Cell 2-representations of finitary 2-categories

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Abstract

We study 2-representations of finitary 2-categories with involution and adjunctions by functors on module categories over finite dimensional algebras. In particular, we define, construct and describe in detail (right) cell 2-representations inspired by Kazhdan-Lusztig cell modules for Hecke algebras. Under some natural assumptions we show that cell 2-representations are strongly simple and do not depend on the choice of a right cell inside a two-sided cell. This reproves and extends the uniqueness result on categorification of Kazhdan-Lusztig cell modules for Hecke algebras of type A from [MS].

1. Introduction and description of the results

The philosophy of categorification, which originated in work of Crane and Frenkel (see [Cr, CF]) some fifteen years ago, is nowadays usually formulated in terms of 2-categories. A *categorification* of an algebra (or category) A is now usually understood as a 2-category $\mathscr{A}$, whose decategorification is A. Therefore a natural problem is to "upgrade" the representation theory of A to a 2-representation theory of $\mathscr{A}$. The latter philosophy has been propagated by Rouquier in [Ro1, Ro2] based on the earlier development in [CR].

Not much is known about the 2-category of 2-representations of an abstract 2-category. Some 2-representations of 2-categories categorifying Kac-Moody algebras were constructed and studied in [Ro2]. On the other hand, there are many examples of 2-representations of various 2-categories in the literature, sometimes without an explicit emphasis on their 2-categorical nature, see for example [Kv, St, KMS, MS, KhLa] and references therein. A different direction of the representation theory of certain classes of 2-categories was investigated in [EO, EGNO].

The 2-categorical philosophy also appears, in a disguised form, in [Kh]. In this article the author defines so-called "categories with full projective functors" and considers "functors naturally commuting with projective functors". The former can be understood as certain "full" 2-representations of a 2-category and the latter as morphisms between these 2-representations.

The aim of the present article is to look at the study of 2-representations of abstract 2-categories from a more systematic and more abstract prospective. Given an algebra A there are two natural ways to construct A-modules. The first way is to fix a presentation for A and construct A-modules using generators and checking relations. The second way is to look at homomorphisms between free A-modules and construct their cokernels. Rouquier's approach to

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2-representation theory from [Ro1, Ro2] goes along the first way. In the present article, we try the second one.

Our main object of study is what we call a *fiat category* $\mathscr{C}$, that is a (strict) 2-category with involution which has finitely many objects, finitely many isomorphism classes of indecomposable 1-morphisms, and finite dimensional spaces of 2-morphisms that are also supposed to contain adjunction morphisms. Our 2-setup is described in detail in Section 2. In Section 3 we study *principal 2-representations* of fiat categories, which are analogues of indecomposable projective modules over an algebra. We give an explicit construction of principal 2-representations and prove a natural analogue of the universal property for them. Adding up all principal representations we obtain the regular $\mathscr{C}$-bimodule, which gives rise to an abelian 2-category $\hat{\mathscr{C}}$ enveloping the original category $\mathscr{C}$. The category $\hat{\mathscr{C}}$ is no longer fiat, but has the advantage of being abelian. We show that every 2-representation of $\mathscr{C}$ extends to a 2-representation of $\hat{\mathscr{C}}$ in a natural way.

Inspired by Kazhdan-Lusztig combinatorics (see [KaLu]), in Section 4 we define, for every fiat category $\mathscr{C}$, the concepts of left, right and two-sided cells and cell 2-representations associated with right cells. We expect cell representations to be the most natural candidates for "simple" 2-representations. We even define a class of 2-representations which we call *strongly simple*, but at the moment, we are unsure of what the most useful definition of simple should be. We describe the algebraic structure of module categories on which a cell 2-representation operates and determine homomorphisms from a cell 2-representation. We also study in detail the combinatorial structure of two natural classes of cells, which we call regular and strongly regular. These turn out to have particularly nice properties and appear in many natural examples. Because of the connection with Kazhdan-Lusztig combinatorics, many constructions in the paper seem quite analogous to the theory of cellular algebras developed in [GL].

Section 5 is devoted to the study of the local structure of cell 2-representations. We show that the essential part of cell 2-representations is governed by the action of 1-morphisms from the associated two-sided cell and describe algebraic properties of cell 2-representations in terms of the cell combinatorics of this two-sided cell.

In Section 6 we define and study the notions of cyclicity and strong simplicity for 2-representations. A 2-representation is called cyclic if it is generated, in the 2-categorical sense of categories with full projective functors in [Kh], by some object M. This means that the natural map from $\hat{\mathscr{C}}$ to our 2-representation, sending F to $\mathrm{F}\, M$ is essentially surjective on objects and surjective on morphisms. A 2-representation is called strongly simple if it is generated, in the 2-categorical sense, by any simple object. We show that all cell 2-representations are cyclic and prove the following main result:

Theorem 1. *Let $\mathscr{C}$ be a fiat category. Then, under some natural technical assumptions, every cell 2-representation of $\mathscr{C}$ associated with a strongly regular right cell is strongly simple. Moreover, under the same assumptions, every two cell 2-representations of $\mathscr{C}$ associated with strongly regular right cells inside the same two-sided cell are equivalent.*

Finally, in Section 7 we give several examples. The prime example is the fiat category of projective functors acting on the principal block (or a direct sum of some, possibly singular, blocks) of the BGG category $\mathcal{O}$ for a semi-simple complex finite dimensional Lie algebra. This example is given by Kazhdan-Lusztig combinatorics and our cells coincide with the classical Kazhdan-Lusztig cells. As an application of Theorem 1 we reprove, extend and strengthen the uniqueness result on categorification of Kazhdan-Lusztig cell modules for Hecke algebras of type A from [MS]. We also present another example of a fiat category $\mathscr{C}_A$ given by projective endofunctors of the module category of a weakly symmetric self-injective finite dimensional associative algebra A. We show that the latter example is "universal" in the sense that, under the same assumptions as mentioned in Theorem 1, every cell 2-representation of a fiat category gives rise to a 2-functor to some $\mathscr{C}_A$.

The 2-categories constructed by Rouquier in [Ro2] and by Khovanov and Lauda in [KhLa] are not fiat categories because of our strong finiteness restrictions. It is not difficult to extend all main constructions and some of the results of the present paper to certain locally finite cases (relaxing either the condition of having finitely many objects or the condition of having finitely many isomorphism classes of 1-morphisms) and to the graded case (when the space of 2-morphisms is graded with finite-dimensional graded components). This would, however, substantially increase technical difficulty and decrease readability of the paper.

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2. 2-setup

2.1 Notation

For a 2-category $\mathscr{C}$, objects of $\mathscr{C}$ will be denoted by $\mathtt{i},\mathtt{j}$ and so on. For $\mathtt{i},\mathtt{j}\in\mathscr{C}$, objects of $\mathscr{C}(\mathtt{i},\mathtt{j})$ (1-morphisms of $\mathscr{C}$) will be called F,G and so on. For $\mathrm{F},\mathrm{G}\in\mathscr{C}(\mathtt{i},\mathtt{j})$, morphisms from F to G (2-morphisms of $\mathscr{C}$) will be written α,β and so on. The identity 1-morphism in $\mathscr{C}(\mathtt{i},\mathtt{i})$ will be denoted $\mathbb{1}_{\mathtt{i}}$ and the identity 2-morphism from F to F will be denoted id_{F}. Composition of 1-morphisms will be denoted by $\circ$, horizontal composition of 2-morphisms will be denoted by $\circ_0$ and vertical composition of 2-morphisms will be denoted by $\circ_1$. We often abbreviate $\mathrm{id}_{\mathrm{F}}\circ_0\alpha$ and $\alpha\circ_0\mathrm{id}_{\mathrm{F}}$ by $\mathrm{F}(\alpha)$ and α_{F}, respectively.

For the rest of the paper we fix an algebraically closed field $\Bbbk$. As we will often consider categories $\mathscr{C}(\mathtt{i},\mathtt{j})$, we will denote the morphism space between X and Y in such a category by $\mathrm{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(X,Y)$ to avoid the awkward looking $\mathscr{C}(\mathtt{i},\mathtt{j})(X,Y)$.

2.2 Finitary 2-categories and 2-representations

In what follows, by a *2-category* we always mean a *strict* 2-category and use the name *bicategory* for the corresponding non-strict structure. Note that any bicategory is biequivalent to a 2-category (see, for example, [Le, 2.3]).

We define a 2-category $\mathscr{C}$ to be $\Bbbk$-*finitary* provided that

(I) $\mathscr{C}$ has finitely many objects;

(II) for every $\mathtt{i},\mathtt{j}\in\mathscr{C}$ the category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is a fully additive (i.e. karoubian) $\Bbbk$-linear category with finitely many isomorphism classes of indecomposable objects and finite dimensional morphism spaces, moreover, horizontal composition of 1-morphisms is biadditive;

(III) for every $\mathtt{i}\in\mathscr{C}$ the object $\mathbb{1}_{\mathtt{i}}\in\mathscr{C}(\mathtt{i},\mathtt{i})$ is indecomposable.

From now on $\mathscr{C}$ will always be a $\Bbbk$-finitary 2-category.

Denote by $\mathfrak{R}_{\Bbbk}$ the 2-category whose objects are categories equivalent to module categories of finite-dimensional $\Bbbk$-algebras, 1-morphisms are functors between objects, and 2-morphisms are natural transformations of functors. We will understand a 2-*representation* of $\mathscr{C}$ to be a strict 2-functor from $\mathscr{C}$ to $\mathfrak{R}_{\Bbbk}$. By [Le, 2.0], 2-representations of $\mathscr{C}$, together with strict 2-natural transformations (i.e. morphisms between 2-representation, given by a collection of functors) and modifications (i.e. morphisms between strict 2-natural transformations, given by natural transformations between the defining functors), form a strict 2-category, which we denote by $\mathscr{C}$-$\mathfrak{mod}$. For simplicity we will identify objects in $\mathscr{C}(\mathtt{i},\mathtt{j})$ with their images under a 2-representation (i.e. we will use module notation).

Example 2. Consider the algebra $D := \mathbb{C}[x]/(x^2)$ of dual numbers. It is easy to check that the endofunctor $\mathrm{F} := D \otimes_{\mathbb{C}} {}_-$ of D-mod satisfies $\mathrm{F}\circ\mathrm{F} \cong \mathrm{F}\oplus\mathrm{F}$. Therefore one can consider the 2-category $\mathscr{S}_2$ defined as follows: $\mathscr{S}_2$ has one object $\mathtt{i} := D$-mod; 1-morphisms of $\mathscr{S}_2$ are all endofunctors of $\mathtt{i}$ which are isomorphic to a direct sum of copies of F and the identity functor; 2-morphisms of $\mathscr{S}_2$ are all natural transformations of functors. The category $\mathscr{S}_2$ is a $\mathbb{C}$-finitary 2-category. It comes together with the *natural* representation (the embedding of $\mathscr{S}_2$ into $\mathfrak{R}_{\mathbb{C}}$).

2.3 Path categories associated to $\mathscr{C}(\mathtt{i},\mathtt{j})$

For $\mathtt{i},\mathtt{j}\in\mathscr{C}$ let $\mathrm{F}_1,\mathrm{F}_2,\dots,\mathrm{F}_r$ be a complete list of pairwise non-isomorphic indecomposable objects in $\mathscr{C}(\mathtt{i},\mathtt{j})$. Denote by $\mathcal{C}_{\mathtt{i},\mathtt{j}}$ the full subcategory of $\mathscr{C}(\mathtt{i},\mathtt{j})$ with objects $\mathrm{F}_1,\mathrm{F}_2,\dots,\mathrm{F}_r$. As $\mathscr{C}$ is $\Bbbk$-finitary, the path algebra of $\mathcal{C}_{\mathtt{i},\mathtt{j}}$ is a finite dimensional $\Bbbk$-algebra. There is a canonical equivalence between the category $\mathcal{C}_{\mathtt{i},\mathtt{j}}^{\mathrm{op}}$-mod and the category of modules over the path algebra of $\mathcal{C}_{\mathtt{i},\mathtt{j}}$.

Example 3. For the category $\mathscr{S}_2$ from Example 2 the category $\mathscr{S}_2(\mathtt{i},\mathtt{i})$ has two indecomposable objects, namely $\mathbb{1}_{\mathtt{i}}$ and F. Realizing exact functors on D-mod as D-bimodules, the functor $\mathbb{1}_{\mathtt{i}}$ corresponds to the bimodule D and the functor F corresponds to the bimodule $D\otimes_{\mathbb{C}} D$. Let $\alpha : D\otimes_{\mathbb{C}} D \to D$ be the unique morphism such that $1\otimes 1\mapsto 1$; $\beta : D\to D\otimes_{\mathbb{C}} D$ be the unique morphism such that $1\mapsto 1\otimes x + x\otimes 1$; and $\gamma : D\otimes_{\mathbb{C}} D\to D\otimes_{\mathbb{C}} D$ be the unique morphism such that $1\otimes 1\mapsto 1\otimes x - x\otimes 1$. Then it is easy to check that the category $\mathcal{C}_{\mathtt{i},\mathtt{i}}$ is given by the following quiver and relations:

$$\begin{tikzcd} \bullet \arrow[loop left, "\gamma"] \arrow[r, bend left, "\alpha"] & \bullet \arrow[l, bend left, "\beta"] \end{tikzcd}, \qquad \begin{array}{c}\gamma^2 = -(\beta\alpha)^2, (\alpha\beta)^2 = 0,\\ \alpha\gamma = \gamma\beta = 0.\end{array}$$

2.4 2-categories with involution

If $\mathscr{C}$ is a $\Bbbk$-finitary 2-category, then an *involution* on $\mathscr{C}$ is a lax involutive object-preserving anti-automorphism $*$ of $\mathscr{C}$. A finitary 2-category $\mathscr{C}$ with involution $*$ is said to have *adjunctions* provided that for any $\mathtt{i},\mathtt{j}\in\mathscr{C}$ and any 1-morphism $\mathrm{F}\in\mathscr{C}(\mathtt{i},\mathtt{j})$ there exist 2-morphisms $\alpha : \mathrm{F}\circ\mathrm{F}^*\to\mathbb{1}_{\mathtt{j}}$ and $\beta : \mathbb{1}_{\mathtt{i}}\to\mathrm{F}^*\circ\mathrm{F}$ such that $\alpha_{\mathrm{F}}\circ_1\mathrm{F}(\beta) = \mathrm{id}_{\mathrm{F}}$ and $\mathrm{F}^*(\alpha)\circ_1\beta_{\mathrm{F}^*} = \mathrm{id}_{\mathrm{F}^*}$. A $\Bbbk$-finitary 2-category with an involution and adjunctions will be called a *fiat category*.

Example 4. The category $\mathscr{S}_2$ from Example 2 is easily seen to be a fiat category.

3. Principal 2-representations

3.1 2-representations $\mathbf{P}_{\mathtt{i}}$

Let $\mathscr{C}$ be a finitary 2-category. For $\mathtt{i},\mathtt{j}\in\mathscr{C}$ denote by $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ the category defined as follows: Objects of $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ are diagrams of the form $\mathrm{F}\xrightarrow{\alpha}\mathrm{G}$, where $\mathrm{F},\mathrm{G}\in\mathscr{C}(\mathtt{i},\mathtt{j})$ are 1-morphisms and α is a 2-morphism. Morphisms of $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ are equivalence classes of diagrams as given by the solid part of the following picture:

$$\begin{tikzcd} \mathrm{F} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathrm{G} \arrow[d, "\beta'"] \arrow[dl, dotted, "\xi"'] \\ \mathrm{F}' \arrow[r, "\alpha'"] & \mathrm{G}' \end{tikzcd}, \qquad \mathrm{F}, \mathrm{F}', \mathrm{G}, \mathrm{G}' \in \mathscr{C}(\mathtt{i}, \mathtt{j}),$$

modulo the ideal generated by all morphisms for which there exists ξ as shown by the dotted arrow above such that $\alpha'\xi = \beta'$. As $\mathscr{C}$ is a finitary category, the category $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ is abelian and equivalent to $\mathcal{C}_{\mathtt{i},\mathtt{j}}^{\mathrm{op}}$-mod, see [Fr].

For $\mathtt{i} \in \mathscr{C}$ define the 2-functor $\mathbf{P}_{\mathtt{i}} : \mathscr{C} \to \mathfrak{R}_{\Bbbk}$ as follows: for $\mathtt{j} \in \mathscr{C}$ set $\mathbf{P}_{\mathtt{i}}(\mathtt{j}) = \overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$. Further, for $\mathtt{k} \in \mathscr{C}$ and $\mathrm{F} \in \mathscr{C}(\mathtt{j},\mathtt{k})$ left horizontal composition with (the identity on) F defines a functor from $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ to $\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})$. We define this functor to be $\mathbf{P}_{\mathtt{i}}(\mathrm{F})$. Given a 2-morphism $\alpha : \mathrm{F} \to \mathrm{G}$, left horizontal composition with α gives a natural transformation from $\mathbf{P}_{\mathtt{i}}(\mathrm{F})$ to $\mathbf{P}_{\mathtt{i}}(\mathrm{G})$. We define this natural transformation to be $\mathbf{P}_{\mathtt{i}}(\alpha)$. From the definition it follows that $\mathbf{P}_{\mathtt{i}}$ is a strict 2-functor from $\mathscr{C}$ to $\mathfrak{R}_{\Bbbk}$. The 2-representation $\mathbf{P}_{\mathtt{i}}$ is called the $\mathtt{i}$-*th principal* 2-representation of $\mathscr{C}$.

For $\mathtt{i},\mathtt{j} \in \mathscr{C}$ and a 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ we denote by P_{F} the projective object $0 \to \mathrm{F}$ of $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$.

3.2 The universal property of $\mathbf{P}_{\mathtt{i}}$

Proposition 5. *Let $\mathbf{M}$ be a 2-representation of $\mathscr{C}$ and $M \in \mathbf{M}(\mathtt{i})$.*

(*a*) *For $\mathtt{j} \in \mathscr{C}$ define the functor $\Phi_{\mathtt{j}}^{M} : \overline{\mathscr{C}}(\mathtt{i},\mathtt{j}) \to \mathbf{M}(\mathtt{j})$ as follows: $\Phi_{\mathtt{j}}^{M}$ sends a diagram $\mathrm{F} \xrightarrow{\alpha} \mathrm{G}$ in $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ to the cokernel of $\mathbf{M}(\alpha)_M$. Then $\Phi^{M} = (\Phi_{\mathtt{j}}^{M})_{\mathtt{j}\in\mathscr{C}}$ is the unique morphism from $\mathbf{P}_{\mathtt{i}}$ to $\mathbf{M}$ given by right exact functors and sending $P_{\mathbb{1}_{\mathtt{i}}}$ to M.*

(*b*) *The correspondence $M \mapsto \Phi^{M}$ is functorial.*

Proof. Claim (a) follows directly from 2-functoriality of $\mathbf{M}$. To prove claim (b) let $f : M \to M'$. Choose now any $\mathrm{F}, \mathrm{G} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ and $\alpha : \mathrm{F} \to \mathrm{G}$. Applying $\mathbf{M}$ to $\mathrm{F} \xrightarrow{\alpha} \mathrm{G}$ gives $\mathbf{M}(\mathrm{F}) \xrightarrow{\mathbf{M}(\alpha)} \mathbf{M}(\mathrm{G})$. Applying the latter to $M \xrightarrow{f} M'$ yields the commutative diagram

$$\begin{tikzcd} \mathbf{M}(\mathrm{F})\,M \arrow[r, "\mathbf{M}(\mathrm{F})\,f"] \arrow[d, "\mathbf{M}(\alpha)_M"'] & \mathbf{M}(\mathrm{F})\,M' \arrow[d, "\mathbf{M}(\alpha)_{M'}"] \\ \mathbf{M}(\mathrm{G})\,M \arrow[r, "\mathbf{M}(\mathrm{G})\,f"] & \mathbf{M}(\mathrm{G})\,M'. \end{tikzcd}$$

This commutative diagram implies that $\{\mathbf{M}(\mathrm{F})\,f : \mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})\}$ extends to a natural transformation from $\Phi_{\mathtt{j}}^{M}$ to $\Phi_{\mathtt{j}}^{M'}$ and claim (b) follows. $\square$

3.3 Connections to categories with full projective functors

Denote by $\mathscr{C}_{\mathtt{i}}$ the full 2-subcategory of $\mathscr{C}$ with object $\mathtt{i}$. Restricting $\mathbf{P}_{\mathtt{i}}$ to $\mathtt{i}$ defines a (unique) principal 2-representation of $\mathscr{C}_{\mathtt{i}}$. As $\mathscr{C}$ is finitary, the identity $\mathbb{1}_{\mathtt{i}}$ is indecomposable and hence so is the projective object $P_{\mathbb{1}_{\mathtt{i}}}$. By definition, for any $\mathrm{F}, \mathrm{G} \in \mathscr{C}_{\mathtt{i}}(\mathtt{i},\mathtt{i})$ the evaluation map

$$\mathrm{Hom}_{\mathscr{C}_{\mathtt{i}}(\mathtt{i},\mathtt{i})}(\mathrm{F},\mathrm{G}) \to \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(\mathrm{F} \circ P_{\mathbb{1}_{\mathtt{i}}}, \mathrm{G} \circ P_{\mathbb{1}_{\mathtt{i}}})$$

is surjective (and, in fact, even bijective). Therefore the category $\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})$ with the designated object $P_{\mathbb{1}_{\mathtt{i}}}$ and endofunctors $\mathbf{P}_{\mathtt{i}}(\mathrm{F})$, $\mathrm{F} \in \mathscr{C}_{\mathtt{i}}(\mathtt{i},\mathtt{i})$, is a category with full projective functors in the sense of [Kh]. The notion of *functors naturally commuting with projective functors* in [Kh] corresponds to morphisms between 2-representations of $\mathscr{C}_{\mathtt{i}}$ in our language. It might be worth pointing out that [Kh] works in the setup of bicategories (without mentioning them).

Similarly, for every $\mathtt{j} \in \mathscr{C}$ and any $\mathrm{F}, \mathrm{G} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ the evaluation map

$$\mathrm{Hom}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(\mathrm{F}, \mathrm{G}) \to \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F} \circ P_{\mathbb{1}_{\mathtt{i}}}, \mathrm{G} \circ P_{\mathbb{1}_{\mathtt{i}}})$$

is surjective (and, in fact, even bijective).

3.4 The regular bimodule

For $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathscr{C}$ and any 1-morphism $\mathrm{F} \in \mathscr{C}(\mathtt{k}, \mathtt{i})$ the right horizontal composition with (the identity on) F gives a functor from $\overline{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ to $\overline{\mathscr{C}}(\mathtt{k}, \mathtt{j})$. For any 1-morphisms $\mathrm{F}, \mathrm{G} \in \mathscr{C}(\mathtt{k}, \mathtt{i})$ and a 2-morphism $\alpha : \mathrm{F} \to \mathrm{G}$ the right horizontal composition with α gives a natural transformation between the corresponding functors. This turns $\overline{\mathscr{C}}(\cdot, \cdot)$ into a 2-bimodule over $\mathscr{C}$. This bimodule is called the *regular bimodule*.

3.5 The abelian envelope of $\mathscr{C}$

Because of the previous subsection, it is natural to expect that one could turn $\overline{\mathscr{C}}$ into a 2-category with the same set of objects as $\mathscr{C}$. Unfortunately, we do not know how to do this as it seems that $\overline{\mathscr{C}}$ contains "too many" objects (and hence only has the natural structure of a bicategory). Instead, we define a biequivalent 2-category $\hat{\mathscr{C}}$ as follows: Objects of $\hat{\mathscr{C}}$ are objects of $\mathscr{C}$. To define 1-morphisms of $\hat{\mathscr{C}}$ consider the regular 2-bimodule $\overline{\mathscr{C}}(\cdot, \cdot)$ over $\mathscr{C}$ just as a left 2-representation. Let $\mathscr{R}$ be the 2-category with same objects as $\mathscr{C}$ and such that for $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ the category $\mathscr{R}(\mathtt{i}, \mathtt{j})$ is defined as the category of all functors from $\bigoplus\limits_{\mathtt{k} \in \overline{\mathscr{C}}} \overline{\mathscr{C}}(\mathtt{k}, \mathtt{i})$ to $\bigoplus\limits_{\mathtt{k} \in \overline{\mathscr{C}}} \overline{\mathscr{C}}(\mathtt{k}, \mathtt{j})$, where morphisms are all natural transformations of functors. We are going to define $\hat{\mathscr{C}}$ as a 2-subcategory of $\mathscr{R}$.

The regular bimodule 2-representation of $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to $\mathscr{R}$ (which is the identity on objects). As usual, for every $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ and any $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ we will denote the image of F under this 2-functor also by F. We define 1-morphisms in $\hat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ as functors in $\mathscr{R}(\mathtt{i}, \mathtt{j})$ of the form $\mathrm{Coker}(\alpha)$, where α is a 2-morphism from F to G for some $\mathrm{F}, \mathrm{G} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$. We define 2-morphisms in $\hat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ as natural transformations between the corresponding cokernel functors coming from commutative diagrams of the following form, where all solid arrows are 2-morphisms in $\mathscr{C}$:

$$\begin{tikzcd}
\mathrm{F} \arrow[r, "\alpha"] \arrow[d, "\xi'"'] & \mathrm{G} \arrow[r, dotted, two heads, "\mathrm{proj}"] \arrow[d, "\xi"] & \mathrm{Coker}(\alpha) \arrow[d, dotted] \\
\mathrm{F}' \arrow[r, "\alpha'"] & \mathrm{G}' \arrow[r, dotted, two heads, "\mathrm{proj}"] & \mathrm{Coker}(\alpha')
\end{tikzcd}$$

Lemma 6. (*a*) *1-morphisms in $\hat{\mathscr{C}}$ are closed with respect to the usual composition of functors in $\mathscr{R}$.*

(*b*) *2-morphisms in $\hat{\mathscr{C}}$ are closed with respect to both horizontal and vertical compositions in $\mathscr{R}$.*

Proof. Let $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathscr{C}$, $\mathrm{F}, \mathrm{G} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, $\mathrm{F}', \mathrm{G}' \in \mathscr{C}(\mathtt{j}, \mathtt{k})$ and $\mathrm{F} \xrightarrow{\alpha} \mathrm{G}$, $\mathrm{F}' \xrightarrow{\alpha'} \mathrm{G}'$ be some 2-morphisms. Then the interchange law for the 2-category $\mathscr{C}$ yields that the following diagram is commutative:

$$\begin{tikzcd} \mathrm{F}'\circ\mathrm{F} \arrow[r, "\mathrm{F}'(\alpha)"] \arrow[d, "\alpha'_{\mathrm{F}}"'] & \mathrm{F}'\circ\mathrm{G} \arrow[d, "\alpha'_{\mathrm{G}}"] \\ \mathrm{G}'\circ\mathrm{F} \arrow[r, "\mathrm{G}'(\alpha)"] & \mathrm{G}'\circ\mathrm{G} \end{tikzcd}$$

This means that

$$\mathrm{Coker}(\alpha')\circ\mathrm{Coker}(\alpha)=\mathrm{Coker}((\alpha'_{\mathrm{G}},\mathrm{G}'(\alpha))),$$

where $(\alpha'_{\mathrm{G}},\mathrm{G}'(\alpha))$ is given by the following diagram:

$$(\mathrm{F}'\circ\mathrm{G})\oplus(\mathrm{G}'\circ\mathrm{F}) \xrightarrow{(\alpha'_{\mathrm{G}},\mathrm{G}'(\alpha))} \mathrm{G}'\circ\mathrm{G}\,.$$

This implies claim (a).

That 2-morphisms are closed with respect to vertical composition follows directly from the definitions. To see that 2-morphisms are closed with respect to horizontal composition, consider the following two commutative diagrams in $\mathscr{C}$:

$$\begin{tikzcd} \mathrm{F}_1 \arrow[r, "\alpha"] \arrow[d, "\xi_1"'] & \mathrm{G}_1 \arrow[d, "\eta_1"] \\ \mathrm{F}'_1 \arrow[r, "\alpha'"] & \mathrm{G}'_1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{F}_2 \arrow[r, "\beta"] \arrow[d, "\xi_2"'] & \mathrm{G}_2 \arrow[d, "\eta_2"] \\ \mathrm{F}'_2 \arrow[r, "\beta'"] & \mathrm{G}'_2 \end{tikzcd}$$

These diagrams induce 2-morphisms between the corresponding cokernels. The horizontal composition of these two morphisms is induced by the following commutative diagram:

$$\begin{tikzcd} \mathrm{F}_1\circ\mathrm{G}_2\oplus\mathrm{G}_1\circ\mathrm{F}_2 \arrow[r, "(\alpha_{\mathrm{G}_2}{,}\mathrm{G}_1(\beta))"] \arrow[d, "\begin{pmatrix}\xi_1\circ_0\eta_2 & 0\\ 0 & \eta_1\circ_0\xi_2\end{pmatrix}"'] & \mathrm{G}_1\circ\mathrm{G}_2 \arrow[d, "\eta_1\circ_0\eta_2"] \\ \mathrm{F}'_1\circ\mathrm{G}'_2\oplus\mathrm{G}'_1\circ\mathrm{F}'_2 \arrow[r, "(\alpha'_{\mathrm{G}'_2}{,}\mathrm{G}'_1(\beta'))"] & \mathrm{G}'_1\circ\mathrm{G}'_2. \end{tikzcd}$$

This proves claim (b) and completes the proof. □

From Lemma 6 it follows that $\hat{\mathscr{C}}$ is a 2-subcategory of $\mathscr{R}$. From the construction it also follows that for any $\mathtt{i},\mathtt{j}\in\mathscr{C}$ the categories $\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ and $\hat{\mathscr{C}}(\mathtt{i},\mathtt{j})$ are equivalent. Furthermore, directly from the definitions we have:

Lemma 7. *There is a unique full and faithful* 2*-functor* $\mathfrak{i}:\mathscr{C}\to\hat{\mathscr{C}}$ *such that for any* $\mathtt{i},\mathtt{j}\in\mathscr{C}$, $\mathrm{F},\mathrm{G}\in\mathscr{C}(\mathtt{i},\mathtt{j})$ *and* $\alpha:\mathrm{F}\to\mathrm{G}$ *we have* $\mathfrak{i}(\mathrm{F})=\mathrm{Coker}(\,0\xrightarrow{0}\mathrm{F}\,)$ *and* $\mathfrak{i}(\alpha)$ *is induces by*

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, "0"'] & \mathrm{F} \arrow[d, "\alpha"] \\ 0 \arrow[r, "0"] & \mathrm{G} \end{tikzcd}\ .$$

As usual, the 2-functor $\mathfrak{i}$ induces the restriction 2-functor $\hat{\mathfrak{i}}:\hat{\mathscr{C}}\text{-}\mathfrak{mod}\to\mathscr{C}\text{-}\mathfrak{mod}$. For the opposite direction we have:

Theorem 8. *Every* 2*-representation of* $\mathscr{C}$ *extends to a* 2*-representation of* $\hat{\mathscr{C}}$.

Proof. Let $\mathbf{M} \in \mathscr{C}\text{-}\mathfrak{mod}$. Abusing notation we will denote the extension of $\mathbf{M}$ to a 2-representation of $\hat{\mathscr{C}}$ also by $\mathbf{M}$. Let $\mathtt{i}, \mathtt{j} \in \mathscr{C}$, $\mathrm{F}, \mathrm{G} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ and $\alpha : \mathrm{F} \to \mathrm{G}$. Then for $\mathrm{Coker}(\alpha) \in \hat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ we define $\mathbf{M}(\mathrm{Coker}(\alpha))$ as $\mathrm{Coker}(\mathbf{M}(\alpha))$.

To define $\mathbf{M}$ on 2-morphisms in $\hat{\mathscr{C}}$, let $\mathrm{F}', \mathrm{G}' \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, $\alpha' : \mathrm{F}' \to \mathrm{G}'$, $\beta : \mathrm{F} \to \mathrm{F}'$ and $\beta' : \mathrm{G} \to \mathrm{G}'$ are such that the diagram

$$\Gamma := \begin{tikzcd} \mathrm{F} \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathrm{G} \arrow[d, "\beta'"] \\ \mathrm{F}' \arrow[r, "\alpha'"] & \mathrm{G}' \end{tikzcd}$$

is commutative. Then a typical 2-morphism γ in $\hat{\mathscr{C}}$ is induced by Γ. Applying $\mathbf{M}$ induces the commutative solid part of the following diagram:

$$\begin{tikzcd} \mathbf{M}(\mathrm{F}) \arrow[r, "\mathbf{M}(\alpha)"] \arrow[d, "\mathbf{M}(\beta)"'] & \mathbf{M}(\mathrm{G}) \arrow[r, dashed, "\mathrm{proj}"] \arrow[d, "\mathbf{M}(\beta')"] & \mathrm{Coker}(\mathbf{M}(\alpha)) \arrow[d, dashed, "\xi"] \\ \mathbf{M}(\mathrm{F}') \arrow[r, "\mathbf{M}(\alpha')"] & \mathbf{M}(\mathrm{G}') \arrow[r, dashed, two heads, "\mathrm{proj}"] & \mathrm{Coker}(\mathbf{M}(\alpha')) \end{tikzcd}$$

Because of the commutativity of the solid part, the diagram extends uniquely to a commutative diagram by the dashed arrows as shown above. Directly from the construction it follows that $\mathbf{M}$ becomes a 2-representation of $\hat{\mathscr{C}}$. □

Because of Theorem 8 it is natural to call $\hat{\mathscr{C}}$ the *abelian envelope* of $\mathscr{C}$. In what follows we will always view 2-representation of $\mathscr{C}$ as 2-representation of $\hat{\mathscr{C}}$ via the construction given by Theorem 8.

4. Cells and cell 2-representations of fiat categories

From now on we assume that $\mathscr{C}$ is a fiat category.

4.1 Orders and cells

Set $\mathcal{C} = \cup_{\mathtt{i},\mathtt{j}} \mathcal{C}_{\mathtt{i},\mathtt{j}}$. Let $\mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l} \in \mathscr{C}$, $\mathrm{F} \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$ and $\mathrm{G} \in \mathcal{C}_{\mathtt{k},\mathtt{l}}$. We will write $\mathrm{F} \leqslant_R \mathrm{G}$ provided that there exists $\mathrm{H} \in \mathscr{C}(\mathtt{j}, \mathtt{l})$ such that G occurs as a direct summand of $\mathrm{H} \circ \mathrm{F}$ (note that this is possible only if $\mathtt{i} = \mathtt{k}$). Similarly, we will write $\mathrm{F} \leqslant_L \mathrm{G}$ provided that there exists $\mathrm{H} \in \mathscr{C}(\mathtt{k}, \mathtt{i})$ such that G occurs as a direct summand of $\mathrm{F} \circ \mathrm{H}$ (note that this is possible only if $\mathtt{j} = \mathtt{l}$). Finally, we will write $\mathrm{F} \leqslant_{LR} \mathrm{G}$ provided that there exists $\mathrm{H}_1 \in \mathscr{C}(\mathtt{k}, \mathtt{i})$ and $\mathrm{H}_2 \in \mathscr{C}(\mathtt{j}, \mathtt{l})$ such that G occurs as a direct summand of $\mathrm{H}_2 \circ \mathrm{F} \circ \mathrm{H}_1$. The relations $\leqslant_L$, $\leqslant_R$ and $\leqslant_{LR}$ are partial preorders on $\mathcal{C}$. The map $\mathrm{F} \mapsto \mathrm{F}^*$ preserves $\leqslant_{LR}$ and swaps $\leqslant_L$ and $\leqslant_R$.

For $\mathrm{F} \in \mathcal{C}$ the set of all $\mathrm{G} \in \mathcal{C}$ such that $\mathrm{F} \leqslant_R \mathrm{G}$ and $\mathrm{G} \leqslant_R \mathrm{F}$ will be called the *right cell* of F and denoted by $\mathcal{R}_{\mathrm{F}}$. The *left cell* $\mathcal{L}_{\mathrm{F}}$ and the *two-sided cell* $\mathcal{LR}_{\mathrm{F}}$ are defined analogously. We will write $\mathrm{F} \sim_R \mathrm{G}$ provided that $\mathrm{G} \in \mathcal{R}_{\mathrm{F}}$ and define $\sim_L$ and $\sim_{LR}$ analogously. These are equivalence relations on $\mathcal{C}$. If $\mathrm{F} \leqslant_L \mathrm{G}$ and $\mathrm{F} \not\sim_L \mathrm{G}$, then we will write $\mathrm{F} <_L \mathrm{G}$ and similarly for $<_R$ and $<_{LR}$.

Example 9. The 2-category $\mathscr{S}_2$ from Example 2 has two right cells $\{\mathbb{1}_{\mathtt{i}}\}$ and $\{\mathrm{F}\}$, which are also left cells and thus two-sided cells as well.

4.2 Annihilators and filtrations

Let $\mathbf{M}$ be a 2-representation of $\mathscr{C}$. For any $\mathtt{i} \in \mathscr{C}$ and any $M \in \mathbf{M}(\mathtt{i})$ consider the *annihilator* $\mathrm{Ann}_{\mathcal{C}}(M) := \{\mathrm{F} \in \mathcal{C} : \mathrm{F}\, M = 0\}$ of M. The set $\mathrm{Ann}_{\mathcal{C}}(M)$ is a coideal with respect to $\leqslant_R$ in the sense that $\mathrm{F} \in \mathrm{Ann}_{\mathcal{C}}(M)$ and $\mathrm{F} \leqslant_R \mathrm{G}$ implies $\mathrm{G} \in \mathrm{Ann}_{\mathcal{C}}(M)$. The *annihilator* $\mathrm{Ann}_{\mathcal{C}}(\mathbf{M}) := \bigcap\limits_{M} \mathrm{Ann}_{\mathcal{C}}(M)$ of $\mathbf{M}$ is a coideal with respect to $\leqslant_{LR}$.

Let $\mathcal{I}$ be a coideal in $\mathcal{C}$ with respect to $\leqslant_{LR}$. For every $\mathtt{i} \in \mathscr{C}$ denote by $\mathbf{M}_{\mathcal{I}}(\mathtt{i})$ the Serre subcategory of $\mathbf{M}(\mathtt{i})$ generated by all simple modules L such that $\mathcal{I} \subset \mathrm{Ann}_{\mathcal{C}}(L)$.

Lemma 10. *By restriction, $\mathbf{M}_{\mathcal{I}}$ is a 2-representation of $\mathscr{C}$.*

Proof. We need to check that $\mathbf{M}_{\mathcal{I}}$ is stable under the action of elements from $\mathcal{C}$. If L is a simple module in $\mathbf{M}_{\mathcal{I}}(\mathtt{i})$ and $\mathrm{F} \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$, then for any $\mathrm{G} \in \mathcal{I}$ the 1-morphism $\mathrm{G} \circ \mathrm{F}$ is either zero or decomposes into a direct sum of 1-morphisms in $\mathcal{I}$ (as $\mathcal{I}$ is a coideal with respect to $\leqslant_{LR}$). This implies $\mathrm{G} \circ \mathrm{F}\, L = 0$. Exactness of G implies that $\mathrm{G}\, K = 0$ for any simple subquotient K of $\mathrm{F}\, L$. The claim follows. $\square$

Assume that for any $\mathtt{i} \in \mathscr{C}$ we fix some Serre subcategory $\mathbf{N}(\mathtt{i})$ in $\mathbf{M}(\mathtt{i})$ such that for any $\mathtt{j} \in \mathscr{C}$ and any $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ we have $\mathrm{F}\, \mathbf{N}(\mathtt{i}) \subset \mathbf{N}(\mathtt{j})$. Then $\mathbf{N}(\mathtt{i})$ is a 2-representation of $\mathscr{C}$ by restriction. It will be called a *Serre 2-subrepresentation* of $\mathbf{M}$. For example, the 2-representation $\mathbf{M}_{\mathcal{I}}$ constructed in Lemma 10 is a Serre 2-subrepresentation of $\mathbf{M}$.

Proposition 11. (*a*) *For any coideal $\mathcal{I}$ in $\mathcal{C}$ with respect to $\leqslant_{LR}$ we have*

$$\mathbf{M}_{\mathcal{I}} = \mathbf{M}_{\mathrm{Ann}_{\mathcal{C}}(\mathbf{M}_{\mathcal{I}})}.$$

(*b*) *For any Serre 2-subrepresentation $\mathbf{N}$ of $\mathbf{M}$ we have*

$$\mathrm{Ann}_{\mathcal{C}}(\mathbf{N}) = \mathrm{Ann}_{\mathcal{C}}(\mathbf{M}_{\mathrm{Ann}_{\mathcal{C}}(\mathbf{N})}).$$

Proof. We prove claim (b). Claim (a) is proved similarly. By definition, for every $\mathtt{i} \in \mathscr{C}$ we have $\mathbf{N}(\mathtt{i}) \subset \mathbf{M}_{\mathrm{Ann}_{\mathcal{C}}(\mathbf{N})}(\mathtt{i})$. This implies $\mathrm{Ann}_{\mathcal{C}}(\mathbf{M}_{\mathrm{Ann}_{\mathcal{C}}(\mathbf{N})}) \subset \mathrm{Ann}_{\mathcal{C}}(\mathbf{N})$. On the other hand, by definition $\mathrm{Ann}_{\mathcal{C}}(\mathbf{N})$ annihilates $\mathbf{M}_{\mathrm{Ann}_{\mathcal{C}}(\mathbf{N})}$, so $\mathrm{Ann}_{\mathcal{C}}(\mathbf{N}) \subset \mathrm{Ann}_{\mathcal{C}}(\mathbf{M}_{\mathrm{Ann}_{\mathcal{C}}(\mathbf{N})})$. This completes the proof. $\square$

Proposition 11 says that $\mathcal{I} \mapsto \mathbf{M}_{\mathcal{I}}$ and $\mathbf{N} \mapsto \mathrm{Ann}_{\mathcal{C}}(\mathbf{N})$ is a Galois correspondence between the partially ordered set of coideals in $\mathcal{C}$ with respect to $\leqslant_{LR}$ and the partially ordered set of Serre 2-subrepresentations of $\mathbf{M}$ with respect to inclusions.

4.3 Annihilators in principal 2-representations

Let $\mathtt{i} \in \mathscr{C}$. By construction, for $\mathtt{j} \in \mathscr{C}$ isomorphism classes of simple modules in $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ are indexed by $\mathcal{C}_{\mathtt{i},\mathtt{j}}$. For $\mathrm{F} \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$ we denote by L_{F} the unique simple quotient of P_{F}.

Lemma 12. *For $\mathrm{F}, \mathrm{G} \in \mathcal{C}$ the inequality $\mathrm{F}\, L_{\mathrm{G}} \neq 0$ is equivalent to $\mathrm{F}^* \leqslant_L \mathrm{G}$.*

Proof. Without loss of generality we may assume $\mathrm{G} \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$ and $\mathrm{F} \in \mathcal{C}_{\mathtt{j},\mathtt{k}}$. Then $\mathrm{F}\, L_{\mathrm{G}} \neq 0$ if and only if there is $\mathrm{H} \in \mathcal{C}_{\mathtt{i},\mathtt{k}}$ such that $\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(P_{\mathrm{H}}, \mathrm{F}\, L_{\mathrm{G}}) \neq 0$. Using $P_{\mathrm{H}} = \mathrm{H}\, P_{\mathbb{1}_{\mathtt{i}}}$ and adjunction we obtain

$$0 \neq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(P_{\mathrm{H}}, \mathrm{F}\, L_{\mathrm{G}}) = \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}^* \circ \mathrm{H}\, P_{\mathbb{1}_{\mathtt{i}}}, L_{\mathrm{G}}).$$

This inequality is equivalent to the claim that $P_{\mathrm{G}} = \mathrm{G}\, P_{\mathbb{1}_{\mathtt{i}}}$ is a direct summand of $\mathrm{F}^* \circ \mathrm{H}\, P_{\mathbb{1}_{\mathtt{i}}}$, that is G is a direct summand of $\mathrm{F}^* \circ \mathrm{H}$. The claim follows. $\square$

Lemma 13. (*a*) *For* $\mathrm{F}, \mathrm{G}, \mathrm{H} \in \mathcal{C}$ *the inequality* $[\mathrm{F}\, L_{\mathrm{G}} : L_{\mathrm{H}}] \neq 0$ *implies* $\mathrm{H} \leqslant_R \mathrm{G}$.
(*b*) *For* $\mathrm{G}, \mathrm{H} \in \mathcal{C}$ *such that* $\mathrm{H} \leqslant_R \mathrm{G}$ *there exists* $\mathrm{F} \in \mathcal{C}$ *such that* $[\mathrm{F}\, L_{\mathrm{G}} : L_{\mathrm{H}}] \neq 0$.

Proof. Without loss of generality we may assume

$$\mathrm{G} \in \mathcal{C}_{\mathtt{i},\mathtt{j}}, \quad \mathrm{F} \in \mathcal{C}_{\mathtt{j},\mathtt{k}} \quad \text{and} \quad \mathrm{H} \in \mathcal{C}_{\mathtt{i},\mathtt{k}}. \tag{1}$$

Then $[\mathrm{F}\, L_{\mathrm{G}} : L_{\mathrm{H}}] \neq 0$ is equivalent to $\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(P_{\mathrm{H}}, \mathrm{F}\, L_{\mathrm{G}}) \neq 0$. Similarly to Lemma 12 we obtain that G must be a direct summand of $\mathrm{F}^* \circ \mathrm{H}$. This means that $\mathrm{H} \leqslant_R \mathrm{G}$, proving (a).

To prove (b) we note that $\mathrm{H} \leqslant_R \mathrm{G}$ implies existence of $\mathrm{F} \in \mathcal{C}$ such that G is a direct summand of $\mathrm{F}^* \circ \mathrm{H}$. We may assume that F, G and H are as in (1). Then, by adjunction, we have

$$0 \neq \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{k})}(\mathrm{F}^*\, P_{\mathrm{H}}, L_{\mathrm{G}}) = \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(P_{\mathrm{H}}, \mathrm{F}\, L_{\mathrm{G}}),$$

which means that $[\mathrm{F}\, L_{\mathrm{G}} : L_{\mathrm{H}}] \neq 0$. This completes the proof. □

Corollary 14. *Let* $\mathrm{F}, \mathrm{G}, \mathrm{H} \in \mathcal{C}$. *If* L_{F} *occurs in the top or in the socle of* $\mathrm{H}\, L_{\mathrm{G}}$, *then* $\mathrm{F} \in \mathcal{R}_{\mathrm{G}}$.

Proof. We prove the claim in the case when L_{F} occurs in the top of $\mathrm{H}\, L_{\mathrm{G}}$, the other case being analogous. As $[\mathrm{H}\, L_{\mathrm{G}} : L_{\mathrm{F}}] \neq 0$, we have $\mathrm{F} \leqslant_R \mathrm{G}$ by Lemma 13. On the other hand, by adjunction, L_{G} occurs in the socle of $\mathrm{H}^*\, L_{\mathrm{F}}$. Hence $[\mathrm{H}^*\, L_{\mathrm{F}} : L_{\mathrm{G}}] \neq 0$ and thus we have $\mathrm{G} \leqslant_R \mathrm{F}$ by Lemma 13. The claim follows. □

Lemma 15. *For any* $\mathrm{F} \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$ *there is a unique (up to scalar) nontrivial homomorphism from* $P_{\mathbb{1}_{\mathtt{i}}}$ *to* $\mathrm{F}^* L_{\mathrm{F}}$. *In particular,* $\mathrm{F}^* L_{\mathrm{F}} \neq 0$.

Proof. Adjunction yields

$$\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(P_{\mathbb{1}_{\mathtt{i}}}, \mathrm{F}^*\, L_{\mathrm{F}}) = \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(P_{\mathrm{F}}, L_{\mathrm{F}}) \cong \Bbbk$$

and the claim follows. □

4.4 Serre 2-subrepresentations of $\mathbf{P}_{\mathtt{i}}$

Let $\mathcal{I}$ be an ideal in $\mathcal{C}$ with respect to $\leqslant_R$, i.e. $\mathrm{F} \in \mathcal{I}$ and $\mathrm{F} \geqslant_R \mathrm{G}$ implies $\mathrm{G} \in \mathcal{I}$. For $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ define $\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}(\mathtt{j})$ as the Serre subcategory of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ generated by L_{F} for $\mathrm{F} \in \mathcal{C}_{\mathtt{i},\mathtt{j}} \cap \mathcal{I}$. Then from Lemma 13 it follows that $\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}$ is a Serre 2-subrepresentation of $\mathbf{P}_{\mathtt{i}}$ and that every Serre 2-subrepresentation of $\mathbf{P}_{\mathtt{i}}$ arises in this way. For $\mathrm{F} \in \mathcal{I} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}$ we denote by $P_{\mathrm{F}}^{\mathcal{I}}$ the maximal quotient of P_{F} in $\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}(\mathtt{j})$. The module $P_{\mathrm{F}}^{\mathcal{I}}$ is a projective cover of L_{F} in $\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}(\mathtt{j})$. Since $\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}$ is a 2-subrepresentation of $\mathbf{P}_{\mathtt{i}}$, for $\mathrm{F} \in \mathcal{C}$ we have

$$\mathrm{F}\, P_{\mathbb{1}_{\mathtt{i}}}^{\mathcal{I}} = \begin{cases} P_{\mathrm{F}}^{\mathcal{I}}, & \mathrm{F} \in \mathcal{I}; \\ 0, & \text{otherwise.} \end{cases}$$

From the definition we have that 2-morphisms in $\mathscr{C}$ surject onto homomorphisms between the various $P_{\mathrm{F}}^{\mathcal{I}}$. The natural inclusion $\mathfrak{i}^{\mathcal{I}} : \mathbf{P}_{\mathtt{i}}^{\mathcal{I}} \to \mathbf{P}_{\mathtt{i}}$ is a morphism of 2-representations, given by the collection of exact inclusions $\mathfrak{i}_{\mathtt{j}}^{\mathcal{I}} : \mathbf{P}_{\mathtt{i}}^{\mathcal{I}}(\mathtt{j}) \to \mathbf{P}_{\mathtt{i}}(\mathtt{j})$.

Note that $\mathcal{C} \setminus \mathcal{I}$ is a coideal in $\mathcal{C}$ with respect to $\leqslant_R$. Hence for any 2-representation $\mathbf{M}$ of $\mathscr{C}$ we have the corresponding Serre 2-subrepresentation $\mathbf{M}_{\mathcal{C}\setminus\mathcal{I}}$ of $\mathbf{M}$.

Proposition 16 Universal property of $\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}$. *Let* $\mathbf{M}$ *a* 2*-representation of* $\mathscr{C}$.

(*a*) *For any morphism* $\Phi : \mathbf{P}_{\mathtt{i}}^{\mathcal{I}} \to \mathbf{M}$ *we have* $\Phi(P_{\mathbb{1}_{\mathtt{i}}}^{\mathcal{I}}) \in \mathbf{M}_{\mathcal{C}\setminus\mathcal{I}}(\mathtt{i})$.

(*b*) *Let* $M \in \mathbf{M}_{\mathcal{C}\setminus\mathcal{I}}(\mathtt{i})$. *For* $\mathtt{j} \in \mathscr{C}$ *let* $\Phi_{\mathtt{j}}^{M} : \mathbf{P}_{\mathtt{i}}^{\mathcal{I}}(\mathtt{j}) \to \mathbf{M}(\mathtt{j})$ *be the unique right exact functor such that for any* $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$ *we have*

$$\Phi_{\mathtt{j}}^{M} : P_{\mathrm{F}}^{\mathcal{I}} \mapsto \mathbf{M}(\mathrm{F})\, M.$$

Then $\Phi^M = (\Phi^M_{\mathtt{j}})_{\mathtt{j}\in\mathscr{C}} : \mathbf{P}_{\mathtt{i}} \to \mathbf{M}$ *is the unique morphism sending* $P^{\mathcal{I}}_{\mathrm{F}}$ *to* M*.*

(*c*) *The correspondence* $M \mapsto \Phi^M$ *is functorial.*

Proof. Claim (a) follows from the fact that $\mathcal{C} \setminus \mathcal{I} \subset \mathrm{Ann}_{\mathcal{C}}(P^{\mathcal{I}}_{\mathbb{1}_{\mathtt{i}}})$. Mutatis mutandis, the rest is Proposition 5. □

4.5 Right cell 2-representations

Fix $\mathtt{i} \in \mathscr{C}$. Let $\mathcal{R}$ be a right cell in $\mathcal{C}$ such that $\mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}} \neq \varnothing$ for some $\mathtt{j} \in \mathscr{C}$.

PROPOSITION 17. (*a*) *There is a unique submodule* $K = K_{\mathcal{R}}$ *of* $P_{\mathbb{1}_{\mathtt{i}}}$ *which has the following properties:*

(*i*) *Every simple subquotient of* $P_{\mathbb{1}_{\mathtt{i}}}/K$ *is annihilated by any* $\mathrm{F} \in \mathcal{R}$*.*

(*ii*) *The module* K *has simple top* $L_{\mathrm{G}_{\mathcal{R}}}$ *for some* $\mathrm{G}_{\mathcal{R}} \in \mathcal{C}$ *and* $\mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}} \neq 0$ *for any* $\mathrm{F} \in \mathcal{R}$*.*

(*b*) *For any* $\mathrm{F} \in \mathcal{R}$ *the module* $\mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}}$ *has simple top* L_{F}*.*

(*c*) *We have* $\mathrm{G}_{\mathcal{R}} \in \mathcal{R}$*.*

(*d*) *For any* $\mathrm{F} \in \mathcal{R}$ *we have* $\mathrm{F}^* \leqslant_L \mathrm{G}_{\mathcal{R}}$ *and* $\mathrm{F} \leqslant_R \mathrm{G}^*_{\mathcal{R}}$*.*

(*e*) *We have* $\mathrm{G}^*_{\mathcal{R}} \in \mathcal{R}$*.*

Proof. Let $\mathrm{F} \in \mathcal{R}$. Let further $\mathtt{j} \in \mathscr{C}$ be such that $\mathrm{F} \in \mathcal{R}\cap\mathcal{C}_{\mathtt{i},\mathtt{j}}$. Then the module $\mathrm{F}\, P_{\mathbb{1}_{\mathtt{i}}}$ is a nonzero indecomposable projective in $\overline{\mathscr{C}(\mathtt{i},\mathtt{j})}$. Hence F does not annihilate $P_{\mathbb{1}_{\mathtt{i}}}$ and thus there is at least one simple subquotient of $P_{\mathbb{1}_{\mathtt{i}}}$ which is not annihilated by F. Let K be the minimal submodule of $P_{\mathbb{1}_{\mathtt{i}}}$ such that every simple subquotient of $P_{\mathbb{1}_{\mathtt{i}}}/K$ is annihilated by F. As $\mathrm{Ann}_{\mathcal{C}}(P_{\mathbb{1}_{\mathtt{i}}}/K)$ is a coideal with respect to $\leqslant_R$, the module $P_{\mathbb{1}_{\mathtt{i}}}/K$ is annihilated by every $\mathrm{G} \in \mathcal{R}$. Similarly, we have that for any simple subquotient L in the top of K and for any $\mathrm{G} \in \mathcal{R}$ we have $\mathrm{G}\, L \neq 0$. This implies that K does not depend on the choice of $\mathrm{F} \in \mathcal{R}$. Then (ai) is satisfied and to complete the proof of (a) we only have to show that K has simple top.

Applying F to the exact sequence $K \hookrightarrow P_{\mathbb{1}_{\mathtt{i}}} \twoheadrightarrow P_{\mathbb{1}_{\mathtt{i}}}/K$ we obtain the exact sequence

$$\mathrm{F}\, K \hookrightarrow \mathrm{F}\, P_{\mathbb{1}_{\mathtt{i}}} \twoheadrightarrow \mathrm{F}\, P_{\mathbb{1}_{\mathtt{i}}}/K.$$

As $\mathrm{F}\, P_{\mathbb{1}_{\mathtt{i}}}/K = 0$, we see that $\mathrm{F}\, K \cong \mathrm{F}\, P_{\mathbb{1}_{\mathtt{i}}}$ is an indecomposable projective and hence has simple top. Applying F to the exact sequence $\mathrm{rad}\, K \hookrightarrow K \twoheadrightarrow \mathrm{top}\, K$ we obtain the exact sequence

$$\mathrm{F}\,\mathrm{rad}\, K \hookrightarrow \mathrm{F}\, K \twoheadrightarrow \mathrm{F}\,\mathrm{top}\, K.$$

As $\mathrm{F}\, K$ has simple top by the above, we obtain that $\mathrm{F}\,\mathrm{top}\, K$ has simple top. By construction, $\mathrm{top}\, K$ is semi-simple and none of its submodules are annihilated by F. Therefore $\mathrm{top}\, K$ is simple, which implies (a) and also (b).

For $\mathrm{F} \in \mathcal{R}$, the projective module P_{F} surjects onto the nontrivial module $\mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}}$ by the above. Hence L_{F} occurs in the top of $\mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}}$ and thus (c) follows from Corollary 14. For $\mathrm{F} \in \mathcal{R}$ we have $\mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}} \neq 0$ and hence (d) follows from Lemma 12.

From (d) we have $\mathrm{G}_{\mathcal{R}} \leqslant_R \mathrm{G}^*_{\mathcal{R}}$. Assume that $\mathrm{G}^*_{\mathcal{R}} \notin \mathcal{R}$ and let $\tilde{\mathcal{R}}$ be the right cell containing $\mathrm{G}^*_{\mathcal{R}}$. By Lemma 15 we have $\mathrm{G}^*_{\mathcal{R}} L_{\mathrm{G}_{\mathcal{R}}} \neq 0$, which implies that $K_{\mathcal{R}} \subset K_{\tilde{\mathcal{R}}}$. If $K_{\mathcal{R}} = K_{\tilde{\mathcal{R}}}$, then $L_{\mathrm{G}_{\mathcal{R}}} = L_{\mathrm{G}_{\tilde{\mathcal{R}}}}$ and hence $\mathcal{R} = \tilde{\mathcal{R}}$, which implies (e). If $K_{\mathcal{R}} \subsetneq K_{\tilde{\mathcal{R}}}$, then from (ai) we have $\mathrm{G}_{\mathcal{R}}\, L_{\mathrm{G}_{\tilde{\mathcal{R}}}} = 0$. As $\mathrm{Ann}_{\mathcal{C}}(L_{\mathrm{G}_{\tilde{\mathcal{R}}}})$ is a coideal with respect to $\leqslant_R$, it follows that $L_{\mathrm{G}_{\tilde{\mathcal{R}}}}$ is annihilated by $\mathrm{G}^*_{\mathcal{R}}$. This contradicts (aii) and hence (e) follows. The proof is complete. □

For simplicity we set $L = L_{\mathrm{G}_{\mathcal{R}}}$ and for $\mathrm{F} \in \mathcal{R}$ define $\mathsf{P}_{\mathrm{F}} := \mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}}$. For $\mathtt{j} \in \mathscr{C}$ denote by $\mathcal{D}_{\mathcal{R},\mathtt{j}}$ the full subcategory of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ with objects P_{G}, $\mathrm{G} \in \mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}$. As each P_{G} is a quotient of

P_{G} and 2-morphisms in $\mathscr{C}$ surject onto homomorphisms between projective modules in $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ (see Subsection 3.3), it follows that 2-morphisms in $\mathscr{C}$ surject onto homomorphisms between the various P_{G}.

Lemma 18. *For every* $\mathrm{F}\in\mathcal{C}$ *and* $\mathrm{G}\in\mathcal{R}$, *the module* $\mathrm{F}\,\mathsf{P}_{\mathrm{G}}$ *is isomorphic to a direct sum of modules of the form* P_{H}, $\mathrm{H}\in\mathcal{R}$.

Proof. Any H occurring as a direct summand of $\mathrm{F}\circ\mathrm{G}$ satisfies $\mathrm{H}\geqslant_R\mathrm{G}$. On the other hand, $\mathrm{H}\,L\neq 0$ implies $\mathrm{H}^*\leqslant_L\mathrm{G}_{\mathcal{R}}$ by Lemma 12. This is equivalent to $\mathrm{H}\leqslant_R\mathrm{G}_{\mathcal{R}}^*$. By Proposition 17(e), we have $\mathrm{G}_{\mathcal{R}}^*\in\mathcal{R}$. Thus $\mathrm{H}\in\mathcal{R}$, as claimed. □

Lemma 19. *For every* $\mathrm{F},\mathrm{H}\in\mathcal{R}\cap\mathcal{C}_{\mathtt{i},\mathtt{j}}$ *we have*

$$\dim\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathsf{P}_{\mathrm{F}},\mathsf{P}_{\mathrm{H}})=[\mathsf{P}_{\mathrm{H}}:L_{\mathrm{F}}].$$

Proof. Let k denote the multiplicity of $\mathrm{G}_{\mathcal{R}}$ as a direct summand of $\mathrm{H}^*\circ\mathrm{F}$. Then for the right hand side we have

$$\begin{array}{rcl}[\mathsf{P}_{\mathrm{H}}:L_{\mathrm{F}}]&=&\dim\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}\,P_{\mathbb{1}_{\mathtt{i}}},\mathrm{H}\,L)\\ \text{(by adjunction)}&=&\dim\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(\mathrm{H}^*\circ\mathrm{F}\,P_{\mathbb{1}_{\mathtt{i}}},L)\\ &=&k.\end{array}$$

At the same time, by adjunction, for the left hand side we have

$$\dim\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}\,L,\mathrm{H}\,L)=\dim\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(\mathrm{H}^*\circ\mathrm{F}\,L,L).\tag{2}$$

From Proposition 17(b) it follows that the right hand side of (2) is at least k. On the other hand,

$$\dim\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}\,L,\mathrm{H}\,L)\leqslant\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}\,P_{\mathbb{1}_{\mathtt{i}}},\mathrm{H}\,L)=[\mathrm{H}\,L:L_{\mathrm{F}}]=k,$$

which completes the proof. □

For $\mathrm{F}\in\mathcal{R}$ consider the short exact sequence

$$\mathrm{Ker}_{\mathrm{F}}\hookrightarrow P_{\mathrm{F}}\twoheadrightarrow\mathsf{P}_{\mathrm{F}},\tag{3}$$

given by Proposition 17(b). Set

$$\mathrm{Ker}_{\mathcal{R},\mathtt{j}}=\bigoplus_{\mathrm{F}\in\mathcal{R}\cap\mathcal{C}_{\mathtt{i},\mathtt{j}}}\mathrm{Ker}_{\mathrm{F}},\qquad P_{\mathcal{R},\mathtt{j}}=\bigoplus_{\mathrm{F}\in\mathcal{R}\cap\mathcal{C}_{\mathtt{i},\mathtt{j}}}P_{\mathrm{F}},\qquad Q_{\mathcal{R},\mathtt{j}}=\bigoplus_{\mathrm{F}\in\mathcal{R}\cap\mathcal{C}_{\mathtt{i},\mathtt{j}}}\mathsf{P}_{\mathrm{F}}.$$

Lemma 20. *The module* $\mathrm{Ker}_{\mathcal{R},\mathtt{j}}$ *is stable under any endomorphism of* $P_{\mathcal{R},\mathtt{j}}$.

Proof. Let $\mathrm{F},\mathrm{H}\in\mathcal{R}\cap\mathcal{C}_{\mathtt{i},\mathtt{j}}$ and $\varphi:P_{\mathrm{F}}\to P_{\mathrm{H}}$ be a homomorphism. It is enough to show that $\varphi(\mathrm{Ker}_{\mathrm{F}})\subset\mathrm{Ker}_{\mathrm{H}}$. Assume this is false. Composing φ with the natural projection onto Q_{H} we obtain a homomorphism from P_{F} to Q_{H} which does not factor through Q_{F}. However, the existence of such homomorphism contradicts Lemma 19. This implies the claim. □

Now we are ready to define the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ of $\mathscr{C}$ corresponding to $\mathcal{R}$. Define $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ to be the full subcategory of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ which consists of all modules M admitting a two step resolution $X_1\to X_0\twoheadrightarrow M$, where $X_1,X_0\in\mathrm{add}(Q_{\mathcal{R},\mathtt{j}})$.

Lemma 21. *The category* $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ *is equivalent to* $\mathcal{D}_{\mathcal{R},\mathtt{j}}^{\mathrm{op}}$-mod.

Proof. Consider first the full subcategory $\mathcal{X}$ of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ which consists of all modules M admitting a two step resolution $X_1\to X_0\twoheadrightarrow M$, where $X_1,X_0\in\mathrm{add}(P_{\mathcal{R}})$. By [Au, Section 5], the category $\mathcal{X}$ is equivalent to $\mathrm{End}_{\mathcal{C}_{\mathtt{i},\mathtt{j}}^{\mathrm{op}}}(P_{\mathcal{R},\mathtt{j}})^{\mathrm{op}}$-mod. By Lemma 20, the algebra $\mathrm{End}_{\mathcal{C}_{\mathtt{i},\mathtt{j}}^{\mathrm{op}}}(Q_{\mathcal{R},\mathtt{j}})$ is the quotient of $\mathrm{End}_{\mathcal{C}^{\mathrm{op}}_{\mathtt{i},\mathtt{j}}}(P_{\mathcal{R},\mathtt{j}})$ by a two-sided ideal. It is easy to see that the standard embedding of $\mathrm{End}_{\mathcal{C}^{\mathrm{op}}_{\mathtt{i},\mathtt{j}}}(Q_{\mathcal{R},\mathtt{j}})^{\mathrm{op}}\text{-mod} \cong \mathcal{D}^{\mathrm{op}}_{\mathcal{R},\mathtt{j}}\text{-mod}$ into $\mathcal{X}$ coincides with $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$. The claim follows. □

Theorem 22 Construction of right cell 2-representations. *Restriction from $\mathbf{P}_{\mathtt{i}}$ defines the structure of a 2-representation of $\mathscr{C}$ on $\mathbf{C}_{\mathcal{R}}$.*

Proof. From Lemma 18 it follows that for any $\mathrm{F} \in \mathcal{C}_{\mathtt{j},\mathtt{k}}$ we have $\mathrm{F}\,\mathbf{C}_{\mathcal{R}}(\mathtt{j}) \subset \mathbf{C}_{\mathcal{R}}(\mathtt{k})$. The claim follows. □

The 2-representation $\mathbf{C}_{\mathcal{R}}$ constructed in Theorem 22 is called the *right cell 2-representation* corresponding to $\mathcal{R}$. Note that the inclusion of $\mathbf{C}_{\mathcal{R}}$ into $\mathbf{P}_{\mathtt{i}}$ is only right exact in general.

Example 23. Consider the category $\mathscr{S}_2$ from Example 2. For the cell representation $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}}$ we have $G_{\{\mathbb{1}_{\mathtt{i}}\}} = \mathbb{1}_{\mathtt{i}}$, which implies that $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}}(\mathtt{i}) = \mathbb{C}\text{-mod}$; $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}}(\mathrm{F}) = 0$ and $\mathbf{C}_{\{\mathbb{1}_{\mathtt{i}}\}}(f) = 0$ for $f = \alpha, \beta, \gamma$. For the cell representation $\mathbf{C}_{\{\mathrm{F}\}}$ we have $G_{\{\mathrm{F}\}} = \mathrm{F}$, which implies that $\mathbf{C}_{\{\mathrm{F}\}}(\mathtt{i}) = D\text{-mod}$, $\mathbf{C}_{\{\mathrm{F}\}}(\mathrm{F}) = \mathrm{F}$ and $\mathbf{C}_{\{\mathrm{F}\}}(f) = f$ for $f = \alpha, \beta, \gamma$.

4.6 Homomorphisms from a cell 2-representation

Consider a right cell $\mathcal{R}$ and let $\mathtt{i} \in \mathscr{C}$ be such that $\mathrm{G}_{\mathcal{R}} \in \mathcal{C}_{\mathtt{i},\mathtt{i}}$. Let further $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{i})$ and $\alpha : \mathrm{F} \to \mathrm{G}_{\mathcal{R}}$ be such that $\mathbf{P}_{\mathtt{i}}(\alpha) : \mathrm{F}\,P_{\mathbb{1}_{\mathtt{i}}} \to \mathrm{G}_{\mathcal{R}}\,P_{\mathbb{1}_{\mathtt{i}}}$ gives a projective presentation of $L_{\mathrm{G}_{\mathcal{R}}}$.

Theorem 24. *Let $\mathbf{M}$ be a 2-representation of $\mathscr{C}$. Denote by $\Theta = \Theta^{\mathbf{M}}_{\mathcal{R}}$ the cokernel of $\mathbf{M}(\alpha)$.*

(a) *The functor Θ is a right exact endofunctor of $\mathbf{M}(\mathtt{i})$.*
(b) *For every morphism Ψ from $\mathbf{C}_{\mathcal{R}}$ to $\mathbf{M}$ we have $\Psi(L_{\mathrm{G}_{\mathcal{R}}}) \in \Theta(\mathbf{M}(\mathtt{i}))$.*
(c) *For every $M \in \Theta(\mathbf{M}(\mathtt{i}))$ there is a unique morphism Ψ^M from $\mathbf{C}_{\mathcal{R}}$ to $\mathbf{M}$ given by a collection of right exact functors such that Ψ^M sends $L_{\mathrm{G}_{\mathcal{R}}}$ to M.*
(d) *The correspondence $M \mapsto \Psi^M$ is functorial in M in the image $\Theta(\mathbf{M}(\mathtt{i}))$ of Θ.*

Proof. Both $\mathbf{M}(\mathrm{F})$ and $\mathbf{M}(\mathrm{G}_{\mathcal{R}})$ are exact functors as $\mathscr{C}$ is a fiat category and $\mathbf{M}$ is a 2-functor. The functor Θ is the cokernel of a homomorphism between two exact functors and hence is right exact by the Snake lemma. This proves claim (a). Claim (b) follows from the definitions.

To prove claims (c) and (d) choose $M \in \Theta(\mathbf{M}(\mathtt{i}))$ such that $M = \Theta\,N$ for some $N \in \mathbf{M}(\mathtt{i})$. Consider the morphism Φ^N given by Proposition 5. As Φ^N is a morphism of 2-representations, $\Phi^N(L_{\mathrm{G}_{\mathcal{R}}}) = \Theta\,N = M$. The restriction Ψ^M of Φ^N to $\mathbf{C}_{\mathcal{R}}$ is a morphism from $\mathbf{C}_{\mathcal{R}}$ to $\mathbf{M}$. Now the existence parts of (c) and (d) follow from Proposition 5. To prove uniqueness, we note that, for every $\mathtt{j} \in \mathscr{C}$, every projective in $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ has the form $\mathrm{F}\,L_{\mathrm{G}_{\mathcal{R}}}$ for some $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})$ and every morphism between projectives comes from a 2-morphism of $\mathscr{C}$ (see Subsection 4.5). As any morphism from $\mathbf{C}_{\mathcal{R}}$ to $\mathbf{M}$ is a natural transformation of 2-functors, the value of this transformation on $L_{\mathrm{G}_{\mathcal{R}}}$ uniquely determines its value on all other modules. This implies the uniqueness claim and completes the proof. □

4.7 A canonical quotient of $P_{\mathbb{1}_{\mathtt{i}}}$ associated with $\mathcal{R}$

Fix $\mathtt{i} \in \mathscr{C}$. Let $\mathcal{R}$ be a right cell in $\mathcal{C}$ such that $\mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}} \neq \varnothing$ for some $\mathtt{j} \in \mathscr{C}$. Denote by $\Delta_{\mathcal{R}}$ the unique minimal quotient of $P_{\mathbb{1}_{\mathtt{i}}}$ such that the composition $K_{\mathcal{R}} \hookrightarrow P_{\mathbb{1}_{\mathtt{i}}} \twoheadrightarrow \Delta_{\mathcal{R}}$ is nonzero.

Proposition 25. *For every $\mathrm{F} \in \mathcal{R}$ the image of the a unique (up to scalar) nonzero homomorphism $\varphi : P_{\mathbb{1}_{\mathtt{i}}} \to \mathrm{F}^*\,L_{\mathrm{F}}$ is isomorphic to $\Delta_{\mathcal{R}}$.*

Proof. The existence of φ is given by Lemma 15. Let Y denote the image of φ. Assume that $\mathrm{F} \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$. For $X \in \{P_{\mathbb{1}_\mathtt{i}}, \Delta_\mathcal{R}\}$ we have, by adjunction,

$$\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(X, \mathrm{F}^* L_\mathrm{F}) = \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}\, X, L_\mathrm{F}) = \Bbbk$$

as $\mathrm{F}\, X$ is a nontrivial quotient of P_F (see Proposition 17). By construction, Y has simple top isomorphic to $L_{\mathbb{1}_\mathtt{i}}$ and, by the above, the latter module occurs in $\mathrm{F}^* L_\mathrm{F}$ with multiplicity one. Since $\Delta_\mathcal{R}$ also has simple top isomorphic to $L_{\mathbb{1}_\mathtt{i}}$, it follows that the image of any nonzero map from $\Delta_\mathcal{R}$ to Y covers the top of Y and hence is surjective. To complete the proof it is left to show that the image of any nonzero map from $\Delta_\mathcal{R}$ to Y is injective.

By construction, $L_{\mathrm{G}_\mathcal{R}}$ is the simple socle of $\Delta_\mathcal{R}$. Let N denote the cokernel of $L_{\mathrm{G}_\mathcal{R}} \hookrightarrow \Delta_\mathcal{R}$. Similarly to the previous paragraph, we have

$$\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(N, \mathrm{F}^* L_\mathrm{F}) = \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}\, N, L_\mathrm{F}) = 0$$

since all composition factors of N are annihilated by F (by Proposition 17(ai)). The claim follows. □

We complete this section with the following collection of useful facts:

Lemma 26. (*a*) *For any* $\mathrm{F}, \mathrm{G} \in \mathcal{C}_{\mathtt{i},\mathtt{j}}$ *we have* $[\mathrm{F}^* L_\mathrm{G} : L_{\mathbb{1}_\mathtt{i}}] \neq 0$ *if and only if* $\mathrm{F} = \mathrm{G}$.
(*b*) *For any* $\mathrm{F} \in \mathcal{C}$ *we have* $\mathrm{F} \sim_{LR} \mathrm{F}^*$.

Proof. Using adjunction, we have

$$\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(P_{\mathbb{1}_\mathtt{i}}, \mathrm{F}^* L_\mathrm{G}) = \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(P_\mathrm{F}, L_\mathrm{G}) = \begin{cases} \Bbbk, & \mathrm{F} = \mathrm{G}; \\ 0, & \text{otherwise}, \end{cases}$$

which proves (a).

To prove (b) let $\mathcal{R}$ be the right cell containing F. Then we have $\mathrm{F} \sim_R \mathrm{G}_\mathcal{R}$ and hence $\mathrm{F}^* \sim_L \mathrm{G}_\mathcal{R}^*$. At the same time $\mathrm{G}_\mathcal{R} \sim_R \mathrm{G}_\mathcal{R}^*$ by Proposition 17(e). Claim (b) follows and the proof is complete. □

4.8 Regular cells

We denote by $\star$ the usual product of binary relations.

Lemma 27. *We have* $\leqslant_{LR} = \leqslant_R \star \leqslant_L = \leqslant_L \star \leqslant_R$.

Proof. Obviously the product of $\leqslant_R$ and $\leqslant_L$ (in any order) is contained in $\leqslant_{LR}$. On the other hand, for $\mathrm{F}, \mathrm{G} \in \mathcal{C}$ we have $\mathrm{F} \leqslant_{LR} \mathrm{G}$ if and only if there exist $\mathrm{H}, \mathrm{K} \in \mathcal{C}$ such that G occurs as a direct summand of $\mathrm{H} \circ \mathrm{F} \circ \mathrm{K}$. This means that there is a direct summand L of $\mathrm{H} \circ \mathrm{F}$ such that G occurs as a direct summand of $\mathrm{L} \circ \mathrm{K}$. By definition, we have $\mathrm{F} \leqslant_R \mathrm{L}$ and $\mathrm{L} \leqslant_R \mathrm{G}$. This implies that $\leqslant_{LR}$ is contained in $\leqslant_R \star \leqslant_L$ and hence $\leqslant_{LR}$ coincides with $\leqslant_R \star \leqslant_L$. Similarly $\leqslant_{LR}$ coincides with $\leqslant_L \star \leqslant_R$ and the claim of the lemma follows. □

A two-sided cell $\mathcal{Q}$ is called *regular* provided that any two different right cells inside $\mathcal{Q}$ are not comparable with respect to the right order. From Lemma 26(b) it follows that $\mathcal{Q}$ is regular if and only if any two different left cells inside $\mathcal{Q}$ are not comparable with respect to the left order. A right (left) cell is called *regular* if it belongs to a regular two-sided cell. An element F is called *regular* if it belongs to a regular two-sided cell.

Proposition 28 Structure of regular two-sided cells. *Let* $\mathcal{Q}$ *be a regular two-sided cell.*

(a) *For any right cell $\mathcal{R}$ in $\mathcal{Q}$ and left cell $\mathcal{L}$ in $\mathcal{Q}$ we have $\mathcal{L}\cap\mathcal{R}\neq\varnothing$.*

(b) *Let $\sim_R^{\mathcal{Q}}$ and $\sim_L^{\mathcal{Q}}$ denote the restrictions of $\sim_R$ and $\sim_L$ to $\mathcal{Q}$, respectively. Then $\mathcal{Q}\times\mathcal{Q}=\sim_R^{\mathcal{Q}}\star\sim_L^{\mathcal{Q}}=\sim_L^{\mathcal{Q}}\star\sim_R^{\mathcal{Q}}$.*

Proof. If $\mathrm{F}\in\mathcal{R}$ and $\mathrm{G}\in\mathcal{L}$, then there exist $\mathrm{H},\mathrm{K}\in\mathcal{C}$ such that G occurs as a direct summand of $\mathrm{H}\circ\mathrm{F}\circ\mathrm{K}$. This means that there exists a direct summand N of $\mathrm{H}\circ\mathrm{F}$ such that G occurs as a direct summand of $\mathrm{N}\circ\mathrm{K}$. Then $\mathrm{N}\geqslant_R\mathrm{F}$ and $\mathrm{N}\leqslant_L\mathrm{G}$. As $\mathrm{F}\sim_{LR}\mathrm{G}$ it follows that $\mathrm{N}\in\mathcal{Q}$. Since $\mathcal{Q}$ is regular, it follows that $\mathrm{N}\in\mathcal{R}$ and $\mathrm{N}\in\mathcal{L}$ proving (a).

To prove (b) consider $\mathrm{F},\mathrm{G}\in\mathcal{Q}$. By (a), there exist $\mathrm{H}\in\mathcal{Q}$ such that $\mathrm{H}\sim_L\mathrm{F}$ and $\mathrm{H}\sim_R\mathrm{G}$. Similarly, there exist $\mathrm{K}\in\mathcal{Q}$ such that $\mathrm{K}\sim_R\mathrm{F}$ and $\mathrm{K}\sim_L\mathrm{G}$. Then we have $(\mathrm{F},\mathrm{G})=(\mathrm{F},\mathrm{H})\star(\mathrm{H},\mathrm{G})$ and $(\mathrm{F},\mathrm{G})=(\mathrm{F},\mathrm{K})\star(\mathrm{K},\mathrm{G})$ proving (b). □

For a regular right cell $\mathcal{R}$ the corresponding module $\Delta_{\mathcal{R}}$ has the following property.

Proposition 29. *Let $\mathcal{R}$ be a regular right cell and M the cokernel of $L_{\mathrm{G}_{\mathcal{R}}}\hookrightarrow\Delta_{\mathcal{R}}$. Then for any composition factor L_{F} of M we have $\mathrm{F}<_R\mathrm{G}_{\mathcal{R}}$ and $\mathrm{F}<_L\mathrm{G}_{\mathcal{R}}$.*

Proof of Proposition 29.. Let $\mathrm{F}\in\mathcal{C}$ be such that L_{F} is a composition factor of M. As $\Delta_{\mathcal{R}}$ is a submodule of $\mathrm{G}_{\mathcal{R}}^*\,L_{\mathrm{G}_{\mathcal{R}}}$ (by Proposition 25), from Lemma 13(a) it follows that $\mathrm{F}\leqslant_R\mathrm{G}_{\mathcal{R}}$.

Consider $\mathcal{I}:=\{\mathrm{H}\in\mathcal{C}:\mathrm{H}\leqslant_R\mathrm{G}_{\mathcal{R}}\}$. Then $\mathcal{I}$ is an ideal with respect to $\leqslant_R$. Assume that $\mathrm{G}_{\mathcal{R}}\in\mathcal{C}_{\mathtt{i},\mathtt{i}}$ and consider the 2-subrepresentation $\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}$ of $\mathbf{P}_{\mathtt{i}}$. Then $\mathrm{F}\in\mathcal{C}_{\mathtt{i},\mathtt{i}}$ and $\Delta_{\mathcal{R}}\in\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}(\mathtt{i})$. Using adjunction, we have:

$$0\neq\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(\mathrm{F}\,P_{\mathbb{1}_{\mathtt{i}}}^{\mathcal{I}},\Delta_{\mathcal{R}})=\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(P_{\mathbb{1}_{\mathtt{i}}}^{\mathcal{I}},\mathrm{F}^*\,\Delta_{\mathcal{R}}).$$

This yields $\mathrm{F}^*\,\Delta_{\mathcal{R}}\neq 0$. The module $\mathrm{F}^*\,\Delta_{\mathcal{R}}$ on the one hand belongs to $\mathbf{P}_{\mathtt{i}}^{\mathcal{I}}(\mathtt{i})$ (by Lemma 13(a)), on the other hand is a quotient of $\mathrm{F}^*\,P_{\mathbb{1}_{\mathtt{i}}}$ (as $\Delta_{\mathcal{R}}$ is a quotient of $P_{\mathbb{1}_{\mathtt{i}}}$ and F^* is exact). The module $\mathrm{F}^*\,P_{\mathbb{1}_{\mathtt{i}}}$ has simple top L_{F^*}. This implies $\mathrm{F}^*\leqslant_R\mathrm{G}_{\mathcal{R}}$ by Lemma 13(a) and thus $\mathrm{F}\leqslant_L\mathrm{G}_{\mathcal{R}}^*\in\mathcal{R}$ (see Proposition 17(e)).

This leaves us with two possibilities: either $\mathrm{F}\not\sim_{LR}\mathrm{G}_{\mathcal{R}}$, in which case we have both $\mathrm{F}<_R\mathrm{G}_{\mathcal{R}}$ and $\mathrm{F}<_L\mathrm{G}_{\mathcal{R}}$, as desired; or $\mathrm{F}\sim_{LR}\mathrm{G}_{\mathcal{R}}$, in which case we have both $\mathrm{F}\sim_L\mathrm{G}_{\mathcal{R}}$ and $\mathrm{F}\sim_R\mathrm{G}_{\mathcal{R}}$ since $\mathcal{R}$ is regular. In the latter case we, however, have $\mathrm{G}_{\mathcal{R}}^*\,L_{\mathrm{F}}\neq 0$ by Lemma 12, which contradicts Proposition 17(ai). This completes the proof. □

A two-sided cell $\mathcal{Q}$ is called *strongly regular* if it is regular and for every left cell $\mathcal{L}$ and right cell $\mathcal{R}$ in $\mathcal{Q}$ we have $|\mathcal{L}\cap\mathcal{R}|=1$. A left (right) cell is *strongly regular* if it is contained in a strongly regular two-sided cell.

Proposition 30 Structure of strongly regular right cells. *Let $\mathcal{R}$ be a strongly regular right cell. Then we have:*

(a) $\mathrm{G}_{\mathcal{R}}\cong\mathrm{G}_{\mathcal{R}}^*$.

(b) *If $\mathrm{F}\in\mathcal{R}$ satisfies $\mathrm{F}\cong\mathrm{F}^*$, then $\mathrm{F}=\mathrm{G}_{\mathcal{R}}$.*

(c) *If $\mathrm{F}\in\mathcal{R}$ and $\mathrm{G}\sim_L\mathrm{F}$ is such that $\mathrm{G}\cong\mathrm{G}^*$, then $\mathrm{G}\,L_{\mathrm{F}}\neq 0$ and every simple occurring both in the top and in the socle of $\mathrm{G}\,L_{\mathrm{F}}$ is isomorphic to L_{F}.*

Proof. Claim (a) follows from the strong regularity of $\mathcal{R}$ and Proposition 17(e). Claim (b) follows directly from the strong regularity of $\mathcal{R}$.

Let us prove claim (c). That $\mathrm{G}\,L_{\mathrm{F}}\neq 0$ follows from Lemma 12. If some L_{H} occurs in the top of $\mathrm{G}\,L_{\mathrm{F}}\neq 0$ then, using adjunction and $\mathrm{G}\cong\mathrm{G}^*$, we get $\mathrm{G}\,L_{\mathrm{H}}\neq 0$. The latter implies $\mathrm{G}\sim_L\mathrm{H}$

by Lemma 12. At the same time $\mathrm{H} \sim_R \mathrm{F}$ by Corollary 14. Hence $\mathrm{H} = \mathrm{F}$ because of the strong regularity of $\mathcal{R}$. This completes the proof. $\square$

5. The 2-category of a two-sided cell

5.1 The quotient associated with a two-sided cell

Let $\mathcal{Q}$ be a two-sided cell in $\mathcal{C}$. Denote by $\mathscr{I}_{\mathcal{Q}}$ the 2-ideal of $\mathscr{C}$ generated by F and id_{F} for all $\mathrm{F} \not\leq_{LR} \mathcal{Q}$. In other words, for every $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ we have that $\mathscr{I}_{\mathcal{Q}}(\mathtt{i}, \mathtt{j})$ is the ideal of $\mathscr{C}(\mathtt{i}, \mathtt{j})$ consisting of all 2-morphisms which factor through a direct sum of 1-morphisms of the form F, where $\mathrm{F} \not\leq_{LR} \mathcal{Q}$. Taking the quotient we obtain the 2-category $\mathscr{C}/\mathscr{I}_{\mathcal{Q}}$.

Lemma 31. *Let $\mathcal{R} \subset \mathcal{Q}$ be a right cell. Then $\mathscr{I}_{\mathcal{Q}}$ annihilates the cell 2-representation $\mathbf{C}_{\mathcal{R}}$. In particular, $\mathbf{C}_{\mathcal{R}}$ carries the natural structure of a 2-representation of $\mathscr{C}/\mathscr{I}_{\mathcal{Q}}$.*

Proof. This follows from the construction and Lemma 12. $\square$

The construction of $\mathscr{C}/\mathscr{I}_{\mathcal{Q}}$ is analogous to constructions from [Be, Os].

5.2 The 2-category associated with $\mathcal{Q}$

Denote by $\mathscr{C}_{\mathcal{Q}}$ the full 2-subcategory of $\mathscr{C}/\mathscr{I}_{\mathcal{Q}}$, closed under isomorphisms, generated by the identity morphisms $\mathbb{1}_{\mathtt{i}}$, $\mathtt{i} \in \mathscr{C}$, and $\mathrm{F} \in \mathcal{Q}$. We will call $\mathscr{C}_{\mathcal{Q}}$ the 2-*category* associated to $\mathcal{Q}$. This category is especially good in the case of a strongly regular $\mathcal{Q}$, as follows from the following statement:

Proposition 32. *Assume $\mathcal{Q}$ is a strongly regular two-sided cell in $\mathcal{C}$. Then $\mathcal{Q}$ remains a two-sided cell for $\mathscr{C}_{\mathcal{Q}}$.*

Proof. Let $\mathrm{F} \in \mathcal{Q}$. Denote by G the unique self-adjoint element in the right cell $\mathcal{R}$ of F. The action of G on the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ is nonzero and hence $\mathrm{G} \neq 0$, when restricted to $\mathbf{C}_{\mathcal{R}}$.

Further, by Proposition 17(b), $\mathrm{G}\, L_{\mathrm{G}}$ has simple top L_{G}. Using Proposition 17(b) again, we thus get $\mathrm{F} \circ \mathrm{G}\, L_{\mathrm{G}} \neq 0$, implying $\mathrm{F} \circ \mathrm{G} \neq 0$, when restricted to $\mathbf{C}_{\mathcal{R}}$. But the restriction of $\mathrm{F} \circ \mathrm{G}$ decomposes into a direct sum of some $\mathrm{H} \in \mathcal{Q}$, which are in the same right cell as G and in the same left cell as F. Since $\mathcal{Q}$ is strongly regular, the only element satisfying both conditions is F. This implies that, when restricted to $\mathbf{C}_{\mathcal{R}}$, F occurs as a direct summand of $\mathrm{F} \circ \mathrm{G}$, which yields $\mathrm{F} \geqslant_R \mathrm{G}$ in $\mathscr{C}_{\mathcal{Q}}$.

Now consider the functor $\mathrm{F}^* \circ \mathrm{F}$. Since $\mathrm{F} \neq 0$, when restricted to $\mathbf{C}_{\mathcal{R}}$, by adjunction we have $\mathrm{F}^* \circ \mathrm{F} \neq 0$, when restricted to $\mathbf{C}_{\mathcal{R}}$, as well. The functor $\mathrm{F}^* \circ \mathrm{F}$ decomposes into a direct sum of functors from $\mathcal{R} \cap \mathcal{R}^* = \{\mathrm{G}\}$. This implies $\mathrm{G} \geqslant_R \mathrm{F}$ in $\mathscr{C}_{\mathcal{Q}}$ and hence $\mathcal{R}$ remains a right cell in $\mathscr{C}_{\mathcal{Q}}$. Using $*$ we get that all left cells in $\mathcal{Q}$ remain left cells in $\mathscr{C}_{\mathcal{Q}}$. Now the claim of the proposition follows from Proposition 28(b). $\square$

The important property of $\mathscr{C}_{\mathcal{Q}}$ is that for strongly regular right cells the corresponding cell 2-representations can be studied over $\mathscr{C}_{\mathcal{Q}}$:

Corollary 33. *Let $\mathcal{Q}$ be a strongly regular two-sided cell of $\mathcal{C}$ and $\mathcal{R}$ be a right cell of $\mathcal{Q}$. Then the restriction of the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ from $\mathscr{C}$ to $\mathscr{C}_{\mathcal{Q}}$ gives the corresponding cell 2-representation for $\mathscr{C}_{\mathcal{Q}}$.*

Proof. Let $\mathtt{i} \in \mathcal{C}$ be such that $\mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{i}} \neq \varnothing$. Denote by $\mathbf{C}_{\mathcal{R}}^{\mathcal{Q}}$ the cell 2-representation of $\mathscr{C}_{\mathcal{Q}}$ associated to $\mathcal{R}$. We will use the upper index $\mathcal{Q}$ for elements of this 2-representation. Consider

$\mathbf{C}_{\mathcal{R}}$ as a 2-representation of $\mathscr{C}_{\mathcal{Q}}$ by restriction. By Theorem 24, we have the morphism of 2-representations $\Psi := \Psi^{L_{\mathrm{G}_{\mathcal{R}}}} : \mathbf{C}_{\mathcal{R}}^{\mathcal{Q}} \to \mathbf{C}_{\mathcal{R}}$ sending $L^{\mathcal{Q}}_{\mathrm{G}_{\mathcal{R}}}$ to $L_{\mathrm{G}_{\mathcal{R}}}$.

Let $\mathtt{j} \in \mathcal{C}$ and $\mathrm{F} \in \mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}$. By Proposition 17(b), the morphism Ψ sends the indecomposable projective module $\mathsf{P}^{\mathcal{Q}}_{\mathrm{F}}$ of $\mathbf{C}^{\mathcal{Q}}_{\mathcal{R}}(\mathtt{j})$ to the indecomposable projective module P_{F} in $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$. As mentioned after Proposition 17, we have that 2-morphisms in $\mathscr{C}_{\mathcal{Q}}$ surject onto the homomorphisms between indecomposable projective modules both in $\mathbf{C}^{\mathcal{Q}}_{\mathcal{R}}(\mathtt{j})$ and $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$.

To prove the claim it is left to show that Ψ is injective, when restricted to indecomposable projective modules in $\mathbf{C}^{\mathcal{Q}}_{\mathcal{R}}(\mathtt{j})$. For this it is enough to show that the Cartan matrices of $\mathbf{C}^{\mathcal{Q}}_{\mathcal{R}}(\mathtt{j})$ and $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ coincide. For indecomposable F and H in $\mathcal{R} \cap \mathcal{C}_{\mathtt{i},\mathtt{j}}$, using adjunction, we have

$$\mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}}, \mathrm{H}\, L_{\mathrm{G}_{\mathcal{R}}}) = \mathrm{Hom}_{\overline{\mathscr{C}}(\mathtt{i},\mathtt{i})}(\mathrm{H}^* \circ \mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}}, L_{\mathrm{G}_{\mathcal{R}}}). \tag{4}$$

and similarly for $\mathbf{C}^{\mathcal{Q}}_{\mathcal{R}}$. The dimension of the right hand side of (4) equals the multiplicity of $\mathrm{G}_{\mathcal{R}}$ as a direct summand of $\mathrm{H}^* \circ \mathrm{F}$. Since this multiplicity is the same for $\mathbf{C}^{\mathcal{Q}}_{\mathcal{R}}$ and $\mathbf{C}_{\mathcal{R}}$, the claim follows. $\square$

5.3 Cell 2-representations for strongly regular cells

In this section we fix a strongly regular two-sided cell $\mathcal{Q}$ in $\mathcal{C}$. We would like to understand combinatorics of the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ for a right cell $\mathcal{R} \subset \mathcal{Q}$. By the previous subsection, for this it is enough to assume that $\mathscr{C} = \mathscr{C}_{\mathcal{Q}}$. We work under this assumption in the rest of this subsection and consider the direct sum $\mathbf{C}$ of all $\mathbf{C}_{\mathcal{R}}$, where $\mathcal{R}$ runs through the set of all right cells in $\mathcal{Q}$. To simplify our notation, by $\mathrm{Hom}_{\mathbf{C}}$ we denote the homomorphism space in an appropriate module category $\mathbf{C}(\mathtt{i})$.

Proposition 34. *Let $\mathcal{Q}$ be as above and* $\mathrm{F}, \mathrm{H} \in \mathcal{Q}$.

(a) *For some $m_{\mathrm{F},\mathrm{H}} \in \{0,1,2,\dots\}$ we have $\mathrm{H}^* \circ \mathrm{F} \cong m_{\mathrm{F},\mathrm{H}}\mathrm{G}$, where $\{\mathrm{G}\} = \mathcal{L}_{\mathrm{H}^*} \cap \mathcal{R}_{\mathrm{F}}$; moreover, $m_{\mathrm{F},\mathrm{F}} \neq 0$.*

(b) *If $\mathrm{F} \sim_R \mathrm{H}$, then $m_{\mathrm{F},\mathrm{H}} = m_{\mathrm{H},\mathrm{F}}$.*

(c) *If $\mathrm{H} = \mathrm{H}^*$ and $\mathrm{F} \sim_R \mathrm{H}$, then $m_{\mathrm{F},\mathrm{F}} = \dim \mathrm{End}_{\mathbf{C}}(\mathrm{F}\, L_{\mathrm{H}})$.*

(d) *If $\mathrm{H} = \mathrm{H}^*$ and $\mathrm{F} \sim_R \mathrm{H}$, then $\mathrm{F} \circ \mathrm{H} \cong m_{\mathrm{H},\mathrm{H}}\mathrm{F}$ and $\mathrm{H} \circ \mathrm{F}^* \cong m_{\mathrm{H},\mathrm{H}}\mathrm{F}^*$*

(e) *If $\mathrm{H} = \mathrm{H}^*$ and $\mathrm{H} \sim_L \mathrm{F}$, then $m_{\mathrm{H},\mathrm{H}} = \dim \mathrm{Hom}_{\mathbf{C}}(\mathsf{P}_{\mathrm{F}}, \mathrm{H}\, L_{\mathrm{F}})$.*

(f) *Assume $\mathrm{G} \in \mathcal{Q}$ and $\mathrm{H} = \mathrm{H}^*$, $\mathrm{G} = \mathrm{G}^*$, $\mathrm{H} \sim_L \mathrm{F}$ and $\mathrm{G} \sim_R \mathrm{F}$. Then*

$$m_{\mathrm{F},\mathrm{F}} m_{\mathrm{G},\mathrm{G}} = m_{\mathrm{F}^*,\mathrm{F}^*} m_{\mathrm{H},\mathrm{H}}.$$

Proof. By our assumptions, every indecomposable direct summand of $\mathrm{H}^* \circ \mathrm{F}$ belongs to the right cell of F and the left cell of H^*, hence is isomorphic to G. Note that $\mathrm{F}^* \circ \mathrm{F}$ is nonzero by adjunction since $\mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}_{\mathrm{F}}}} \neq 0$. This implies claim (a) and claim (c) follows from Proposition 17(b) using adjunction.

If $\mathrm{F} \sim_R \mathrm{H}$, then $\mathrm{H}^* \circ \mathrm{F} \cong m_{\mathrm{F},\mathrm{H}}\mathrm{G}_{\mathcal{R}_{\mathrm{F}}}$ by (a). By Proposition 30(a), the functor $\mathrm{G}_{\mathcal{R}_{\mathrm{F}}}$ is self-adjoint. Hence $\mathrm{H}^* \circ \mathrm{F}$ is self-adjoint, which implies claim (b).

Set $m = m_{\mathrm{H},\mathrm{H}}$. Similarly to the proof of claim (a), we have $\mathrm{F} \circ \mathrm{H} \cong k\mathrm{F}$ for some $k \in \{1,2,\dots\}$. Using associativity, we obtain

$$\begin{aligned} k^2\mathrm{F} = k(\mathrm{F} \circ \mathrm{H}) = (k\mathrm{F}) \circ \mathrm{H} = (\mathrm{F} \circ \mathrm{H}) \circ \mathrm{H} = \\ = \mathrm{F} \circ (\mathrm{H} \circ \mathrm{H}) = \mathrm{F} \circ (m\mathrm{H}) = m(\mathrm{F} \circ \mathrm{H}) = mk\mathrm{F}. \end{aligned}$$

This implies claim (d) and claim (e) follows by adjunction.

Claim (f) follows from the following computation:

$$m_{\mathrm{F},\mathrm{F}}m_{\mathrm{G},\mathrm{G}}\mathrm{F} \overset{(\mathrm{d})}{=} m_{\mathrm{F},\mathrm{F}}(\mathrm{F}\circ\mathrm{G}) = \mathrm{F}\circ(m_{\mathrm{F},\mathrm{F}}\mathrm{G}) \overset{(\mathrm{a})}{=} \mathrm{F}\circ(\mathrm{F}^*\circ\mathrm{F}) =$$
$$= (\mathrm{F}\circ\mathrm{F}^*)\circ\mathrm{F} \overset{(\mathrm{a})}{=} (m_{\mathrm{F}^*,\mathrm{F}^*}\mathrm{H})\circ\mathrm{F} = m_{\mathrm{F}^*,\mathrm{F}^*}(\mathrm{H}\circ\mathrm{F}) \overset{(\mathrm{d})}{=} m_{\mathrm{F}^*,\mathrm{F}^*}m_{\mathrm{H},\mathrm{H}}\mathrm{F}.$$

This completes the proof. □

As a corollary we obtain that the Cartan matrix of the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ is symmetric.

Corollary 35. *Assume that $\mathcal{R}$ is a strongly regular right cell. Then for any $\mathrm{F},\mathrm{H}\in\mathcal{R}$ we have $[\mathsf{P}_{\mathrm{F}} : L_{\mathrm{H}}] = [\mathsf{P}_{\mathrm{H}} : L_{\mathrm{F}}]$.*

Proof. We have $[\mathsf{P}_{\mathrm{F}} : L_{\mathrm{H}}] = \dim \mathrm{Hom}_{\mathbf{C}_{\mathcal{R}}}(\mathsf{P}_{\mathrm{H}},\mathsf{P}_{\mathrm{F}})$. Using adjunction and Proposition 34(a), the latter equals $m_{\mathrm{H},\mathrm{F}}$. Now the claim follows from Proposition 34(b). □

Corollary 36. *Let $\mathrm{F},\mathrm{H}\in\mathcal{Q}$ be such that $\mathrm{H}=\mathrm{H}^*$ and $\mathrm{F}\sim_L\mathrm{H}$. Then the module P_{F} is a direct summand of $\mathrm{H}\,L_{\mathrm{F}}$ and $m_{\mathrm{F},\mathrm{F}}\leqslant m_{\mathrm{H},\mathrm{H}}$.*

Proof. From Proposition 34 we have:

$$m_{\mathrm{F},\mathrm{F}} = \dim \mathrm{End}_{\mathbf{C}}(\mathsf{P}_{\mathrm{F}}),\quad m_{\mathrm{H},\mathrm{H}} = \dim \mathrm{Hom}_{\mathbf{C}}(\mathsf{P}_{\mathrm{F}},\mathrm{H}\,L_{\mathrm{F}}).$$

Hence to prove the corollary we just need to show that P_{F} is a direct summand of $\mathrm{H}\,L_{\mathrm{F}}$.

By Proposition 34, the module $\mathrm{F}\circ\mathrm{F}^*\,L_{\mathrm{F}}$ decomposes into a direct sum of $m_{\mathrm{F}^*,\mathrm{F}^*}$ copies of the module $\mathrm{H}\,L_{\mathrm{F}}$. Hence it is enough to show that P_{F} is a direct summand of $\mathrm{F}\circ\mathrm{F}^*\,L_{\mathrm{F}}$.

Let $\mathcal{R}$ be the right cell of F. We know that $\mathrm{F}^*\,L_{\mathrm{F}}\neq 0$. Using adjunction and Lemma 12, we obtain that every simple quotient of $\mathrm{F}^*\,L_{\mathrm{F}}\neq 0$ is isomorphic to $L_{\mathrm{G}_{\mathcal{R}}}$. Hence $\mathrm{F}^*\,L_{\mathrm{F}}$ surjects onto $L_{\mathrm{G}_{\mathcal{R}}}$ and, applying F, we have that $\mathrm{F}\circ\mathrm{F}^*\,L_{\mathrm{F}}$ surjects onto P_{F}. Now the claim follows from projectivity of P_{F}. □

Corollary 37. *For every $\mathrm{F}\in\mathcal{Q}$ the projective module P_{F} is injective.*

Proof. Let $\mathcal{R}$ be the right cell of F. Since the functorial actions of F and F^* on $\mathbf{C}_{\mathcal{R}}$ are biadjoint, they preserve both the additive category of projective modules and the additive category of injective modules. Now take any injective module I and let L_{H} be some simple occurring in its top. Applying H^* we get an injective module such that $L_{\mathrm{G}_{\mathcal{R}}}$ occurs in its top. Applying now F we get an injective module in which the projective module $\mathsf{P}_{\mathrm{F}}\cong\mathrm{F}\,L_{\mathrm{G}_{\mathcal{R}}}$ is a quotient. Hence P_{F} splits off as a direct summand in this module and thus is injective. This completes the proof. □

Corollary 38. *Let $\mathrm{F},\mathrm{H}\in\mathcal{Q}$ and $\mathcal{R}$ be the right cell of F.*

(a) *We have $\mathrm{F}^*\,L_{\mathrm{F}}\cong\mathsf{P}_{\mathrm{G}_{\mathcal{R}}}$.*

(b) *The module $\mathrm{H}\,L_{\mathrm{F}}$ is either zero or both projective and injective.*

Proof. Similarly to the proof of Corollary 36 one shows that the module $P_{\mathrm{G}_{\mathcal{R}}}$ is a direct summand of $\mathrm{F}^*\,L_{\mathrm{F}}$, so to prove claim (a) we have to show that $\mathrm{F}^*\,L_{\mathrm{F}}$ is indecomposable. We will show that $\mathrm{F}^*\,L_{\mathrm{F}}$ has simple socle. Since F annihilates all simple modules in $\mathbf{C}_{\mathcal{R}}$ but $L_{\mathrm{G}_{\mathcal{R}}}$, using adjunction it follows that every simple submodule in the socle of $\mathrm{F}^*\,L_{\mathrm{F}}$ is isomorphic to $L_{\mathrm{G}_{\mathcal{R}}}$. On the other hand, using adjunction and Proposition 17(b) we obtain that the homomorphism space from $L_{\mathrm{G}_{\mathcal{R}}}$ to $\mathrm{F}^*\,L_{\mathrm{F}}$ is one-dimensional. This means that $\mathrm{F}^*\,L_{\mathrm{F}}$ has simple socle and proves claim (a).

Assume that $\mathrm{H}\,L_{\mathrm{F}}\neq 0$. Then, by Lemma 12, we have $\mathrm{F}^*\sim_R\mathrm{H}$ (since $\mathcal{Q}$ is strongly regular). Let $\mathrm{G}\in\mathcal{R}$ be such that $\mathrm{G}\sim_L\mathrm{H}$. Then, by Proposition 34(a), we have $\mathrm{G}\circ\mathrm{F}^*\cong m_{\mathrm{F}^*,\mathrm{G}^*}\mathrm{H}$. So, to prove claim (b) it is enough to show that $m_{\mathrm{F}^*,\mathrm{G}^*} \neq 0$ and that $\mathrm{G} \circ \mathrm{F}^* \, L_{\mathrm{F}}$ is both projective and injective. By claim (a), we have $\mathrm{F}^* \, L_{\mathrm{F}} \cong P_{\mathrm{G}_{\mathcal{R}}}$. Since $\mathrm{G} \, L_{\mathrm{G}_{\mathcal{R}}} \neq 0$ by Proposition 17(b) and G is exact, it follows that $\mathrm{G} \circ \mathrm{F}^* \, L_{\mathrm{F}} \neq 0$ and hence $m_{\mathrm{F}^*,\mathrm{G}^*} \neq 0$. Further, $\mathrm{G} \, P_{\mathrm{G}_{\mathcal{R}}}$ is projective as $P_{\mathrm{G}_{\mathcal{R}}}$ is projective and G is biadjoint to G^*. Finally, $\mathrm{G} \, P_{\mathrm{G}_{\mathcal{R}}}$ is injective by Corollary 37. Claim (b) follows and the proof is complete. □

Corollary 39. *Let* $\mathrm{F}, \mathrm{H} \in \mathcal{Q}$ *be such that* $\mathrm{H} = \mathrm{H}^*$ *and* $\mathrm{F} \sim_L \mathrm{H}$. *Then* $m_{\mathrm{F},\mathrm{F}} | m_{\mathrm{H},\mathrm{H}}$.

Proof. Let $\mathcal{R}$ be the right cell of F. By Lemma 12, H annihilates all simples of $\mathbf{C}_{\mathcal{R}}$ but L_{F}. This and Corollary 38(b) imply that $\mathrm{H} \, L_{\mathrm{F}} = k \mathsf{P}_{\mathrm{F}}$ for some $k \in \mathbb{N}$. On the one hand, using Propositions 17 and 34 we have

$$(\mathrm{F}^* \circ \mathrm{H}) \, L_{\mathrm{F}} = k \mathrm{F}^* \mathsf{P}_{\mathrm{F}} = k(\mathrm{F}^* \circ \mathrm{F}) L_{\mathrm{G}_{\mathcal{R}}} = k m_{\mathrm{F},\mathrm{F}} \mathrm{G}_{\mathcal{R}} \, L_{\mathrm{G}_{\mathcal{R}}} = k m_{\mathrm{F},\mathrm{F}} \, \mathsf{P}_{\mathrm{G}_{\mathcal{R}}}. \tag{5}$$

On the other hand, we have $\mathrm{F}^* \sim_R \mathrm{H}$ and thus, using Proposition 34(d) and Corollary 38(a), we have:

$$(\mathrm{F}^* \circ \mathrm{H}) \, L_{\mathrm{F}} = m_{\mathrm{H},\mathrm{H}} \mathrm{F}^* \, L_{\mathrm{F}} = m_{\mathrm{H},\mathrm{H}} \mathsf{P}_{\mathrm{G}_{\mathcal{R}}}. \tag{6}$$

The claim follows comparing (5) and (6). □

6. Cyclic and simple 2-representations of fiat categories

6.1 Cyclic 2-representations

Let $\mathscr{C}$ be a fiat category, $\mathbf{M}$ a 2-representation of $\mathscr{C}$, $\mathtt{i} \in \mathscr{C}$ and $M \in \mathbf{M}(\mathtt{i})$. We will say that M *generates* $\mathbf{M}$ if for any $\mathtt{j} \in \mathscr{C}$ and $X, Y \in \mathbf{M}(\mathtt{j})$ there are $\mathrm{F}, \mathrm{G} \in \hat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$ such that $\mathrm{F} \, M \cong X$, $\mathrm{G} \, M \cong Y$ and the evaluation map $\mathrm{Hom}_{\hat{\mathscr{C}}(\mathtt{i},\mathtt{j})}(\mathrm{F}, \mathrm{G}) \to \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(\mathrm{F} \, M, \mathrm{G} \, M)$ is surjective. The 2-representation $\mathbf{M}$ is called *cyclic* provided that there exists $\mathtt{i} \in \mathscr{C}$ and $M \in \mathbf{M}(\mathtt{i})$ such that M generates $\mathbf{M}$. Examples of cyclic 2-representations of $\mathscr{C}$ are given by the following:

Proposition 40. (*a*) *For any* $\mathtt{i} \in \mathscr{C}$ *the* 2*-representation* $\mathbf{P}_{\mathtt{i}}$ *is cyclic and generated by* $P_{\mathbb{1}_{\mathtt{i}}}$.
(*b*) *For any right cell* $\mathcal{R}$ *of* $\mathcal{C}$ *the cell* 2*-representation* $\mathbf{C}_{\mathcal{R}}$ *is cyclic and generated by* $L_{\mathrm{G}_{\mathcal{R}}}$.

Proof. Let $\mathtt{j} \in \mathscr{C}$, $X, Y \in \mathbf{P}_{\mathtt{i}}(\mathtt{j})$ and $f : X \to Y$. Taking some projective presentations of X and Y yields the following commutative diagram with exact rows:

$$\begin{tikzcd} X_1 \arrow[r, "h"] \arrow[d, "f''"'] & X_0 \arrow[r, two heads] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y_1 \arrow[r, "g"] & Y_0 \arrow[r, two heads] & Y \end{tikzcd} \tag{7}$$

Now X_1, X_0, Y_1, Y_0 are projective in $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$ and we may assume that $X_1 = \mathrm{F}_1 \, P_{\mathbb{1}_{\mathtt{i}}}$, $X_0 = \mathrm{F}_0 \, P_{\mathbb{1}_{\mathtt{i}}}$, $Y_1 = \mathrm{G}_1 \, P_{\mathbb{1}_{\mathtt{i}}}$ and $Y_0 = \mathrm{G}_0 \, P_{\mathbb{1}_{\mathtt{i}}}$ for some $\mathrm{F}_1, \mathrm{F}_0, \mathrm{G}_1, \mathrm{G}_0 \in \mathscr{C}(\mathtt{i}, \mathtt{j})$. From the definition of $\mathbf{P}_{\mathtt{i}}$ we then obtain that g, h, f' and f'' are given by 2-morphisms between the corresponding 1-morphisms (which we denote by the same symbols).

It follows that X equals to the image of $P_{\mathbb{1}_{\mathtt{i}}}$ under $\mathrm{H}_1 := \mathrm{Coker}(\mathrm{F}_1 \xrightarrow{h} \mathrm{F}_0) \in \hat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$. Similarly, Y equals to the image of $P_{\mathbb{1}_{\mathtt{i}}}$ under $\mathrm{H}_2 := \mathrm{Coker}(\mathrm{G}_1 \xrightarrow{g} \mathrm{G}_0) \in \hat{\mathscr{C}}(\mathtt{i}, \mathtt{j})$. Finally, f is induced by the diagram

$$\begin{tikzcd} \mathrm{F}_1 \arrow[r, "h"] \arrow[d, "f''"'] & \mathrm{F}_0 \arrow[d, "f'"] \\ \mathrm{G}_1 \arrow[r, "g"] & \mathrm{G}_0. \end{tikzcd}$$

Claim (a) follows.

To prove claim (b) we view every $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ as the corresponding full subcategory of $\mathbf{P}_{\mathtt{i}}(\mathtt{j})$. Let $X, Y \in \mathbf{C}_{\mathcal{R}}(\mathtt{j})$ and $f : X \to Y$. From the proof of claim (a) we have the commutative diagram (7) as described above. Our proof of claim (b) will proceed by certain manipulations of this diagram. Denote by $\mathcal{I}$ the ideal of $\mathcal{C}$ with respect to $\leqslant_R$ generated by $\mathcal{R}$ and set $\mathcal{I}' := \mathcal{I} \setminus \mathcal{R}$.

To start with, we modify the left column of (7). Let X_1' and Y_1' denote the trace of all projective modules of the form P_{G}, $\mathrm{G} \notin \mathcal{I}'$, in X_1 and Y_1, respectively. Consider some minimal projective covers $\hat{X}_1 \twoheadrightarrow X_1'$ and $\hat{Y}_1 \twoheadrightarrow X_1'$ of X_1' and Y_1', respectively. Let $\mathrm{can} : \hat{X}_1 \twoheadrightarrow X_1' \hookrightarrow X_1$ and $\mathrm{can}' : \hat{Y}_1 \twoheadrightarrow Y_1' \hookrightarrow Y_1$ denote the corresponding canonical maps and set $\hat{h} = h \circ \mathrm{can}$ and $\hat{g} = g \circ \mathrm{can}'$. Then the cokernel of both can and can' has only composition factors of the form L_{F}, $\mathrm{F} \in \mathcal{I}'$. By construction, the image of $f'' \circ \mathrm{can}$ is contained in the image of can'. Hence, using projectivity of $\hat{X}_1$, the map f'' lifts to a map $\hat{f}'' : \hat{X}_1 \to \hat{Y}_1$ such that the following diagram commutes:

$$
\begin{tikzcd}
\hat{X}_1 \arrow[r, dotted, "\mathrm{can}"'] \arrow[d, "\hat{f}''"'] \arrow[rr, bend left, "\hat{h}"] & X_1 \arrow[r, dotted, "h"'] \arrow[d, dotted, "f''"] & X_0 \arrow[r, two heads] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\hat{Y}_1 \arrow[r, dotted, "\mathrm{can}'"] \arrow[rr, bend right, "\hat{g}"'] & Y_1 \arrow[r, dotted, "g"] & Y_0 \arrow[r, two heads] & Y
\end{tikzcd}
\tag{8}
$$

The difference between (7) and (8) is that the rows of the solid part of (8) are no longer exact but might have homology in the middle. By construction, all simple subquotients of these homologies have the form L_{F}, $\mathrm{F} \in \mathcal{I}'$. Further, all projective direct summands appearing in (8) have the form P_{F} for $\mathrm{F} \notin \mathcal{I}'$.

Denote by $\tilde{X}_1$, $\tilde{X}_0$, $\tilde{Y}_1$ and $\tilde{Y}_0$ the submodules of $\hat{X}_1$, X_0, $\hat{Y}_1$ and Y_0, respectively, which are uniquely defined by the following construction: The corresponding submodules contain all direct summands of the form P_{F} for $\mathrm{F} \notin \mathcal{R}$; and for each direct summand of the form P_{F}, $\mathrm{F} \in \mathcal{R}$, the corresponding submodules contain the submodule $\mathrm{Ker}_{\mathrm{F}}$ of P_{F} as defined in (3). By construction and Lemma 20, we have $\hat{h} : \tilde{X}_1 \to \tilde{X}_0$, $\hat{g} : \tilde{Y}_1 \to \tilde{Y}_0$, $f' : \tilde{X}_0 \to \tilde{Y}_0$ and $\hat{f}'' : \tilde{X}_1 \to \tilde{Y}_1$. Since $X, Y \in \mathbf{C}_{\mathcal{R}}$, the images (on diagram (8)) of $\tilde{X}_0$ and $\tilde{Y}_0$ in X and Y, respectively, are zero. Hence, taking quotients gives the following commutative diagram:

$$
\begin{tikzcd}
\hat{X}_1/\tilde{X}_1 \arrow[r, "\tilde{h}"] \arrow[d, "\tilde{f}''"'] & X_0/\tilde{X}_0 \arrow[r, two heads] \arrow[d, "\tilde{f}'"] & X \arrow[d, "f"] \\
\hat{Y}_1/\tilde{Y}_1 \arrow[r, "\tilde{g}"] & Y_0/\tilde{Y}_0 \arrow[r, two heads] & Y,
\end{tikzcd}
\tag{9}
$$

where $\tilde{h}$, $\tilde{g}$, $\tilde{f}''$ and $\tilde{f}'$ denote the corresponding induced maps.

By our construction of (9) and definition of $\mathbf{C}_{\mathcal{R}}$, all indecomposable modules appearing in the left square of (9) are projective in $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ (and hence, by definition of $\mathbf{C}_{\mathcal{R}}$, have the form $\mathrm{F}\, L_{\mathrm{G}_{\mathcal{R}}}$ for some $\mathrm{F} \in \mathcal{R}$). Moreover, all simples of the form L_{F}, $\mathrm{F} \in \mathcal{I}'$, become zero in $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ (since $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$ is defined as a Serre subquotient and simples L_{F}, $\mathrm{F} \in \mathcal{I}'$, belong to the kernel). This implies that both rows of (9) are exact in $\mathbf{C}_{\mathcal{R}}(\mathtt{j})$. As mentioned in the proof of claim (a), the maps g, h, f' and f'' on diagram (7) are given by 2-morphisms in $\mathscr{C}$. Similarly, the maps $\hat{g}$, $\hat{h}$ and $\hat{f}''$ on diagram (8) are given by 2-morphisms in $\mathscr{C}$ as well. By construction of (9), the maps $\tilde{h}$, $\tilde{g}$, $\tilde{f}''$ and $\tilde{f}'$ are induced by $\hat{h}$, $\hat{g}$, $\hat{f}''$ and f', respectively. Now the proof of claim (b) is completed similarly to the proof of claim (a). □

6.2 Simple 2-representations

A (nontrivial) 2-representation $\mathbf{M}$ of $\mathscr{C}$ is called *quasi-simple* provided that it is cyclic and generated by a simple module. From Proposition 40(b) it follows that every cell 2-representation is quasi-simple. A (nontrivial) 2-representation $\mathbf{M}$ of $\mathscr{C}$ is called *strongly simple* provided that it is cyclic and generated by any simple module. It turns out that for strongly regular right cells strong simplicity of cell 2-representations behaves well with respect to restrictions.

PROPOSITION 41. *Let $\mathcal{Q}$ be a strongly regular two-sided cell and $\mathcal{R}$ a right cell in $\mathcal{Q}$. Then the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ of $\mathscr{C}$ is strongly simple if and only if its restriction to $\mathscr{C}_{\mathcal{Q}}$ is strongly simple.*

To prove this we will need the following general lemma:

LEMMA 42. *Let $\mathcal{Q}$ be two-sided cell and $\mathbf{M}$ a 2-representations of $\mathscr{C}$. Let $\mathrm{H} \in \hat{\mathscr{C}}$ be such that for any $\mathrm{F} \in \mathcal{C}$ the inequality $\mathrm{Hom}_{\hat{\mathscr{C}}}(\mathrm{F}, \mathrm{H}) \neq 0$ implies $\mathrm{F} <_{LR} \mathcal{Q}$. Then for any $\mathrm{G} \in \mathcal{Q}$ the functor $\mathbf{M}(\mathrm{G})$ annihilates the image of $\mathbf{M}(\mathrm{H})$.*

Proof. Let $\mathrm{H} = \mathrm{Coker}(\alpha)$, where $\alpha : \mathrm{H}' \to \mathrm{H}''$ is a 2-morphism in $\mathscr{C}$. From Lemma 12, applied to an appropriate $\mathbf{P}_{\mathtt{i}}$, it follows that $\mathrm{G}(\alpha)$ is surjective. This implies that $\mathrm{G} \circ \mathrm{H} = 0$ and yields the claim. □

Proof of Proposition 41.. Let $\mathrm{H} \in \mathcal{C}$ be such that $\mathrm{H} <_{LR} \mathcal{Q}$. Then there exists a 1-morphism F in $\mathscr{C}$ and a 2-morphism $\alpha : \mathrm{F} \to \mathrm{H}$ in $\mathscr{C}$ such that every indecomposable direct summand of $\mathbf{C}_{\mathcal{R}}(\mathrm{F})$ has the form $\mathbf{C}_{\mathcal{R}}(\mathrm{G})$ for some $\mathrm{G} \in \mathcal{Q}$ and the cokernel of $\mathbf{C}_{\mathcal{R}}(\alpha)$ satisfies the condition that for any $\mathrm{K} \in \mathcal{C}$ the existence of a nonzero homomorphism from $\mathbf{C}_{\mathcal{R}}(\mathrm{K})$ to $\mathrm{Coker}(\mathbf{C}_{\mathcal{R}}(\alpha))$ implies $\mathrm{K} <_{LR} \mathcal{Q}$. By Lemma 42, every 1-morphism in $\mathcal{Q}$ annihilates the image of $\mathrm{Coker}(\mathbf{C}_{\mathcal{R}}(\alpha))$. Since every simple in $\mathbf{C}_{\mathcal{R}}$ is not annihilated by some 1-morphism in $\mathcal{Q}$, we have that the image of $\mathrm{Coker}(\mathbf{C}_{\mathcal{R}}(\alpha))$ is zero and hence $\mathrm{Coker}(\mathbf{C}_{\mathcal{R}}(\alpha))$ is the zero functor. This means that $\mathbf{C}_{\mathcal{R}}(\mathrm{H})$ is a quotient of $\mathbf{C}_{\mathcal{R}}(\mathrm{F})$, which implies the claim. □

The following theorem is our main result (and a proper formulation of Theorem 1 from Section 1).

THEOREM 43 Strong simplicity of cell 2-representations. *Let $\mathcal{Q}$ be a strongly regular two-sided cell. Assume that*

$$\text{the function} \quad \begin{array}{ccc} \mathcal{Q} & \longrightarrow & \mathbb{N}_0 \\ \mathrm{F} & \mapsto & m_{\mathrm{F},\mathrm{F}} \end{array} \quad \text{is constant on left cells of } \mathcal{Q}. \tag{10}$$

Then we have:

(a) *For any right cell $\mathcal{R}$ in $\mathcal{Q}$ the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ is strongly simple.*

(b) *If $\mathcal{R}$ and $\mathcal{R}'$ are two right cells in $\mathcal{Q}$, then the cell 2-representations $\mathbf{C}_{\mathcal{R}}$ and $\mathbf{C}_{\mathcal{R}'}$ are equivalent.*

Proof. Let $\mathrm{F}, \mathrm{G} \in \mathcal{R}$ and $\mathrm{H} \in \mathcal{Q}$ be such that $\mathrm{H} \in \mathcal{R}_{\mathrm{F}^*} \cap \mathcal{L}_{\mathrm{G}}$. The module $\mathrm{H}\, L_{\mathrm{F}}$ is nonzero by Lemma 12 and projective by Corollary 38(b). From Proposition 34 and Corollary 38(b) it follows that $\mathrm{H}\, L_{\mathrm{F}} \cong k\mathsf{P}_{\mathrm{G}}$ for some $k \in \mathbb{N}$. Hence, by adjunction,

$$m_{\mathrm{H},\mathrm{H}} = \dim \mathrm{End}_{\mathbf{C}_{\mathcal{R}}}(\mathrm{H}\, L_{\mathrm{F}}) = \dim \mathrm{End}_{\mathbf{C}_{\mathcal{R}}}(k\mathsf{P}_{\mathrm{G}}) = k^2 \dim \mathrm{End}_{\mathbf{C}_{\mathcal{R}}}(\mathsf{P}_{\mathrm{G}}) = k^2 m_{\mathrm{G},\mathrm{G}}.$$

On the other hand, $\mathrm{H} \sim_L \mathrm{G}$ and thus $m_{\mathrm{H},\mathrm{H}} = m_{\mathrm{G},\mathrm{G}}$ by our assumption (10), which implies $k = 1$. This means that every $\mathrm{H} \in \mathcal{R}_{\mathrm{F}^*}$ maps L_{F} to an indecomposable projective module.

To prove (a) it is left to show that 2-morphisms in $\mathscr{C}$ surject onto homomorphisms between indecomposable projective modules. By adjunction, it is enough to show that for any $\mathrm{H}, \mathrm{J} \in \mathcal{R}_{\mathrm{F}^*}$ the space of 2-morphisms from $\mathrm{H}^* \circ \mathrm{J}$ to the identity surjects onto homomorphisms from the projective module $\mathrm{H}^* \circ \mathrm{J}\, L_{\mathrm{F}}$ to L_{F}. For the latter homomorphism space to be nonzero, the functor $\mathrm{H}^* \circ \mathrm{J}$ should decompose into a direct sum of copies of $\mathrm{K} \in \mathcal{R}_{\mathrm{F}^*}$ such that $\mathrm{K} \cong \mathrm{K}^*$ (see Proposition 34(a)). By additivity, it is enough to show that there is a 2-morphism from K to the identity such that its evaluation at L_{F} is nonzero. We have $\mathrm{K}\, L_{\mathrm{F}} \neq 0$ by Lemma 12, which implies that the evaluation at L_{F} of the adjunction morphism from $\mathrm{K} \circ \mathrm{K}$ to the identity is nonzero. We have $\mathrm{K} \circ \mathrm{K} \cong m_{\mathrm{K},\mathrm{K}}\mathrm{K} \neq 0$ by Proposition 34(a). By additivity, the nonzero adjunction morphism restricts to a morphism from one of the summands such that the evaluation at L_{F} remains nonzero. Claim (a) follows.

To prove (b), consider the cell 2-representations $\mathbf{C}_{\mathcal{R}}$ and $\mathbf{C}_{\mathcal{R}'}$. Without loss of generality we may assume that $\mathcal{Q}$ is the unique maximal two-sided cell with respect to $\leqslant_{LR}$. Let $\mathrm{G} := \mathrm{G}_{\mathcal{R}}$ and denote by F the unique element in $\mathcal{R}' \cap \mathcal{L}_{\mathrm{G}}$. Then $\mathrm{G}\, L_{\mathrm{F}} \neq 0$ by Lemma 12. Moreover, from the proof of (a) we know that $\mathrm{G}\, L_{\mathrm{F}}$ is an indecomposable projective module and hence has simple top.

Assume that $\mathtt{i} \in \mathscr{C}$ is such that $\mathrm{G} \in \mathscr{C}(\mathtt{i}, \mathtt{i})$. Let K be a 1-morphism in $\mathscr{C}$ and $\alpha : \mathrm{K} \to \mathrm{G}$ be a 2-morphism such that $\mathbf{P}_{\mathtt{i}}(\alpha)$ is a projective presentation of L_{G}. Denote by $\hat{\mathrm{G}} \in \hat{\mathscr{C}}$ the cokernel of α.

Lemma 44. *The module $\hat{\mathrm{G}}\, L_{\mathrm{F}}$ surjects onto L_{F}.*

Proof. It is enough to prove that $\hat{\mathrm{G}}\, L_{\mathrm{F}} \neq 0$. Since $\mathrm{G}\, L_{\mathrm{F}}$ has simple top, it is enough to show that for any indecomposable 1-morphism M and any 2-morphism $\beta : \mathrm{M} \to \mathrm{G}$ which is not an isomorphism, the morphism $\beta_{L_{\mathrm{F}}}$ is not surjective.

The statement is obvious if $\mathrm{M}\, L_{\mathrm{F}} = 0$. If $\mathrm{M}\, L_{\mathrm{F}} \neq 0$, we have $\mathrm{M} \leqslant_R \mathrm{F}^*$ by Lemma 12. Hence either $\mathrm{M} \sim_R \mathrm{G}$ or $\mathrm{M} <_R \mathrm{G}$. If $\mathrm{M} = \mathrm{G}$, then β is a radical endomorphism of G, hence nilpotent (as $\mathscr{C}$ is a fiat category). This means that $\beta_{L_{\mathrm{F}}}$ is nilpotent and thus is not surjective. If $\mathrm{M} \in \mathcal{R} \setminus \mathrm{G}$, then $\mathrm{M}^* \notin \mathcal{R}$ and hence $\mathrm{M}^* L_{\mathrm{F}} = 0$ by Lemma 12. By adjunction this implies that L_{F} does not occur in the top of $\mathrm{M}\, L_{\mathrm{F}}$, which means that $\beta_{L_{\mathrm{F}}}$ cannot be surjective. This implies the claim for all $\mathrm{M} \in \mathcal{Q}$.

Consider now the remaining case $\mathrm{M} <_R \mathrm{G}$ and assume that $\beta_{L_{\mathrm{F}}}$ is surjective. Let M'' be a 1-morphism and $\gamma : \mathrm{M}'' \to \mathrm{M}$ be a 2-morphism such that γ gives the trace in M of all 1-morphisms J satisfying $\mathrm{J} \not<_R \mathcal{R}$. Denote by M' and G' the cokernels of γ and $\beta \circ_1 \gamma$, respectively. Then both M' and G' are in $\hat{\mathscr{C}}$. Let $\beta' : \mathrm{M}' \to \mathrm{G}'$ be the 2-morphism induced by β. Any direct summand of M'' which does not annihilate L_{F} has the form $\hat{\mathrm{M}}$ for some $\hat{\mathrm{M}} \in \mathcal{Q}$ because of our construction and maximality of $\mathcal{Q}$. Hence from the previous paragraph it follows that the map $\beta'_{L_{\mathrm{F}}}$ is still surjective. On the other hand, because of our construction of M'', an application of Lemma 42 gives $\mathrm{G} \circ \mathrm{M}' = 0$ in $\hat{\mathscr{C}}$. At the same time, the nonzero module $\mathrm{G}'\, L_{\mathrm{F}}$ is a quotient of $\mathrm{G}\, L_{\mathrm{F}}$ and hence has simple top L_{F}. This implies $\mathrm{G} \circ \mathrm{G}'\, L_{\mathrm{F}} \neq 0$. Therefore, applying G to the epimorphism

$$\beta'_{L_{\mathrm{F}}} : \mathrm{M}'\, L_{\mathrm{F}} \twoheadrightarrow \mathrm{G}'\, L_{\mathrm{F}}$$

annihilates the left hand side and does not annihilate the right hand side. This contradicts the right exactness of G and the claim follows. □

By (a), any extension of L_{F} by any other simple in $\mathbf{C}_{\mathcal{R}'}$ comes from some 2-morphism in $\mathscr{C}$. Hence this extension cannot appear in $\hat{\mathrm{G}}\, L_{\mathrm{F}}$ by construction of $\hat{\mathrm{G}}$. This and Lemma 44 imply $\hat{\mathrm{G}}\, L_{\mathrm{F}} \cong L_{\mathrm{F}}$.

Therefore, by Theorem 24, there is a unique homomorphism $\Psi : \mathbf{C}_{\mathcal{R}} \to \mathbf{C}_{\mathcal{R}'}$ of 2-representations, which maps L_{G} to L_{F}. From claim (a) it follows that Ψ maps indecomposable projectives to indecomposable projectives. Restrict Ψ to $\mathscr{C}_{\mathcal{Q}}$. Then from the proof of (a) we have that for any $\mathrm{H}_1, \mathrm{H}_2 \in \mathcal{R}$ we have

$$\dim \mathrm{Hom}_{\mathbf{C}_{\mathcal{R}}}(\mathsf{P}_{\mathrm{H}_1}, \mathsf{P}_{\mathrm{H}_2}) = \dim \mathrm{Hom}_{\mathbf{C}_{\mathcal{R}'}}(\Psi\, \mathsf{P}_{\mathrm{H}_1}, \Psi\, \mathsf{P}_{\mathrm{H}_2}).$$

Moreover, both spaces are isomorphic to $\mathscr{C}_{\mathcal{Q}}(\mathrm{H}_1, \mathrm{H}_2)$. From (a) and construction of Ψ it follows that Ψ induces an isomorphism between $\mathrm{Hom}_{\mathbf{C}_{\mathcal{R}}}(\mathsf{P}_{\mathrm{H}_1}, \mathsf{P}_{\mathrm{H}_2})$ and $\mathrm{Hom}_{\mathbf{C}_{\mathcal{R}'}}(\Psi\, \mathsf{P}_{\mathrm{H}_1}, \Psi\, \mathsf{P}_{\mathrm{H}_2})$. This means that Ψ induces an equivalence between the additive categories of projective modules in $\mathbf{C}_{\mathcal{R}}$ and $\mathbf{C}_{\mathcal{R}'}$. Since Ψ is right exact, this implies that Ψ is an equivalence of categories and completes the proof. $\square$

7. Examples

7.1 Projective functors on the regular block of the category $\mathcal{O}$

Let $\mathfrak{g}$ denote a semi-simple complex finite dimensional Lie algebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathcal{O}_0$ the principal block of the BGG-category $\mathcal{O}$ for $\mathfrak{g}$ (see [Hu]). If W denotes the Weyl group of $\mathfrak{g}$, then simple objects in $\mathcal{O}_0$ are simple highest weight modules $L(w)$, $w \in W$, of highest weight $w \cdot 0 \in \mathfrak{h}^*$. Denote by $P(w)$ the indecomposable projective cover of $L(w)$ and by $\Delta(w)$ the corresponding Verma module.

Let $\mathscr{S} = \mathscr{S}_{\mathfrak{g}}$ denote the (strict) 2-category defined as follows: it has one object $\mathtt{i}$ (which we identify with $\mathcal{O}_0$); its 1-morphisms are projective functors on $\mathcal{O}_0$, that is functors isomorphic to direct summands of tensoring with finite dimensional $\mathfrak{g}$-modules (see [BG]); and its 2-morphisms are natural transformations of functors. For $w \in W$ denote by θ_w the unique (up to isomorphism) indecomposable projective functor on $\mathcal{O}_0$ sending $P(e)$ to $P(w)$. Then $\{\theta_w : w \in W\}$ is a complete and irredundant list of representatives of isomorphism classes of indecomposable projective functors. Since $\mathcal{O}_0$ is equivalent to the category of modules over a finite-dimensional associative algebra, all spaces of 2-morphisms in $\mathscr{S}$ are finite dimensional. From [BG] we also have that $\mathscr{S}$ is stable under taking adjoint functors. It follows that $\mathscr{S}$ is a fiat category. The split Grothendieck ring $[\mathscr{S}]_{\oplus}$ of $\mathscr{S}$ is isomorphic to the integral group ring $\mathbb{Z}W$ such that the basis $\{[\theta_w] : w \in W\}$ of $[\mathscr{S}]_{\oplus}$ corresponds to the Kazhdan-Lusztig basis of $\mathbb{Z}W$. We refer the reader to [Ma] for an overview and more details on this category.

Left and right cells of $\mathscr{S}$ are given by the Kazhdan-Lusztig combinatorics for W (see [KaLu]) and correspond to Kazhdan-Lusztig left and right cells in W, respectively. Namely, for $x, y \in W$ the functors θ_x and θ_y belong to the same left (right or two-sided) cell as defined in Subsection 4.1 if and only if x and y belong to the same Kazhdan-Lusztig left (right or two-sided) cell, respectively. This is an immediate consequence of the multiplication formula for elements of the Kazhdan-Lusztig basis (see [KaLu]). In particular, from [Lu] it follows that all cells for $\mathscr{S}$ are regular. If $\mathfrak{g} \cong \mathfrak{sl}_n$, then W is isomorphic to the symmetric group S_n. Robinson-Schensted correspondence associates to every $w \in S_n$ a pair $(\alpha(w), \beta(w))$ of standard Young tableaux of the same shape (see [Sa, Section 3.1]). Elements $x, y \in S_n$ belong to the same Kazhdan-Lusztig right or left cell if and only if $\alpha(x) = \alpha(y)$ and $\beta(x) = \beta(y)$, respectively (see [KaLu]). It follows that in the case $\mathfrak{g} \cong \mathfrak{sl}_n$ all cells for $\mathscr{S}$ are strongly regular.

The 2-category $\mathscr{S}$ comes along with the *defining 2-representation*, that is the natural action of $\mathscr{S}$ on $\mathcal{O}_0$. Various 2-representations of $\mathscr{S}$ were constructed, as subquotients of the defining representation, in [KMS] and [MS] (see also [Ma] for a more detailed overview). In particular, in [MS] for every Kazhdan-Lusztig right cell $\mathcal{R}$ there is a construction of the corresponding *cell module*. The later is obtained by restricting the action of $\mathscr{S}$ to the full subcategory of $\mathcal{O}_0$ consisting of all modules M admitting a presentation $X_1 \to X_0 \twoheadrightarrow M$, where every indecomposable direct summand of both X_0 and X_1 is isomorphic to $\theta_w L(d)$, where $w \in \mathcal{R}$ and d is the Duflo involution in $\mathcal{R}$. Similarly to the proof of Theorem 43 one shows that this cell module is equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{R}}$ of $\mathscr{S}$.

Let $\mathcal{Q}$ be a strongly regular two-sided cell for $\mathscr{S}$. In this case from [Ne, Theorem 5.3] it follows that the condition (10) is satisfied for $\mathcal{Q}$. Hence from Theorem 43 we obtain that cell 2-representations of $\mathscr{S}$ for right cells inside a given two-sided cell are equivalent. This reproves, strengthens and extends the similar result [MS, Theorem 18], originally proved in the case $\mathfrak{g} \cong \mathfrak{sl}_n$.

7.2 Projective functors between singular blocks of $\mathcal{O}$

The 2-category $\mathscr{S}_{\mathfrak{g}}$ from the previous subsection admits the following natural generalization. For every parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ let $W_{\mathfrak{p}} \subset W$ be the corresponding parabolic subgroup. Fix some dominant and integral weight $\lambda_{\mathfrak{p}}$ such that $W_{\mathfrak{p}}$ coincides with the stabilizer of $\lambda_{\mathfrak{p}}$ with respect to the dot action (to show the connection with the previous subsection we take $\lambda_{\mathfrak{b}} = 0$). Let $\mathcal{O}_{\lambda_{\mathfrak{p}}}$ denote the corresponding block of the category $\mathcal{O}$.

Consider the 2-category $\mathscr{S}^{\mathrm{sing}} = \mathscr{S}^{\mathrm{sing}}_{\mathfrak{g}}$ defined as follows: its objects are the categories $\mathcal{O}_{\lambda_{\mathfrak{p}}}$, where $\mathfrak{p}$ runs through the (finite!) set of parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$, its 1-morphisms are all projective functors between these blocks, its 2-morphisms are all natural transformations of functors. Similarly to the previous subsection, the 2-category $\mathscr{S}^{\mathrm{sing}}$ is a fiat-category. The category $\mathscr{S}$ from the previous subsection is just the full subcategory of $\mathscr{S}^{\mathrm{sing}}$ with the object $\mathcal{O}_0$. A deformed version of $\mathscr{S}^{\mathrm{sing}}$ (which has infinite-dimensional spaces of 2-morphisms and hence is not fiat) was considered in [Wi].

Let us describe in more detail the structure of $\mathscr{S}^{\mathrm{sing}}$ in the smallest nontrivial case of $\mathfrak{g} = \mathfrak{sl}_2$. In this case we have two parabolic subalgebras, namely $\mathfrak{b}$ and $\mathfrak{g}$. Using the usual identification of $\mathfrak{h}$ with $\mathbb{C}$ we set $\lambda_{\mathfrak{b}} = 0$ and $\lambda_{\mathfrak{g}} = -1$. The objects of $\mathscr{S}^{\mathrm{sing}}$ are thus $\mathtt{i} = \mathcal{O}_0$ and $\mathtt{j} = \mathcal{O}_{-1}$.

The category $\mathscr{S}^{\mathrm{sing}}(\mathtt{j},\mathtt{j})$ contains a unique (up to isomorphism) indecomposable object, namely $\mathbb{1}_{\mathtt{j}}$, the identity functor on $\mathtt{j}$. The category $\mathscr{S}^{\mathrm{sing}}(\mathtt{i},\mathtt{j})$ contains a unique (up to isomorphism) indecomposable object, namely the functor θ^{on} of translation onto the wall. The category $\mathscr{S}^{\mathrm{sing}}(\mathtt{j},\mathtt{i})$ contains a unique (up to isomorphism) indecomposable object, namely the functor θ^{out} of translation out of the wall. The category $\mathscr{S}^{\mathrm{sing}}(\mathtt{i},\mathtt{i})$ contains exactly two (up to isomorphism) non-isomorphic indecomposable objects, namely the identity functor $\mathbb{1}_{\mathtt{i}}$ and the functor $\theta := \theta^{\mathrm{out}} \circ \theta^{\mathrm{on}}$ of translation through the wall.

It is easy to see that there are exactly two two-sided cells: one containing only the functor $\mathbb{1}_{\mathtt{i}}$, and the other one containing all other functors. The right cells of the latter two-sided cell are $\{\mathbb{1}_{\mathtt{j}}, \theta^{\mathrm{out}}\}$ and $\{\theta, \theta^{\mathrm{on}}\}$. The left cells of the latter two-sided cell are $\{\mathbb{1}_{\mathtt{j}}, \theta^{\mathrm{on}}\}$ and $\{\theta, \theta^{\mathrm{out}}\}$. All cells are strongly regular. The values of the function $m_{\mathrm{F},\mathrm{F}}$ from (10) are given by:

F	$\mathbb{1}_{\mathtt{i}}$	$\mathbb{1}_{\mathtt{j}}$	θ^{on}	θ^{out}	θ
$m_{\mathrm{F},\mathrm{F}}$	1	1	1	2	2

In particular, the condition (10) is satisfied.

The cell 2-representation corresponding to the right cell $\{\mathbb{1}_{\mathtt{i}}\}$ is given by the following picture (with the obvious action of the identity 1-morphisms):

$$\begin{tikzcd}
\mathbb{C}\text{-mod} \arrow[loop above, "\theta=0"] \arrow[r, bend left, "\theta^{\mathrm{on}}=0"] & 0. \arrow[l, bend left, "\theta^{\mathrm{out}}=0"]
\end{tikzcd}$$

By Theorem 43, the cell 2-representations for the right cells $\{\mathbb{1}_{\mathtt{j}}, \theta^{\mathrm{out}}\}$ and $\{\theta, \theta^{\mathrm{on}}\}$ are equivalent and strongly simple. Consider the algebra $D := \mathbb{C}[x]/(x^2)$ of dual numbers with the fixed subalgebra $\mathbb{C}$ consisting of scalars. The cell 2-representation for the right cell $\{\mathbb{1}_{\mathtt{j}}, \theta^{\mathrm{out}}\}$ is given (up to isomorphism of functors) by the following picture:

$$\begin{tikzcd}
D\text{-mod} \arrow[loop above, "\theta=D\otimes_-"] \arrow[r, bend left, "\theta^{\mathrm{on}}=\mathrm{Res}^D_{\mathbb{C}}"] & \mathbb{C}\text{-mod}. \arrow[l, bend left, "\theta^{\mathrm{out}}=\mathrm{Ind}^D_{\mathbb{C}}"]
\end{tikzcd}$$

7.3 Projective functors for finite-dimensional algebras

The last example admits a straightforward abstract generalization outside category $\mathcal{O}$. Let $A = A_{\mathtt{1}} \oplus A_{\mathtt{2}} \oplus \cdots \oplus A_{\mathtt{k}}$ be a weakly symmetric self-injective finite-dimensional algebra over an algebraically closed field $\Bbbk$ with a fixed decomposition into a direct sum of connected components (here *weakly symmetric* means that the top and the socle of every projective module are isomorphic). Let $\mathscr{C}_A$ denote the 2-category with objects $\mathtt{1}, \mathtt{2}, \dots, \mathtt{k}$, which we identify with the corresponding $A_{\mathtt{i}}$-mod. For $\mathtt{i}, \mathtt{j} \in \{\mathtt{1}, \mathtt{2}, \dots, \mathtt{k}\}$ define $\mathscr{C}_A(\mathtt{i}, \mathtt{j})$ as the full fully additive subcategory of the category of all functors from $A_{\mathtt{i}}$-mod to $A_{\mathtt{j}}$-mod, generated by all functors isomorphic to tensoring with $A_{\mathtt{i}}$ (in the case $\mathtt{i} = \mathtt{j}$) and tensoring with all projective $A_{\mathtt{j}}$-$A_{\mathtt{i}}$ bimodules (i.e. bimodules of the form $A_{\mathtt{j}} e \otimes_{\Bbbk} f A_{\mathtt{i}}$ for some idempotents $e \in A_{\mathtt{j}}$ and $f \in A_{\mathtt{i}}$) for all $\mathtt{i}$ and $\mathtt{j}$. Functors, isomorphic to tensoring with projective bimodules will be called *projective functors*.

Lemma 45. *The category $\mathscr{C}_A$ is a fiat category.*

Proof. The only nontrivial condition to check is that the left and the right adjoints of a projective functor are again projective and isomorphic. For any A-module M and idempotents $e, f \in A$, using adjunction and projectivity of fA we have

$$\begin{aligned}
\mathrm{Hom}_A(Ae \otimes_{\Bbbk} fA, M) &= \mathrm{Hom}_{\Bbbk}(fA, \mathrm{Hom}_A(Ae, M)) \\
&= \mathrm{Hom}_{\Bbbk}(fA, eM) \\
&= \mathrm{Hom}_{\Bbbk}(fA, eA \otimes_A M) \\
&= \mathrm{Hom}_{\Bbbk}(fA, \Bbbk) \otimes_{\Bbbk} eA \otimes_A M \\
&= (fA)^* \otimes_{\Bbbk} eA \otimes_A M
\end{aligned}$$

Since A is self-injective, $(fA)^*$ is projective. Since A is weakly symmetric, $(fA)^* \cong Af$. This implies that tensoring with $Af \otimes_{\Bbbk} eA$ is right adjoint to tensoring with $Ae \otimes_{\Bbbk} fA$. The claim follows. $\square$

The category $\mathscr{C}_A$ has a unique maximal two-sided cell $\mathcal{Q}$ consisting of all projective functors. This cell is regular. Right and left cells inside $\mathcal{Q}$ are given by fixing primitive idempotents occurring on the left and on the right in projective functors, respectively. In particular, they are in bijection with simple A-A-bimodules and hence $\mathcal{Q}$ is strongly regular. The value of the function

$m_{\mathrm{F},\mathrm{F}}$ on $Ae\otimes_{\Bbbk} fA$ is given by the dimension of $eA\otimes_A Ae \cong eAe$, in particular, the function $m_{\mathrm{F},\mathrm{F}}$ is constant on left cells. From Theorem 43 we thus again obtain that all cell 2-representations of $\mathscr{C}$ corresponding to right cells in $\mathcal{Q}$ are strongly simple and isomorphic.

The category $\mathscr{S}_{\mathfrak{sl}_2}^{\mathrm{sing}}$ from the previous subsection is obtained by taking $\mathtt{k}=2$, $A_{\mathtt{1}}=\mathbb{C}$ and $A_{\mathtt{2}}=D$. In the general case we have the following:

Proposition 46. *Let $\mathscr{C}$ be a fiat category, $\mathcal{Q}$ a strongly regular two-sided cell of $\mathcal{Q}$ and $\mathcal{R}$ a right cell in $\mathcal{Q}$. For $\mathtt{i}\in\mathscr{C}$ let $A_{\mathtt{i}}$ be such that $\mathbf{C}_{\mathcal{R}}(\mathtt{i})\cong A_{\mathtt{i}}$-mod and $A=\oplus_{\mathtt{i}\in\mathscr{C}}A_{\mathtt{i}}$. Assume that the condition* (10) *is satisfied. Then $\mathbf{C}_{\mathcal{R}}$ gives rise to a 2-functor from $\mathscr{C}_{\mathcal{Q}}$ to $\mathscr{C}_A$.*

Proof. We identify $\mathbf{C}_{\mathcal{R}}(\mathtt{i})$ with $A_{\mathtt{i}}$-mod. That A is self-injective follows from Corollary 38. That A is weakly symmetric follows by adjunction from Lemma 12 and strong regularity of $\mathcal{Q}$. Hence, to prove the claim we only need to show that for any $\mathrm{F}\in\mathcal{Q}$ the functor $\mathbf{C}_{\mathcal{R}}(\mathrm{F})$ is a projective endofunctor of A-mod.

As $\mathbf{C}_{\mathcal{R}}(\mathrm{F})$ is exact, it is given by tensoring with some bimodule, say B. Since $\mathbf{C}_{\mathcal{R}}(\mathrm{F})$ kills all simples but one, say L, and sends L to an indecomposable projective, say P (by Theorem 43), the bimodule B has simple top (as a bimodule) and hence is a quotient of some projective bimodule.

By exactness of $\mathbf{C}_{\mathcal{R}}(\mathrm{F})$, the dimension of B equals the dimension of P times the multiplicity of L in A. This is exactly the dimension of the corresponding indecomposable projective bimodule. The claim follows. $\square$

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