

SIMPLE MODULES OVER FACTORPOWERS

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ABSTRACT. In this paper we study complex representations of the factorpower $\mathcal{FP}^+(G, M)$ of a finite group G acting on a finite set M . This includes the finite monoid $\mathcal{FP}^+(S_n)$, which can be seen as a kind of a “balanced” generalization of the symmetric group S_n inside the semigroup of all binary relations. We describe all irreducible representations of $\mathcal{FP}^+(G, M)$ and relate them to irreducible representations of certain inverse semigroups. In particular, irreducible representations of $\mathcal{FP}^+(S_n)$ are related to irreducible representations of the maximal factorizable submonoid of the dual symmetric inverse monoid. We also show that in the latter cases irreducible representations lead to an interesting combinatorial problem in the representation theory of S_n , which, in particular, is related to Foulkes’ conjecture. Finally, we show that all simple $\mathcal{FP}^+(G, M)$ -modules are unitarizable and that tensor products of simple $\mathcal{FP}^+(G, M)$ -modules are completely reducible.

1. INTRODUCTION AND PRELIMINARIES

Let us first fix some notation. For a semigroup S we denote by $E(S)$ the set of idempotents of S ; and by \mathcal{D} , \mathcal{L} , \mathcal{R} , \mathcal{H} , \mathcal{J} the corresponding Green’s relations on S .

Let S be a finite semigroup acting on a finite set M by (everywhere defined) transformations. Consider the power semigroup $\mathcal{P}(S)$, which consists of all subsets of S with the natural multiplication $A \cdot B = \{a \cdot b | a \in A, b \in B\}$ for $A, B \in \mathcal{P}(S)$. Define on $\mathcal{P}(S)$ the binary relation \sim_M in the following way: $A \sim_M B$ provided that $\{a(m) | a \in A\} = \{b(m) | b \in B\}$. It is straightforward to verify that \sim_M is a congruence on $\mathcal{P}(S)$ and hence we can consider the corresponding quotient $\mathcal{FP}(S, M) = \mathcal{P}(S) / \sim_M$. The semigroup $\mathcal{FP}(S, M)$ has an isolated zero element, namely the class of the empty set. Denote by $\mathcal{FP}^+(S, M)$ the complement to this zero element. The semigroup $\mathcal{FP}^+(S, M)$ is called the *factorpower* of the action of S on M . This construction was proposed and first studied in [GM1]. In general, there might exist many different nonequivalent actions of S on different sets, which give rise to different (nonisomorphic) factor powers. In [D1, D2] all factorpowers of a finite group G are classified up to isomorphism. In the present paper we would like to study complex representations of these factorpowers.

There is one very special factorpower, which is studied much better than all others and has some really exciting properties. For a positive integer n let $\mathcal{FP}^+(S_n)$ denote the factorpower of the symmetric group S_n with respect to its natural action on the set $\mathbf{N} = \mathbf{N}_n = \{1, 2, \dots, n\}$. This semigroup was studied in [GM1, GM2, GM3, GM4, Ma1, Ma2]. From the definition it follows that $\mathcal{FP}^+(S, M)$ is a subsemigroup of the semigroup $\mathfrak{B}(M)$ of all binary relations on M , in particular, $\mathcal{FP}^+(S_n)$ is a subsemigroup of \mathfrak{B}_n , the semigroup of all binary relations on \mathbf{N} . The semigroup $\mathcal{FP}^+(S_n)$ has the following properties (see [GM2]):

- (I) Idempotents of $\mathcal{FP}^+(S_n)$ are exactly the equivalence relations on \mathbf{N} , in particular, asymptotically almost all idempotents of \mathfrak{B}_n do not belong to $\mathcal{FP}^+(S_n)$ in the sense $\frac{|E(\mathcal{FP}^+(S_n))|}{|E(\mathfrak{B}_n)|} \rightarrow 0$, $n \rightarrow \infty$.
- (II) The semigroup $\mathcal{FP}^+(S_n)$ contains asymptotically almost all elements of \mathfrak{B}_n in the sense that $\frac{|\mathcal{FP}^+(S_n)|}{|\mathfrak{B}_n|} \rightarrow 1$, $n \rightarrow \infty$.
- (III) The semigroup $\mathcal{FP}^+(S_n)$ is the maximum subsemigroup of \mathfrak{B}_n which contains S_n and whose idempotents are exactly the equivalence relations.
- (IV) The semigroup $\mathcal{FP}^+(S_n)$ is a natural quotient of the semigroup of doubly-stochastic $n \times n$ matrices.
- (V) $\mathcal{FP}^+(S_n)$ is a universal factorpower in the sense that every factorpower of a finite group acting on a finite set is a subsemigroup of $\mathcal{FP}^+(S_n)$ for some n .

In [GM3, Ma1, Ma2] one can find classifications of maximal nilpotent subsemigroups and automorphisms of $\mathcal{FP}^+(S_n)$, as well as a description of Green's relations.

Being such a natural generalization of the symmetric group one could expect $\mathcal{FP}^+(S_n)$ to have further interesting properties. The motivation for the present paper was an attempt to understand complex representations of $\mathcal{FP}^+(S_n)$. There is a well-developed abstract theory for the study of irreducible complex representations of semigroups, see [Mu, CP, GMS, GM5]. Trying to apply it to the study of complex representations of $\mathcal{FP}^+(S_n)$ revealed that $\mathcal{FP}^+(S_n)$ and, more generally, factorpowers of finite groups have some nice properties. For example, they all have an involution and all regular \mathcal{D} -classes of such factorpowers are in fact inverse. This observation allows for a straightforward application of the general theory, which results in a nice description of irreducible representations of $\mathcal{FP}^+(G, M)$, where G is a group, in terms of certain induced G -modules. This is worked out in Section 2. As a corollary we obtain that the quotient of $\mathbb{C}[\mathcal{FP}^+(S_n)]$ modulo the Jacobson ideal is isomorphic to the semigroup algebra of the maximal factorizable submonoid \mathcal{F}_n^* of the dual symmetric inverse monoid \mathcal{I}_n^* which was introduced in [FL]. In particular, irreducible representations of $\mathcal{FP}^+(S_n)$ and \mathcal{F}_n^* are the same (at least as S_n -modules, but also as modules over the traces of the corresponding \mathcal{D} -classes).

It further turns out that the representation theory of $\mathcal{FP}^+(S_n)$ (and also that of \mathcal{FI}_n^*) has an interesting relation to the representation theory of the symmetric group S_n . The group S_n is the group of units of $\mathcal{FP}^+(S_n)$. In particular, every representation of $\mathcal{FP}^+(S_n)$ becomes a representation of S_n via restriction. In the study of representations of semigroups a special role is played by modules, associated with regular \mathcal{L} -classes. Every regular \mathcal{L} -class of $\mathcal{FP}^+(S_n)$ contains a unique idempotent. This idempotent is an equivalence relation on \mathbf{N} and the sizes of the blocks of this relation determine a partition of n , say λ . We will show that, after restriction to S_n , the representation of $\mathcal{FP}^+(S_n)$ associated with our \mathcal{L} -class is isomorphic to the permutation module \mathcal{M}^λ (see [Sa, 2.1]). The module \mathcal{M}^λ is usually realized via tabloids of shape λ . If these tabloids have rows of the same length, permutations of such rows induce automorphisms of \mathcal{M}^λ , which give rise to the action of a certain group G on \mathcal{M}^λ via automorphisms. This group G turns out to be the (unique) maximal subgroup of $\mathcal{FP}^+(S_n)$, contained in our \mathcal{L} -class. Considering the isotypic components of \mathcal{M}^λ with respect to the action of G leads to a finer decomposition of \mathcal{M}^λ as an S_n -module. The elements of this decomposition are simple $\mathcal{FP}^+(S_n)$ -modules and we raise the problem of determining the multiplicities of Specht modules in these simple $\mathcal{FP}^+(S_n)$ -modules. For the permutation module \mathcal{M}^λ the latter are given by Kostka numbers, see [Sa]. Solution to this problem in a special case would give the answer to the famous Foulkes' conjecture, [Fo].

In Section 3 we study some further properties of simple modules over factorpowers. We show that there is a contravariant exact involutive equivalence on the category of $\mathcal{FP}^+(G, M)$ -modules, that all simple $\mathcal{FP}^+(G, M)$ -modules are unitarizable and that tensor products of simple $\mathcal{FP}^+(G, M)$ -modules are completely reducible.

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2. IRREDUCIBLE COMPLEX REPRESENTATIONS OF THE SEMIGROUP $\mathcal{FP}^+(G, M)$

2.1. Green's relations on $\mathcal{FP}^+(G, M)$. Let G be a finite group acting on a finite set $M = \{m_1, m_2, \dots, m_n\}$. For $A \subset G$ let \overline{A} denote the equivalence class of A with respect to the equivalence relation \sim_M defining $\mathcal{FP}^+(G, M)$. For $m \in M$ set $A_m = \{\alpha(m) | \alpha \in A\}$. From the definitions for $A, B \subset G$ we have that $\overline{A} = \overline{B}$ implies $A_m = B_m$ for all m . Thus the element \overline{A} of $\mathcal{FP}^+(G, M)$ can be represented as the following array:

$$(1) \quad \overline{A} = \begin{pmatrix} m_1 & m_2 & \dots & m_n \\ A_{m_1} & A_{m_2} & \dots & A_{m_n} \end{pmatrix}.$$

Observe that not an arbitrary sequence of subsets of M appears in the second row of the above presentation for some elements of $\mathcal{FP}^+(G, M)$. All A_m 's are non-empty by definition. Moreover, for any $m \in M$ and for any $x \in A_m$ there exists $\sigma \in G$ such that $\sigma(m) = x$ and $\sigma(m') \in A_{m'}$ for all $m' \in M$. These two conditions are, obviously, also sufficient.

Using this notation the multiplication in $\mathcal{FP}^+(G, M)$ can be written as follows:

$$\begin{pmatrix} m_1 & \cdots & m_n \\ A_{m_1} & \cdots & A_{m_n} \end{pmatrix} \cdot \begin{pmatrix} m_1 & \cdots & m_n \\ B_{m_1} & \cdots & B_{m_n} \end{pmatrix} = \begin{pmatrix} m_1 & \cdots & m_n \\ \bigcup_{x \in B_{m_1}} A_x & \cdots & \bigcup_{x \in B_{m_n}} A_x \end{pmatrix}$$

The anti-involution $\sigma \mapsto \sigma^{-1}$ on G extends in the natural way to the anti-involution $\pi \mapsto \pi^*$ on $\mathcal{FP}^+(G, M)$. If $\pi = \overline{A}$ for some $A \subset G$, then $\pi^* = \overline{\{\sigma^{-1} | \sigma \in A\}}$.

The principal result of [Ma2] asserts that for $\pi, \tau \in \mathcal{FP}^+(S_n)$ the condition $\pi \mathcal{L} \tau$ is equivalent to the condition $\pi = \sigma \tau$ for some $\sigma \in S_n$. Applying \star we have that $\pi \mathcal{R} \tau$ if and only if $\pi = \tau \sigma$ for some $\sigma \in S_n$. We start with extending this result to any $\mathcal{FP}^+(G, M)$.

Proposition 1. *Let $\pi, \tau \in \mathcal{FP}^+(G, M)$. Then we have the following:*

- (a) $\pi \mathcal{L} \tau$ if and only if there exists $\sigma \in G$ such that $\pi = \sigma \tau$.
- (b) $\pi \mathcal{R} \tau$ if and only if there exists $\sigma \in G$ such that $\pi = \tau \sigma$.
- (c) $\pi \mathcal{H} \tau$ if and only if there exists $\sigma, \sigma' \in G$ such that $\pi = \sigma \tau = \tau \sigma'$.
- (d) $\pi \mathcal{D} \tau$ if and only if there exist $\sigma, \sigma' \in G$ such that $\pi = \sigma \tau \sigma'$.
- (e) $\mathcal{D} = \mathcal{J}$.

Proof. As $\mathcal{FP}^+(G, M)$ is finite by definition, the statement (e) is well-known, see for example [Ho, Proposition 2.1.5]. The statements (c) and (d) follow immediately from the statements (a) and (b). The statement (b) follows from the statement (a) applying the anti-involution \star . Hence to complete the proof we have to prove the statement (a).

Let $A, B \subset G$ be non-empty and assume that $\overline{A} \mathcal{L} \overline{B} \overline{A}$. Let $\tau \in B$ be any element. Since $\tau = \{\tau\}$ is invertible in $\mathcal{FP}^+(G, M)$, we have $\overline{B} \overline{A} \mathcal{L} \tau^{-1} \overline{B} \overline{A}$. In particular, $\overline{A} \mathcal{L} \tau^{-1} \overline{B} \overline{A}$ and hence there exists some $C \subset G$ such that $\overline{A} = \overline{C} \tau^{-1} \overline{B} \overline{A}$.

Assume that \overline{A} is given by (1) and $\tau^{-1} \overline{B} \overline{A}$ is given by a similar formula with sets X_{m_i} 's in the second row. For $B' = \tau^{-1} B$ we observe that B' contains the identity element e of G . This means that $A_{m_i} \subset X_{m_i}$ for all i . Assume that $A_{m_i} \subsetneq X_{m_i}$ for some i , in particular, $|X_{m_i}| > |A_{m_i}|$. Write $\overline{C} \tau^{-1} \overline{B} \overline{A}$ in the form (1) with sets Y_{m_i} 's in the second row, and let $\sigma \in C$ be arbitrary. Then $|\sigma(X_{m_i})| = |X_{m_i}|$ as σ is an invertible transformation on M , and thus

$$|Y_{m_i}| \geq |\sigma(X_{m_i})| = |X_{m_i}| > |A_{m_i}|.$$

This implies that $Y_{m_i} \neq A_{m_i}$ contradicting to $\overline{A} = \overline{C} \tau^{-1} \overline{B} \overline{A}$.

Thus $A_{m_i} = X_{m_i}$ for all i and hence $\tau^{-1} \overline{B} \overline{A} = \overline{A}$, implying $\overline{B} \overline{A} = \tau \overline{A}$. This completes the proof. \square

2.2. Idempotents and regular \mathcal{D} -classes in $\mathcal{FP}^+(G, M)$. Every subgroup H of G acts on M via restriction. Two subgroups of G will be called *orbit-equivalent* if they have the same orbits on M .

Lemma 2. *Let H be a subgroup of G , and $\mathcal{O}(H)$ denote the set of all subgroups of G , which are orbit equivalent to H . Then $\mathcal{O}(H)$ is partially ordered with respect to inclusion and contains a unique maximal element, namely, the maximal with respect to inclusions set H' in \overline{H} .*

Proof. If X is an orbit of H on M , then every element of H preserves X . Hence, by definition, every element of H' preserves X as well. On the other hand, the set of all elements from G , which preserve all orbits of H obviously forms a subgroup, say \hat{H} . Finally, if $\sigma \in \hat{H}$, then for any $m \in M$ the element $\sigma(m) \in \{\tau(m) \mid \tau \in H\}$. Hence $\hat{H} \subset H'$ and thus $\hat{H} = H'$. The claim follows. \square

The maximal element in $\mathcal{O}(H)$ will be called an *orbit-maximal* subgroup of G .

Corollary 3. *The idempotents in $\mathcal{FP}^+(G, M)$ are precisely the classes of orbit-maximal subgroups of G .*

Proof. If H is a subgroup of G , then \overline{H} is obviously and idempotent. From Lemma 2 we have that \overline{H} is an orbit-maximal subgroup of G .

Conversely, let $A \subset G$ be a non-empty subset such that it is maximal with respect to inclusions in the class \overline{A} and assume that $\overline{A}A = \overline{A}$. The latter equality means that A is closed with respect to compositions and hence is a subgroup of G since G is finite. The claim follows. \square

Remark 4. If $G = S_n$ with the natural action on \mathbf{N} , then the orbit-maximal subgroups of G are $S_{N_1} \times \cdots \times S_{N_k}$, where $\mathbf{N} = N_1 \cup \cdots \cup N_k$ is a partition of \mathbf{N} into a disjoint union of nonempty subsets. Thus the idempotents of $\mathcal{FP}^+(S_n)$ correspond to equivalence relations on \mathbf{N} , see [GM2, Theorem 3].

Let H be some orbit-maximal subgroup of G and \mathcal{D}_H denote the regular \mathcal{D} -class of $\mathcal{FP}^+(G, M)$ containing H . For the study of irreducible representations of $\mathcal{FP}^+(G, M)$ it is of principal importance to understand the structure of the *trace* $\hat{\mathcal{D}}_H = \mathcal{D}_H \cup \{0\}$ of \mathcal{D}_H , which is a semigroup with multiplication defined as follows:

$$\pi \cdot \tau = \begin{cases} \pi\tau, & \pi, \tau, \pi\tau \in \mathcal{D}_H; \\ 0, & \text{otherwise.} \end{cases}$$

It turns out that the semigroup $\hat{\mathcal{D}}_H$ has a very nice structure.

Proposition 5. *Let H and H' be two orbit-maximal subgroups of G .*

- (a) *We have $H' \in \mathcal{D}_H$ if and only if H and H' are conjugated in G .*
- (b) *The semigroup $\hat{\mathcal{D}}_H$ is an inverse semigroup.*

Proof. The “if” part of (a) follows directly from Proposition 1(d). To prove the “only if” part assume that $H' \in \mathcal{D}_H$. Then, by Proposition 1(d), there exist $\sigma, \tau \in G$ such that $H' = \sigma H \tau$. As H' has the identity element, we get $\sigma^{-1} \tau^{-1} \in H$ and hence

$$H' = \sigma H \tau = \sigma(H \sigma^{-1} \tau^{-1}) \tau = \sigma H \sigma^{-1},$$

completing the proof of (a).

To prove (b) we assume that H and H' are different orbit-maximal conjugated subgroups of G . Then there exists at least one H -orbit which intersects at least two H' -orbits and vice versa. Assume that H is given by (1) and write $H'H$ in the same form with subsets X_{m_i} 's in the second row. As H' contains the identity element, we will have $A_{m_i} \subset X_{m_i}$ for all i and $A_{m_i} \neq X_{m_i}$ for at least one i . This obviously implies that $H'H \neq \tau H \sigma$ for any $\tau, \sigma \in G$. Hence $H'H \notin \mathcal{D}_H$ by Proposition 1(d), which yields $H'H = 0$ in $\hat{\mathcal{D}}_H$. Analogously $HH' = 0$ in $\hat{\mathcal{D}}_H$. Hence $\hat{\mathcal{D}}_H$ is a regular semigroup with commuting idempotents, hence inverse (see for example [Ho, Theorem 5.1.1]). \square

One of the most natural regular \mathcal{D} -classes of a semigroup is the group of units. For $\mathcal{FP}^+(G, M)$ the latter has the following structure:

Proposition 6. *The mapping $\varphi : \tau \mapsto \overline{\{\tau\}}$, $\tau \in G$, is an epimorphism from G to the group of units of $\mathcal{FP}^+(G, M)$. The kernel of φ coincides with the kernel K of the action of G on M .*

Proof. Obviously, φ is a homomorphism from G to the group of units of $\mathcal{FP}^+(G, M)$. The fact that the kernel of this homomorphism coincides with K follows from the definition of \sim_M .

Let e denote the identity element of G . From the definition of \sim_M we have $\bar{e} = K$. Moreover, for any $A \subset G$ and for any $\tau \in A$ we have $\tau K \subset \bar{A}$. If $A \neq \tau K$ for any $\tau \in A$, then there exist $\tau, \sigma \in G$ and $m \in M$ such that $\tau(m) \neq \sigma(m)$. This means that, representing \bar{A} in the form (1), at least one of A_{m_i} 's in the second row will contain more than one element. This yields that such \bar{A} cannot be invertible in $\mathcal{FP}^+(G, M)$, which implies that φ is surjective. \square

Proposition 6 motivates the following statement:

Corollary 7. *Let K denote the kernel of the action of G on M . Then the semigroups $\mathcal{FP}^+(G, M)$ and $\mathcal{FP}^+(G/K, M)$ are canonically isomorphic.*

Proof. Let $A \subset G$. Then from the proof of Proposition 6 we have that \bar{A} is a union of cosets from G/K . It is straightforward to verify that mapping \bar{A} to the corresponding union of cosets from G/K defines the necessary canonical isomorphism from the semigroup $\mathcal{FP}^+(G, M)$ to the semigroup $\mathcal{FP}^+(G/K, M)$. \square

Corollary 7 says that without loss of generality in what follows we may assume that the action of G on M is faithful.

We complete this subsection with a description of maximal subgroups in $\mathcal{FP}^+(G, M)$.

Proposition 8. *Let H be an orbit-maximal subgroup of G and $N_G(H)$ its normalizer in G . Then the mapping $\varphi : X \mapsto \overline{X}$ is an isomorphism from $N_G(H)/H$ to the maximal subgroup of $\mathcal{FP}^+(G, M)$ whose identity element is H .*

Proof. The fact that φ is a homomorphism follows from definitions. The kernel of φ consists of all cosets which contain only those $\tau \in N_G(H)$ whose action preserves all orbits of H . Since H is orbit maximal, it follows that the kernel of φ coincides with H . Thus φ is a monomorphism.

Finally, the maximal subgroup of $\mathcal{FP}^+(G, M)$ with identity H coincides with the \mathcal{H} -class containing H . Hence, by Proposition 1(c), any element from this maximal subgroup has the form $\overline{\sigma H} = \overline{H\tau}$ for some $\sigma, \tau \in G$. Since H is orbit-maximal, from Lemma 2 it follows that we thus must have the equality $\sigma H = H\tau$, that is $H = \sigma^{-1}H\tau$. This implies $\sigma\tau^{-1} \in H$ and thus $H = \sigma^{-1}H\sigma$, that is $\sigma \in N_G(H)$. This means that $\overline{\sigma H}$ belongs to the image of φ and hence φ is surjective. \square

2.3. Simple modules over essentially inverse semigroups. A finite semigroup S will be called *essentially inverse* provided that the trace of every regular \mathcal{D} -class of S is an inverse semigroup. For example, if G is a finite group acting on a finite set M , then the semigroup $\mathcal{FP}^+(G, M)$ is essentially inverse by Proposition 5. It turns out that for essentially inverse semigroups the description of simple modules (over complex numbers) can be substantially simplified in comparison to the classical approach for all finite semigroups, see [Mu], [GMS], [GM5, Chapter 11]. In the following we mostly follow the approach of [GM5, Chapter 11] and [GMS] using the language of bimodules and functors.

Let S be an essentially inverse monoid and $\mathcal{D}_1, \dots, \mathcal{D}_k$ be a complete list of regular \mathcal{D} -classes in S . For every $i = 1, \dots, k$ fix some idempotent $e_i \in \mathcal{D}_i$ and let \mathcal{L}_i and \mathcal{H}_i denote the \mathcal{D} -class and the \mathcal{H} -class of e_i , respectively. Then $G_i = \mathcal{H}_i$ is the maximal subgroup of S corresponding to the idempotent e_i . Finally, let V_i denote the formal \mathbb{C} -span of all elements in \mathcal{L}_i .

Lemma 9. *The assignment*

$$s \cdot x \cdot g = \begin{cases} sxg, & sxg \in \mathcal{L}_i; \\ 0, & \text{otherwise;} \end{cases}$$

where $x \in \mathcal{L}_i$, $s \in S$ and $g \in G_i$, defines on V_i the structure of an $S - G_i$ -bimodule.

Proof. The left ideal Se_i is obviously stable with respect to the left multiplication with elements from S and right multiplication with elements from G_i . The same holds for the subset $Se_i \setminus \mathcal{L}_i$ of Se_i . Now the claim follows from the associativity of the multiplication in S . \square

Now we are ready to classify all simple S -modules.

- Theorem 10.** (a) Let $i \in \{1, \dots, k\}$ and X be a simple G_i -module. Then the S -module $L(i, X) = V_i \otimes_{G_i} X$ is simple.
 (b) Every simple S -module has the form $L(i, X)$ for some $i \in 1, \dots, k$ and some simple G_i -module X .
 (c) Modules $L(i, X)$ and $L(j, Y)$ are isomorphic if and only if $i = j$ and $X \cong Y$.

Proof. Mutatis mutandis [GM5, Theorem 11.3.1]. \square

Denote by A the associative algebra $\bigoplus_{i=1}^k \mathbb{C}[G_i]$. Then $V = \bigoplus_{i=1}^k V_i$ has the natural structure of a $\mathbb{C}[S] - A$ -bimodule. As immediate corollaries from Theorem 10 we have the following two statement:

Corollary 11. The functor $V \otimes_{A-}$ is an equivalence from the category of left A -modules to the category of all semisimple left $\mathbb{C}[S]$ -modules.

Proof. This follows directly from Theorem 10. \square

Corollary 12. Let J denote the Jacobson radical of $\mathbb{C}[S]$, and n_i denote the number of idempotents in \mathcal{D}_i , $i = 1, \dots, k$. Then we have

$$\mathbb{C}[S]/J \cong \bigoplus_{i=1}^k \text{Mat}_{n_i}(\mathbb{C}[G_i]),$$

where $\text{Mat}_{n_i}(\mathbb{C}[G_i])$ is the algebra of $n_i \times n_i$ matrices with coefficients from $\mathbb{C}[G_i]$.

Proof. That the two algebras are Morita equivalent follows from Corollary 11. It remains to compare the dimensions of simple modules, which is a straightforward calculation. \square

2.4. Simple modules over factorpowers. Here we combine the results of the two previous subsections to give a description of simple modules over $\mathcal{FP}^+(G, M)$. The abstract theory sounds completely satisfactory for the rough description of these modules. However, we observe that G is the group of units in $\mathcal{FP}^+(G, M)$ (note that we already assume that the action of G on M is faithful) and make the main emphasis on the problem of understanding the structure of simple $\mathcal{FP}^+(G, M)$ -modules, when considered as G -modules by restriction.

Let H be an orbit-maximal subgroup of G . By Theorem 10, the simple $\mathcal{FP}^+(G, M)$ -modules corresponding to the idempotent H are indexed by simple modules over the maximal subgroup of $\mathcal{FP}^+(G, M)$, corresponding to H . By Proposition 8, this maximal subgroup is

$N_G(H)/H$. If X is a simple $N_G(H)/H$ -module, we denote the corresponding simple $\mathcal{FP}^+(G, M)$ -module by $L(H, X)$. Denote also by V_H the $\mathcal{FP}^+(G, M) - N_G(H)/H$ -bimodule from Subsection 2.3, which corresponds to the idempotent H . Combining the above results we thus obtain:

Theorem 13. (a) *After restriction to G the $G - N_G(H)/H$ -bimodule V_H is isomorphic to the bimodule $\mathbb{C}[G/H]$.*

(b) *Let X be a simple $N_G(H)/H$ -module. Then, after restriction to G , the module $L(H, X)$ becomes isomorphic to $\mathbb{C}[G/H] \otimes_{N_G(H)/H} X$.*

Proof. The statement (b) is an immediate corollary from the statement (a) and Theorem 10. To prove (a) we observe that, by Proposition 1(a), every element of the \mathcal{L} -class of H has the form $\overline{\tau H}$, $\tau \in G$. Moreover, since H is orbit-maximal, we have $\overline{\tau H} = H$ if and only if $\tau \in H$. This identifies the \mathcal{L} -class of H with G/H . It is straightforward to check that the necessary actions of G and $N_G(H)/H$ correspond under this identification. \square

We note that the $\mathcal{FP}^+(G, M)$ -module structure on $\mathbb{C}[G/H]$ reduces to the G -module structure via Proposition 1(a).

Remark 14. Theorem 13 shows another nice property of factorpowers, namely that on the semigroup level they capture a lot of the intrinsic group structure of the original group, which in our case is the structure of certain representations induced from subgroups.

Remark 15. The construction in Theorem 13 is a special case of the general fact that if H is a subgroup of G , then the set G/H carries a natural left action of G given by the left multiplication and the natural right action of $N_G(H)$ given by the right multiplication. These two actions of course commute with each other.

2.5. Simple modules over $\mathcal{FP}^+(S_n)$. In the special case of the natural action of the symmetric group S_n on \mathbf{N} the ingredients in Theorem 13 turn out to be very classical objects. Therefore we consider this special case in details.

Let ρ be an equivalence relation on \mathbf{N} . Then ρ defines a decomposition of \mathbf{N} into a disjoint union $N_1 \cup \dots \cup N_k$ of nonempty subsets. The orbit-maximal subgroup of S_n with orbits N_1, \dots, N_k is of course the direct product $S_\rho = S_{N_1} \times \dots \times S_{N_k}$. The normalizer $N_{S_n}(S_\rho)$ is generated by S_ρ and permutations which arbitrarily permute those blocks of $N_1 \cup \dots \cup N_k$ which have the same cardinality. For example, if $|N_1| = \dots = |N_k| = m$, the group $N_{S_n}(S_\rho)$ is isomorphic to the wreath products $S_m \wr (S_k, \mathbf{N}_k)$. In the general case the group $N_{S_n}(S_\rho)$ is the direct product of such wreath products, which correspond to each fixed cardinality of blocks in $N_1 \cup \dots \cup N_k$.

The bimodule $\mathbb{C}[S_n/S_\rho]$ is by definition nothing else than the permutation module for S_n associated with the Young subgroup S_ρ , see [Sa,

2.1]. If λ is the partition of n describing the cardinalities of blocks in ρ , then $\mathbb{C}[S_N/S_\rho]$ has a natural basis consisting of tabloids of shape λ . The group $N_{S_n}(S_\rho)/S_\rho$ act on $\mathbb{C}[S_N/S_\rho]$ by automorphisms permuting rows of the same length in all tabloids. Assume that for $i = 1, \dots, n$ the relation ρ has k_i blocks of cardinality i . Then

$$N_{S_n}(S_\rho)/S_\rho \cong S_{k_1} \times S_{k_2} \times \cdots \times S_{k_n}$$

where we use the convention that S_0 is the group with one element. Set $\mathbf{k} = (k_1, \dots, k_n)$ and call $\mathbf{l} = (l_1, \dots, l_n)$ a *partition* of \mathbf{k} provided that every l_i is a partition of the corresponding k_i (denoted $\mathbf{l} \vdash \mathbf{k}$). Then simple $N_{S_n}(S_\rho)/S_\rho$ -modules are just outer tensor products of Specht modules over the symmetric components and thus are indexed by all possible partitions $\mathbf{l} \vdash \mathbf{k}$. We denote these modules by $S^{\mathbf{l}}$.

Corollary 16. *We have the following decomposition of left $\mathcal{FP}^+(S_n)$ -modules:*

$$\mathbb{C}[S_N/S_\rho] \cong \bigoplus_{\mathbf{l} \vdash \mathbf{k}} \dim(S^{\mathbf{l}}) L(S_\rho, S^{\mathbf{l}}).$$

Proof. The $\mathcal{FP}^+(S_n)$ -module $\mathbb{C}[S_N/S_\rho]$ is obtained via tensor induction from the regular $N_{S_n}(S_\rho)/S_\rho$ -module. In the latter each $S^{\mathbf{l}}$ appears with multiplicity $\dim(S^{\mathbf{l}})$. Now the claim follows from Corollary 11. \square

Restricting back to S_n we get from Corollary 16 a finer decomposition of the permutation module into a direct sum of submodules. This decomposition is very natural from the $\mathcal{FP}^+(S_n)$ -point of view. It is thus reasonable to ask what is the structure of these submodules $L(S_\rho, S^{\mathbf{l}})$, when we consider them as S_n -modules.

Problem 17. *For $\lambda \vdash n$ determine the multiplicity of the Specht module S^λ in the S_n -module $L(S_\rho, S^{\mathbf{l}})$.*

For the original permutation module the answer is well-known and given by Kostka numbers, see [Sa, 2.11]. Some special cases of Problem 17 are computed in [Ch].

Remark 18. Problem 17 seems to be very difficult in the general case. Actually a very special case of it is closely related (and would give an answer) to the Foulkes' conjecture, see [Fo]. This conjecture can be formulated as follows: Let $n = km$, where $k < m$, and consider two equivalence relations ρ_1 and ρ_2 on \mathbf{N} , the first having k blocks with m elements each, and the second having m blocks with k elements each. Let triv denote the trivial module (for any group). Then conjecture says that the multiplicity of S^λ in $L(S_{\rho_1}, \text{triv})$ does not exceed the multiplicity of S^λ in $L(S_{\rho_2}, \text{triv})$. Although Foulkes' conjecture is proved in some special cases (see for example [Do, MN, Py] and references therein), the general case seems to be wide open.

2.6. Connection with \mathcal{F}_n^* . Let \mathcal{F}_n^* denote the set of all binary relations on \mathbf{N}_n which have the form $\rho\sigma$, where ρ is an equivalence relation and $\sigma \in S_n$. Given $\rho_1\sigma_1, \rho_2\sigma_2 \in \mathcal{F}_n^*$ we define

$$\rho_1\sigma_1 \bullet \rho_2\sigma_2 = \overline{\rho_1\sigma_1\rho_2\sigma_1^{-1}}\sigma_1\sigma_2,$$

where $\overline{\rho_1\sigma_1\rho_2\sigma_1^{-1}}$ denotes the minimal equivalence relation, containing the binary relation $\rho_1\sigma_1\rho_2\sigma_1^{-1}$ (the latter one is the product of two equivalence relations ρ_1 and $\sigma_1\rho_2\sigma_1^{-1}$). This makes \mathcal{F}_n^* into an inverse monoid, which is called the *maximal factorizable submonoid of the dual symmetric inverse monoid \mathcal{I}_n^** , see [FL, Section 3]. Our main result in this subsection is the following:

Theorem 19. *Let J denote the Jacobson radical of $\mathbb{C}[\mathcal{FP}^+(S_n)]$. Then $\mathbb{C}[\mathcal{FP}^+(S_n)]/J \cong \mathbb{C}[\mathcal{F}_n^*]$.*

Proof. Let $\lambda \vdash n$. For $i = 1, \dots, n$ denote by $k_{\lambda,i}$ the number of entries of λ , which are equal to i . Set

$$n_\lambda = \frac{n!}{\prod_{i=1}^n k_{\lambda,i}! \cdot i!^{k_{\lambda,i}}}$$

and $G_\lambda = S_{k_{\lambda,1}} \times S_{k_{\lambda,2}} \times \dots \times S_{k_{\lambda,n}}$.

Since $\mathbb{C}[\mathcal{FP}^+(S_n)]$ is essentially inverse, using Corollary 12 and the description of Green's relations on $\mathcal{FP}^+(S_n)$ (see [Ma2] or Subsections 2.1 and 2.2) we have:

$$(2) \quad \mathbb{C}[\mathcal{FP}^+(S_n)]/J \cong \bigoplus_{\lambda \vdash n} \text{Mat}_{n_\lambda}(\mathbb{C}[G_\lambda]).$$

Since \mathcal{F}_n^* is an inverse semigroup, the algebra $\mathbb{C}[\mathcal{F}_n^*]$ is semi-simple, see for example [GM5, Theorem 11.5.3]. In particular, the Jacobson radical of $\mathbb{C}[\mathcal{F}_n^*]$ is zero. Applying Corollary 12 and the description of Green's relations on \mathcal{F}_n^* (see [FL, Section 3]) we have:

$$(3) \quad \mathbb{C}[\mathcal{F}_n^*] \cong \bigoplus_{\lambda \vdash n} \text{Mat}_{n_\lambda}(\mathbb{C}[G_\lambda]).$$

The statement now follows by combining (2) and (3). \square

As a standard corollary we have:

Corollary 20. *There exists an algebra monomorphism $\mathbb{C}[\mathcal{F}_n^*] \hookrightarrow \mathbb{C}[\mathcal{FP}^+(S_n)]$.*

Problem 21. *Construct some explicit algebra monomorphism $\mathbb{C}[\mathcal{F}_n^*] \hookrightarrow \mathbb{C}[\mathcal{FP}^+(S_n)]$.*

Remark 22. Theorem 19 basically says that the algebras $\mathbb{C}[\mathcal{FP}^+(S_n)]$ and $\mathbb{C}[\mathcal{F}_n^*]$ have “the same” simple modules. This means the following: as the regular \mathcal{D} -classes of $\mathcal{FP}^+(S_n)$ and the \mathcal{D} -classes of \mathcal{F}_n^* are

both indexed by partitions of n , we have an explicit bijection between these \mathcal{D} -classes, which induces a natural bijection between simple modules. Using the construction of simple modules given in Subsection 2.3 one sees that the corresponding simple modules over $\mathbb{C}[\mathcal{FP}^+(S_n)]$ and $\mathbb{C}[\mathcal{F}_n^*]$ are isomorphic when considered as S_n -modules (in particular they have the same dimension).

3. DUALITY, UNITARIZABILITY AND TENSOR PRODUCTS

Let G be a finite group faithfully acting on a finite set M . Set $A = A(G, M) = \mathbb{C}[\mathcal{FP}^+(G, M)]$. In this section we will try to get some information about the category $A\text{-mod}$ of all finite-dimensional left A -modules.

3.1. Contravariant duality on the module category. The anti-involution \star on $\mathcal{FP}^+(G, M)$ extends to an anti-involution \star on A in the natural way. For $V \in A\text{-mod}$ define an A -module structure on $\text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ via $(af)(v) := f(a^\star v)$ (note that the fact that \star is an *anti*-involution guarantees that we indeed get a *left* A -module). This defines a contravariant functor on $A\text{-mod}$. To distinguish it from the usual duality $*$ between $A\text{-mod}$ and $\text{mod-}A$, we will denote the functor we just defined by \natural .

Proposition 23. *The functor \natural is an exact involutive contravariant equivalence, which preserves the isomorphism classes of simple A -modules.*

Proof. That \natural is exact and contravariant follows from the definition. That \natural is involutive follows from the natural isomorphism

$$\text{Hom}_{\mathbb{C}}(\text{Hom}_{\mathbb{C}}(V, \mathbb{C}), \mathbb{C}) \cong V.$$

As $\natural \circ \natural$ is isomorphic to the identity functor, we also get that \natural is an equivalence. Hence we are left to check that \natural preserves the isomorphism classes of simple A -modules.

To prove the latter recall from Subsection 2.4 that every simple A -module has the form $L(H, X)$, where H is an orbit-maximal subgroup of G and X is a simple $N_G(H)/H$ -module. Since \natural is an equivalence, the module $L(H, X)^\natural$ is simple. Let us determine this module exactly. First we observe that $\pi^\star = \pi$ for any idempotent $\pi \in \mathcal{FP}^+(G, M)$ (as any subgroup of G is obviously invariant under taking the inverses of all elements). From this and the definition of \natural it thus follows that some idempotent π of $\mathcal{FP}^+(G, M)$ annihilates $L(H, X)^\natural$ if and only if π annihilates $L(H, X)$. This, in particular, implies that $L(H, X)^\natural$ is associated with the same \mathcal{D} -class as $L(H, X)$. Therefore $L(H, X)^\natural \cong L(H, X')$ for some simple $N_G(H)/H$ -module X' .

To determine the module X' we can compute its character (as an $N_G(H)/H$ -module) using the definition of \natural . First of all we observe that the module X' coincides, as a vector space, with the image of the

idempotent \overline{H} (note that the latter is stable under \star). In particular, we get $\dim(X) = \dim(X')$ and can restrict our computation to the corresponding subspaces of $L(H, X)$ and $L(H, X)^\natural$. Further, from the definition of \natural we see that the action of every $g \in N_G(H)/H$ gets substituted by the action of g^{-1} . However, the group $N_G(H)/H$ is a direct product of finite symmetric groups. In every finite symmetric group the elements g and g^{-1} are conjugated. This implies that X and X' have the same characters and hence must be isomorphic. Thus $L(H, X)^\natural \cong L(H, X)$, which completes the proof. \square

3.2. Unitarizability of simple modules. The aim of this subsection is to prove that every simple $\mathcal{FP}^+(G, M)$ -module is unitarizable with respect to the anti-involution \star on $\mathcal{FP}^+(G, M)$ in the ordinary sense. The motivation for this question was the combination of the two facts: that $\mathcal{FP}^+(G, M)$ is essentially inverse (Subsection 2.3), and that simple (even all) modules over inverse semigroups are unitarizable ([GM5, 11.5]). As $\mathcal{FP}^+(G, M)$ is not inverse in general, the algebra A is not semi-simple and hence all A -modules cannot be unitarizable in the ordinary sense. However, one could expect that at least simple modules are. We will prove this in the present subsection.

Let H be an orbit-maximal subgroup of G and X be a simple $N_G(H)/H$ -module. Let further $H_1 = H, H_2, \dots, H_k$ be a list of all pairwise different subgroups of G , conjugated to H . For $i = 1, \dots, k$ set $\varepsilon_i = \overline{H_i}$. Then from Proposition 1(d) it follows that $\varepsilon_1, \dots, \varepsilon_k$ is a list of all pairwise different idempotents in the regular \mathcal{D} -class of $\mathcal{FP}^+(G, M)$, containing the idempotent \overline{H} . Finally, for $i = 2, \dots, k$ fix some $g_i \in G$ such that $H_i = g_i^{-1}Hg_i$, and set $g_1 = e$.

Lemma 24. *For $i = 1, \dots, k$ let V_i denote the image of ε_i on $L(H, X)$. Then $L(H, X) = \bigoplus_{i=1}^k V_i$.*

Proof. As $\hat{\mathcal{D}}_H$ is an inverse semigroup by Proposition 5(b), the product of any two different idempotents in it is zero. If $v_i \in V_i$ and $\lambda_i \in \mathbb{C}$ are such that $\sum_i \lambda_i v_i = 0$, applying ε_j and using the above we get $\lambda_j v_j = 0$ and thus for every j we either have $\lambda_j = 0$ or $v_j = 0$. This means that the sum of all the V_i 's is direct.

On the other hand, let $0 \neq v \in L(H, X)$, $v_i = \varepsilon_i v$ and $v' = \sum_i v_i$. Then $\varepsilon_i(v - v') = 0$ for all i . Set

$$W = \{w \in L(H, X) \mid \varepsilon_i w = 0 \text{ for all } i\}.$$

Let $w \in W$, $\pi \in \mathcal{FP}^+(G, M)$ and $i \in \{1, \dots, k\}$. If $\varepsilon_i \pi \notin \mathcal{D}_H$, we have $\varepsilon_i \pi w = 0$ by definition. If $\varepsilon_i \pi \in \mathcal{D}_H$, then the left class of $\varepsilon_i \pi$ contains some ε_j and hence $\varepsilon_i \pi = \tau \varepsilon_j$ for some $\tau \in G$ by Proposition 1(a). This again implies that $\varepsilon_i \pi w = 0$. Thus W is a submodule of $L(H, X)$, different from $L(H, X)$ (as $\varepsilon_1 \neq 0$). Since $L(H, X)$ is simple, we get $W = 0$ and hence $v - v' = 0$. This proves that $L(H, X) = \sum_{i=1}^k V_i$ and completes the proof of the lemma. \square

Let $(\cdot, \cdot)_1$ be any Hermitian scalar product on V_1 . For $v, w \in V_1$ set

$$(v, w) = \sum_{g \in N_G(H)/H} (g(v), g(w))_1$$

and note that this new Hermitian scalar product on V_1 is invariant with respect to the action of $N_G(H)/H$. For $i \neq j$ and arbitrary $v \in V_i$, $w \in V_j$ we set $(v, w) = 0$. Finally, for $v \in V_i$ we note that $\varepsilon_i = g_i^{-1} \varepsilon_1 g_i$ by our choice of g_i and hence $g_i(v) \in V_1$. Thus for $v, w \in V_i$ we may set $(v, w) = (g_i(v), g_i(w))$ and extend (\cdot, \cdot) to the whole $L(H, X)$ by skew-bilinearity. Now we are in position to show that the $\mathcal{FP}^+(G, M)$ -module $L(H, X)$ is *unitarizable* in the following sense:

Proposition 25. (\cdot, \cdot) is a Hermitian scalar product on $L(H, X)$ and

$$(4) \quad (\sigma(v), w) = (v, \sigma^*(w))$$

for all $\sigma \in \mathcal{FP}^+(G, M)$ and $v, w \in L(H, X)$.

Proof. The product (\cdot, \cdot) is bilinear and skew-symmetric by construction and its restriction to every V_i is positive definite as $(\cdot, \cdot)_1$ is positive definite. Hence the fact that (\cdot, \cdot) is positive definite follows from Lemma 24. This shows that (\cdot, \cdot) is a Hermitian scalar product on $L(H, X)$.

Let now $\sigma \in \mathcal{FP}^+(G, M)$, $v \in V_i$ and $w \in V_j$. If $\varepsilon_j \sigma \varepsilon_i \notin \mathcal{D}_H$, then $(\sigma(v), w) = (v, \sigma^*(w)) = 0$ follows immediately from the definitions and thus (4) holds. Hence we may assume $\sigma' = \varepsilon_j \sigma \varepsilon_i \in \mathcal{D}_H$ and even more that $\sigma = \sigma'$. Then $\sigma(v) \in V_j$ and hence we have $(\sigma(v), w) = (g_j \sigma(v), g_j(w))$. From $\sigma = \varepsilon_j \sigma \varepsilon_i \in \mathcal{D}_H$ it follows that $g_j \sigma g_i^{-1} \in N_G(H)/H$. Using the invariance of (\cdot, \cdot) with respect to the action of $N_G(H)/H$ on V_1 we get $(g_j \sigma(v), g_j(w)) = (g_i(v), g_i \sigma^*(w))$, which implies the necessary equality (4) and completes the proof. \square

Remark 26. The existence and uniqueness (in the correct sense) of some (not necessarily positive definite) scalar product on $L(H, X)$ having the property (4) follows easily from Subsection 3.1 and [MT, Theorem 1]. However, the fact that this scalar product is positive definite (established in Proposition 25) will be crucial for the next subsection.

3.3. Complete reducibility of tensor products. If S is a semi-group and X and Y are two S -modules, then the tensor product $X \otimes_{\mathbb{C}} Y$ has a natural structure of an S -module via the diagonal action $s(x \otimes y) = s(x) \otimes s(y)$. Tensor products of certain modules over various generalizations of the symmetric group appear in the literature, especially in connection to Schur-Weyl dualities (see [So, KM]). The main result of this subsection is the following statement:

Theorem 27. *Let H and H' be two orbit-maximal subgroups of G and X and X' be two simple modules over $N_G(H)/H$ and $N_G(H')/H'$ respectively. Then the $\mathcal{FP}^+(G, M)$ -module $L(H, X) \otimes_{\mathbb{C}} L(H', X')$ is completely reducible.*

Proof. By Proposition 25, both $L(H, X)$ and $L(H', X')$ are unitarizable. Let (\cdot, \cdot) and $(\cdot, \cdot)'$ denote the corresponding Hermitian scalar products. Let further v_1, \dots, v_k and w_1, \dots, w_m be some orthonormal bases in $L(H, X)$ and $L(H', X')$ with respect to (\cdot, \cdot) and $(\cdot, \cdot)'$ respectively. Let $\langle \cdot, \cdot \rangle$ denote the Hermitian scalar product on the vector space $L(H, X) \otimes_{\mathbb{C}} L(H', X')$ for which $\{v_i \otimes w_j\}$ is an orthonormal basis.

For $\sigma \in \mathcal{FP}^+(G, M)$ set $\sigma_{i,j} = (\sigma(v_i), v_j)$ and $\sigma'_{a,b} = (\sigma(w_a), w_b)'$. Then unitarizability of $L(H, X)$ and $L(H', X')$ means that $\sigma_{i,j} = \overline{(\sigma^*)_{j,i}}$ and $\sigma'_{a,b} = \overline{(\sigma^*)'_{b,a}}$ for all $\sigma, \sigma', i, j, a, b$, where $\bar{\cdot}$ denotes the complex conjugation. Using this we have

$$\langle \sigma(v_i \otimes w_a), v_j \otimes w_b \rangle = \sigma_{i,j} \sigma'_{a,b}$$

for all appropriate i, j, a, b and, moreover,

$$\sigma_{i,j} \sigma'_{a,b} = \overline{(\sigma^*)_{j,i}} \cdot \overline{(\sigma^*)'_{b,a}} = \overline{(\sigma^*)_{j,i} (\sigma^*)'_{b,a}},$$

which implies that the equality (4) holds for the form $\langle \cdot, \cdot \rangle$ on $L(H, X) \otimes_{\mathbb{C}} L(H', X')$. In particular, the module $L(H, X) \otimes_{\mathbb{C}} L(H', X')$ is unitarizable.

Now if $W \subset L(H, X) \otimes_{\mathbb{C}} L(H', X')$ is a subspace invariant under the action of $\mathcal{FP}^+(G, M)$, using (4) one shows that the orthogonal complement W^\perp is also invariant (and is really a complement as our scalar product is positive definite). This implies that $L(H, X) \otimes_{\mathbb{C}} L(H', X')$ is completely reducible. \square

The following natural question might be very difficult since, as far as the author knows, the complete answer to it is not yet known even for the symmetric group. However, a satisfactory ‘‘semigroup’’ answer could be a reduction of this problem to the analogous problem for the symmetric group.

Problem 28. *Determine the multiplicity of each $L(H, X)$ in the decomposition of $L(H', X') \otimes_{\mathbb{C}} L(H'', X'')$.*

Remark 29. Tensoring with $L(H, X)$ defines an exact endofunctor on $\mathcal{FP}^+(G, M)$ -mod. It would be interesting to understand this functor and its adjoints.

Remark 30. Proposition 25 and Theorem 27 generalize (with the same proof) to arbitrary essentially inverse semigroup with involution, which induces the usual involution $a \mapsto a^{-1}$ on all inverse \mathcal{D} -classes.

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