ for all $(i, i+1) \in \mathfrak{A}_n$ and for all possible l. Now we are ready to formulate our main result.

Theorem 19. *The stabilizer in $\mathcal{S}_n$ of any canonical $\mathcal{R}$–cross-section consists of* $\mathrm{id} \in \mathcal{S}_n$ *and* $\mathfrak{s}_{n-1,n} \prod_{(i,i+1)\in\mathfrak{A}_n} \mathfrak{s}_{i,i+1} \in \mathcal{S}_n$.

- *(i) For $n = 1$ we have one trivial $\mathcal{R}$–cross-section.*
- *(ii) For $n = 2$ we have only one $\mathcal{R}$–cross-section, moreover, it is canonical and consists of $\alpha_{1,2}$ and* id.
- *(iii) For $n = 3$ we have 3 different $\mathcal{R}$–cross-sections, namely Λ, $\mathfrak{s}_{1,2}\Lambda\mathfrak{s}_{1,2}$, and $\mathfrak{s}_{1,3}\Lambda\mathfrak{s}_{1,3}$, where Λ is the canonical cross-section described in Proposition 9*(iii).
- *(iv) For $n = 4$ we have 12 different $\mathcal{R}$–cross-sections, each of which is $\mathcal{S}_4$-conjugated to the regular canonical $\mathcal{R}$–cross-section.*
- *(v) For $n = 5$ we have $2 \cdot 5!$ different $\mathcal{R}$–cross-sections, each of which is $\mathcal{S}_5$-conjugated to a canonical $\mathcal{R}$–cross-section from Proposition 9*(v) *satisfying $\alpha_{1,2}^{(1)} = \mathrm{id}$.*

(vi) *For* $n = 6$ *we have* $2 \cdot 6!$ *different* $\mathcal{R}$*–cross-sections, each of which is* $\mathcal{S}_6$*-conjugated to a canonical* $\mathcal{R}$*–cross-section from Proposition 9*(vi) *satisfying* $\alpha_{1,2}^{(1)} = \mathrm{id}$ *and* $\alpha_{1,2}^{(2)} = \mathrm{id}$.

(vii) *For* $n \geq 7$ *we have* $n!$ *different* $\mathcal{R}$*–cross-sections. Half of them are* $\mathcal{S}_n$*-conjugated to the regular canonical* $\mathcal{R}$*–cross-section, and half of them are* $\mathcal{S}_n$*-conjugated to the alternating canonical* $\mathcal{R}$*–cross-section.*

Proof. Since every $\mathcal{R}$–cross-section is $\mathcal{S}_n$-conjugated to a canonical $\mathcal{R}$–cross-section by Proposition 16, to complete the classification we have to determine which canonical $\mathcal{R}$–cross-sections of $\mathfrak{B}_n$ are $\mathcal{S}_n$-conjugated to each other. Let Λ be a canonical $\mathcal{R}$–cross-section and $\sigma \in \mathcal{S}_n \subset \mathfrak{B}_n$. Assume that σ stabilizes Λ. In particular, $\sigma^{-1}\Lambda\sigma$ is a canonical $\mathcal{R}$–cross-section, which forces $\sigma(n) \in \{n-1, n\}$, $\sigma(n-1) \in \{n-1, n\}$, $\sigma(n-2) \in \{n-3, n-2\}$, and so on. This implies that σ must stabilize either $\alpha_{1,2}$ (if n is even) or $\alpha_{2,3}$ (if n is odd). A direct calculation shows that this is possible if and only if $\sigma = \mathrm{id}$ or $\sigma = \mathfrak{s}_{n-1} \prod_{(i,i+1)\in\mathfrak{A}_n} \mathfrak{s}_i$.

Consider the case $n \geq 7$. From Proposition 11 and Proposition 12 we have that there are exactly $2^{\lfloor \frac{n}{2} \rfloor}$ canonical $\mathcal{R}$–cross-sections. The arguments from the previous paragraph imply that the number of canonical conjugates of a given canonical $\mathcal{R}$–cross-section is exactly $2^{\lfloor \frac{n}{2} \rfloor - 1}$, and the $\mathcal{S}_n$-orbit of every canonical $\mathcal{R}$–cross-section has size $\frac{n!}{2}$. Using the parity arguments it is also easy to show that the regular and the alternating canonical $\mathcal{R}$–cross-sections are not $\mathcal{S}_n$-conjugated (in Theorem 20 below we even prove that they are not isomorphic). This completes the proof in the case $n \geq 7$. The case $n \leq 6$ can be treated using analogous arguments and Proposition 9. This is left to the reader. □

We remark once more that the classification of $\mathcal{L}$–cross-sections in $\mathfrak{B}_n$ is obtained by applying * to the statement of Theorem 19.

Theorem 20. *For all* $n \geq 6$ *the regular* $\mathcal{R}$*–cross-section* Λ *and the alternating* $\mathcal{R}$*–cross-section* Γ *in* $\mathfrak{B}_n$ *are not isomorphic as monoids.*

To prove Theorem 20 we will need the following lemma

Lemma 21. *Let* $\varphi : \Lambda \to \Gamma$ *be an isomorphism. Then for every* $\alpha \in \Lambda$ *we have* $\mathrm{rank}(\alpha) = \mathrm{rank}(\varphi(\alpha))$ *and* $\mathrm{strank}(\alpha) = \mathrm{strank}(\varphi(\alpha))$.

Proof. For $k = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$ denote by Λ_k and Γ_k the set of all elements of corank $2k$ in Λ and Γ, respectively. Obviously, $\varphi : \Lambda_0 \to \Gamma_0$. Now we claim that Λ_1 is the unique irreducible system of generators for Λ (as a monoid), and Γ_1 is the unique irreducible system of generators for Γ. We prove the statement for Λ and for Γ the arguments are the same. That Λ_1 generates Λ we know from Proposition 4(iii). If $\eta_1, \eta_2 \in \Lambda_1$ are such that $\eta_1\eta_2 \in \Lambda_1$ then Proposition 4(i) implies $\eta_1\eta_2 = \eta_1$. Hence Λ_1 is irreducible. On the other hand, if N is an irreducible system of generators for Λ, then N generates, in particular, Λ_1, which means that $N \cap \Lambda_1$ generates Λ_1. By Proposition 4(iii) we have that $N \cap \Lambda_1$ generates Λ and hence coincides with N by the minimality of N. The minimality of Λ_1 now implies $N = \Lambda_1$.

From the above arguments we deduce that $\varphi : \Lambda_1 \to \Gamma_1$ is a bijection (since both sets have the same cardinality). It follows that if $\alpha \in \Lambda_k$ then $\varphi(\alpha) \in \Gamma_m$ for some $m \leq k$. The same argument, applied to φ^{-1}, an isomorphism from Γ to Λ, yields $\alpha \in \varphi^{-1}(\varphi(\alpha)) \in \Lambda_k$ implies $k \leq m$. Thus $k = m$ and so $\varphi : \Lambda_k \to \Gamma_k$ for all k.

Let $\alpha^i = \pi_\alpha \in \Lambda$ be an idempotent. Then $\varphi(\pi_\alpha) = \pi_{\varphi(\alpha)}$ and thus $\mathrm{strank}(\alpha) = \mathrm{strank}(\varphi(\alpha))$ follows from the equality $\mathrm{rank}(\pi_\alpha) = \mathrm{rank}(\varphi(\pi_\alpha))$ proved above. □

Proof of Theorem 20. Suppose that $\varphi : \Lambda \to \Gamma$ is an isomorphism of the regular cross-section Λ onto the alternating cross-section Γ. We will write $\alpha_{i,j}$ for the elements from Λ and $\tilde{\alpha}_{i,j}$ for the corresponding elements from Γ. In particular, $\tilde{\alpha}_{i,j} = \alpha_{i,j}$ for all $(i, j) \in \mathfrak{A}_n$ by the definition of Γ.

Let $n \geq 6$ be even. Then $\alpha_{1,2}$ (resp. $\tilde{\alpha}_{1,2}$) is the unique element in Λ (resp. Γ) satisfying $\mathrm{rank}(\alpha_{1,2}) = n - 2$ and $\mathrm{strank}(\alpha_{1,2}) = 0$. From Lemma 21 it follows that $\varphi(\alpha_{1,2}) = \tilde{\alpha}_{1,2}$. A direct calculation shows that in Λ we have the relation $\alpha_{1,2}\alpha_{2,3} = \alpha_{4,5}\alpha_{1,2}$. Applying φ gives

$$\tilde{\alpha}_{1,2}\varphi(\alpha_{2,3}) = \varphi(\alpha_{4,5})\tilde{\alpha}_{1,2}. \tag{4.1}$$

There are exactly 5 elements in Γ which have rank $n - 2$ and stable rank 2, namely $\tilde{\alpha}_{2,3}$, $\tilde{\alpha}_{1,4}$, $\tilde{\alpha}_{1,3}$, $\tilde{\alpha}_{2,4}$, and $\tilde{\alpha}_{3,4}$. Hence, by Lemma 21, $\varphi(\alpha_{2,3})$ must coincide with one of these elements. We consider all these cases separately.

Case 1: $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{2,3}$. Then (4.1) implies $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{4,5}$ by a direct calculation. Further, we also have the following relation in Λ:

$$\alpha_{2,3}\alpha_{2,3} = \alpha_{4,5}\alpha_{2,3}, \tag{4.2}$$

which implies $\tilde{\alpha}_{2,3}\tilde{\alpha}_{2,3} = \varphi(\alpha_{4,5})\tilde{\alpha}_{2,3}$. However, a direct calculation shows that in this case we have $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{4,6}$, a contradiction.

Case 2: $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{1,4}$. Then (4.1) implies $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{3,6}$, and (4.2) implies $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{3,5}$ by a direct calculation. A contradiction.

Case 3: $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{1,3}$. Then (4.1) implies $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{3,5}$, and (4.2) implies $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{4,6}$ by a direct calculation. A contradiction.

Case 4: $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{2,4}$. Then (4.1) implies $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{4,6}$, and (4.2) implies $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{3,5}$ by a direct calculation. A contradiction.

Case 5: $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{3,4}$. A direct calculation gives us $\alpha_{2,3}\alpha_{1,2} = \alpha_{1,4}\alpha_{1,2}$. Applying φ we have $\tilde{\alpha}_{3,4}\tilde{\alpha}_{1,2} = \varphi(\alpha_{1,4})\tilde{\alpha}_{1,2}$. This implies $\varphi(\alpha_{1,4}) = \tilde{\alpha}_{1,2} = \varphi(\alpha_{1,2})$, which is not possible since φ is bijective.

Hence φ can not exist for even $n \geq 6$.

Now let $n \geq 7$ be odd. In this case we have exactly 3 elements in Λ (resp. Γ) of rank $n-2$ and stable rank 1, namely $\alpha_{1,2}$, $\alpha_{2,3}$ and $\alpha_{1,3}$ (resp. $\tilde{\alpha}_{1,2}$, $\tilde{\alpha}_{2,3}$ and $\tilde{\alpha}_{1,3}$). Using Lemma 21, we obtain that one of the following 6 cases must occur.

Case 1: $\varphi(\alpha_{1,2}) = \tilde{\alpha}_{1,2}$*,* $\varphi(\alpha_{1,3}) = \tilde{\alpha}_{1,3}$ *and* $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{2,3}$. We have the following relations in Λ:

$$\alpha_{1,2}\alpha_{1,2} = \alpha_{3,4}\alpha_{1,2}, \tag{4.3}$$

$$\alpha_{3,4}\alpha_{3,4} = \alpha_{5,6}\alpha_{3,4}, \tag{4.4}$$

$$\alpha_{2,3}\alpha_{3,4} = \alpha_{5,6}\alpha_{2,3}. \tag{4.5}$$

Applying φ and using a direct calculation, we obtain $\varphi(\alpha_{3,4}) = \tilde{\alpha}_{3,5}$ from (4.3), $\varphi(\alpha_{5,6}) = \tilde{\alpha}_{4,6}$ from (4.4), and $\varphi(\alpha_{5,6}) = \tilde{\alpha}_{5,7}$ from (4.5), a contradiction.

Case 2: $\varphi(\alpha_{1,2}) = \tilde{\alpha}_{1,3}$*,* $\varphi(\alpha_{1,3}) = \tilde{\alpha}_{1,2}$ *and* $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{2,3}$. Applying φ and using a direct calculation, we obtain $\varphi(\alpha_{3,4}) = \tilde{\alpha}_{2,4}$ from (4.3), $\varphi(\alpha_{5,6}) = \tilde{\alpha}_{5,7}$ from (4.4), and $\varphi(\alpha_{5,6}) = \tilde{\alpha}_{4,6}$ from (4.5), a contradiction.

Case 3: $\varphi(\alpha_{1,2}) = \tilde{\alpha}_{2,3}$*,* $\varphi(\alpha_{1,3}) = \tilde{\alpha}_{1,3}$ *and* $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{1,2}$. We have the following relations in Λ:

$$\alpha_{2,3}\alpha_{2,3} = \alpha_{4,5}\alpha_{2,3}, \tag{4.6}$$

$$\alpha_{1,2}\alpha_{2,3} = \alpha_{4,5}\alpha_{1,2}. \tag{4.7}$$

Applying φ and using a direct calculation, we obtain $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{3,5}$ from (4.6) and $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{1,4}$ from (4.7), a contradiction.

Case 4: $\varphi(\alpha_{1,2}) = \tilde{\alpha}_{1,3}$*,* $\varphi(\alpha_{1,3}) = \tilde{\alpha}_{2,3}$ *and* $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{1,2}$. Applying φ and using a direct calculation, we obtain $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{3,5}$ from (4.6) and $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{2,5}$ from (4.7), a contradiction.

Case 5: $\varphi(\alpha_{1,2}) = \tilde{\alpha}_{2,3}$*,* $\varphi(\alpha_{1,3}) = \tilde{\alpha}_{1,2}$ *and* $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{1,3}$. Applying φ and using a direct calculation, we obtain $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{2,4}$ from (4.6) and $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{1,5}$ from (4.7), a contradiction.

Case 6: $\varphi(\alpha_{1,2}) = \tilde{\alpha}_{1,2}$*,* $\varphi(\alpha_{1,3}) = \tilde{\alpha}_{2,3}$ *and* $\varphi(\alpha_{2,3}) = \tilde{\alpha}_{1,3}$. Applying φ and using a direct calculation, we obtain $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{2,4}$ from (4.6) and $\varphi(\alpha_{4,5}) = \tilde{\alpha}_{3,4}$ from (4.7), a contradiction.

Hence φ can not exist for odd $n \geq 7$ either. This completes the proof. □

We remark that for $n \leq 4$ all $\mathcal{R}$–cross-sections in $\mathfrak{B}_n$ are conjugat and hence isomorphic. For $n = 5$ the isomorphism of all canonical (and hence of all) $\mathcal{R}$–cross-sections in $\mathfrak{B}_5$ can be shown by a direct calculation.

5 On $\mathcal{D}$– and $\mathcal{H}$–cross-sections in $\mathfrak{B}_n$

It is of course a natural question what one can say about the $\mathcal{D}$– and $\mathcal{H}$–cross-sections in $\mathfrak{B}_n$. For $\mathcal{H}$–cross-sections the answer is very easy.

Proposition 22. *For $n = 1, 2, 3$ the semigroup $\mathfrak{B}_n$ contains a unique $\mathcal{H}$–cross-section. This cross-section consists of all idempotents in $\mathfrak{B}_n$. For $n \geq 4$ the semigroup $\mathfrak{B}_n$ does not contain any $\mathcal{H}$–cross-section.*

Proof. For $n = 1, 2, 3$ the statement is easily checked. For $n \geq 4$ we first observe that an $\mathcal{H}$–cross-section must contain all idempotents of the semigroup. A direct calculation shows that for $n = 4$ the semigroup, generated by all idempotents, is not an $\mathcal{H}$–cross-section. Moreover, it contains $\mathcal{H}$–classes with more than one element, which shows that no $\mathcal{H}$–cross-sections exist. Using the canonical embedding $\mathfrak{B}_4 \hookrightarrow \mathfrak{B}_n$, this also implies the same statement for all $n > 4$. □

On the other hand, for $\mathcal{D}$–cross-sections we will now show that the problem of their classification contains, as a sub-problem, the problem of classification of all $\mathcal{D}$–cross-sections in the symmetric inverse semigroup $\mathcal{IS}_m$, where $m = \lfloor \frac{n}{2} \rfloor$. The latter problem is still open, see [GM].

Proposition 23. *Let $m = \lfloor \frac{n}{2} \rfloor$. Let Γ be a $\mathcal{D}$–cross-section in $\mathcal{IS}_m$. For $f \in \Gamma$ define $\alpha_f \in \mathfrak{B}_n$ as follows:*

1. $(n - 2i + 1) \equiv_{\alpha_f} (n - 2f(i) + 1)'$ *if* $i \in \mathrm{dom}(f)$*;*
2. $(n - 2i + 2) \equiv_{\alpha_f} (n - 2f(i) + 2)'$ *if* $i \in \mathrm{dom}(f)$*;*
3. $(n - 2i + 1) \equiv_{\alpha_f} (n - 2i + 2)$ *if* $i \notin \mathrm{dom}(f)$*;*
4. $(n - 2i + 1)' \equiv_{\alpha_f} (n - 2i + 2)'$ *if* $i \notin \mathrm{ran}(f)$*;*
5. $1 \equiv_{\alpha_f} 1'$ *if n is odd.*

Then $\{\alpha_f : f \in \Gamma\}$ is a $\mathcal{D}$–cross-section in $\mathfrak{B}_n$.

Proof. From the definition of α_f by a direct calculation it follows that $f \mapsto \alpha_f$ is a homomorphism, which implies that $\{\alpha_f : f \in \Gamma\}$ is a semigroup. Using Theorem 1(iv) one easily checks that $\{\alpha_f : f \in \Gamma\}$ is a $\mathcal{D}$–cross-section. The statement follows. □

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