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On simple modules over q-analog for the Virasoro algebra

Volodymyr Mazorchuk

Abstract

We construct and investigate a family of simple modules and describe the support of a simple graded module over q-analog for the Virasoro algebra.

1 Introduction

A q-analog for the Virasoro algebra $\mathcal{V}_q$ was introduced in the recent paper by Kirkman, Procesi and Small ([5]) as one of the possible generalizations of Virasoro algebra (see [3, 4, 9, 10] for more). This algebra is defined as an algebra of inner derivations of the localized quantum plane $\mathbb{C}_q[X, Y, X^{-1}, Y^{-1}]$. It was shown that $\mathcal{V}_q$ is simple infinite-dimensional Lie algebra that has two essential central extensions. The construction studied in [5] enable one to obtain more new simple infinite-dimensional Lie algebras. Few is known about representations of $\mathcal{V}_q$.

The aim of this paper is to construct and investigate a family of simple modules over $\mathcal{V}_q$. This family can be viewed as some analogue of the intermediate series modules for classical Virasoro algebra and higher rank Virasoro algebras ([6, 11, 12]). There appear some difficulties in the investigation of such modules since $\mathcal{V}_q$ has no analogue of a Cartan subalgebra in classical sense. This caused us to consider some extension $\mathfrak{A}$ of $\mathcal{V}_q$ and to work with modules over $\mathfrak{A}$. Fortunately, all simple weight $\mathfrak{A}$-modules remain simple after the restriction on $\mathcal{V}_q$. We also establish the theorem describing the support of a simple weight module over $\mathfrak{A}$ which seems to be a generalization of the results obtained for simple finite-dimensional algebras in [1] and for rank 2 Virasoro algebras in [7].

Let us briefly describe the structure of the paper. In section 2 we collect all necessary notations and preliminary results. In section 3 we define an algebra $\mathfrak{A}$ and investigate the dependence between $\mathfrak{A}$-modules and $\mathcal{V}_q$-modules. In section 4 we construct a family of simple $\mathfrak{A}$-modules $V(a, b)$ and prove that all of them remain simple after the restriction on $\mathcal{V}_q$. Finally, in section 5 we describe the support of a simple weight $\mathfrak{A}$-module.

2 Preliminaries

Let $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers. We will also denote by $\mathbb{Z}_+$ the set of non-negative integers. For a Lie algebra $\mathfrak{G}$ we will denote by $U(\mathfrak{G})$ its universal enveloping algebra.

Set $P = \mathbb{Z}^2 \setminus \{(0,0)\}$. Fix a non-zero non-root of unity $q \in \mathbb{C}$ ($q \neq 0$, $q^n \neq 1$, $n \in \mathbb{N}$). Let $\mathcal{V}_q$ denotes the Lie algebra with the base $e(x)$, $x = (x_1, x_2) \in P$ and with Lie brackets

$$[e(x), e(y)] = (q^{x_2 y_1} - q^{x_1 y_2}) e(x+y).$$

Main properties of $\mathcal{V}_q$ obtained in [5] are collected in the following proposition:

Proposition 1. *1. $\mathcal{V}_q$ is simple infinite-dimensional Lie algebra.*

2. $\mathcal{V}_q$ is isomorphic to the Lie algebra of inner derivations of the skew polynomial ring $\mathbb{C}_q[X, Y, X^{-1}, Y^{-1}]$ ([2]).

3. The second cohomology $H^2(\mathcal{V}_q, \mathbb{C})$ is 2-dimensional with the basis $f_i(a,b)$, $i = 1,2$ such that $f_i(a,b) = 0$ for $a \neq -b$ and

$$f_1((h,k),(-h,-k)) = q^{-hk} h, \text{ and } f_2((h,k),(-h,-k)) = q^{-hk} k.$$

4. $[e(x), e(y)] = 0$ if and only if $x = ky$ for some rational k.

An algebra $\mathcal{V}_q$ possess a natural $\mathbb{Z}^2$-grading: for $x \in \mathbb{Z}^2$ we put $(\mathcal{V}_q)_x = \langle e(x) \rangle$ ($(\mathcal{V}_q)_0 = 0$). It follows immediately from the definition that $[(\mathcal{V}_q)_x, (\mathcal{V}_q)_y] \subset (\mathcal{V}_q)_{x+y}$ for any $x, y \in \mathbb{Z}^2$. This grading induces a grading of $U = U(\mathcal{V}_q)$ in a natural way. Indeed, for $x \in \mathbb{Z}^2$ we set U_x to be linear span of the elements $e(y_1)e(y_2)\dots e(y_k)$, $\quad k \in \mathbb{N}$ with $y_1 + y_2 + \dots + y_k = x$.

An U-module V will be called graded module if

$$V = \bigoplus_{x \in \mathbb{Z}^2} V^x$$

such that $U(y)V^x \subset V^{x+y}$ for any $x \in \mathbb{Z}^2$, $y \in P$.

3 Extended algebra $\mathfrak{A}$

One can easily show that all elements $e(x)$, $x \in P$ are not diagonalizable in the adjoint representation and thus $\mathcal{V}_q$ has no reasonable Cartan subalgebra. For our convenience we will extend $\mathcal{V}_q$ by the derivations as it usually done in constructing affine Kac-Moody Lie algebras ([8]).

Consider the algebra

$$\mathfrak{A} = \mathcal{V}_q \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$$

with the Lie brackets defined on d_1 and d_2 in the following way:

$$[d_i, e(x)] = x_i e(x), \quad i = 1,2;\ x = (x_1, x_2) \in P,, \qquad [d_1, d_2] = 0.$$

Lemma 1. $\mathfrak{A}$ *is a Lie algebra.*

Follows by straightforward calculation.

We will call the subalgebra $\mathfrak{H} = \langle d_1, d_2 \rangle$ Cartan subalgebra of $\mathfrak{A}$. For $\lambda \in \mathfrak{H}^*$ and an $\mathfrak{A}$-module V set

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}.$$

In the case $V_\lambda \neq 0$ we will call it the weight space corresponding to the weight λ. We will call V a weight module provided

$$V = \bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda.$$

For a weight $\mathfrak{A}$-module V we set $\operatorname{Supp} V$ to be the set of all weights of V (i.e. $\lambda \in \mathfrak{H}^*$, $V_\lambda \neq 0$). Clearly, $\mathfrak{A}$ is a weight module under the adjoint action. Set Δ to be the set of all non-zero weights of this module $\mathfrak{A}$. Elements from Δ will be called roots. For $x \in P$ let α_x be the root corresponding to the element $e(x)$.

Lemma 2. *Let V be an indecomposable weight $\mathfrak{A}$-module and $\lambda \in \operatorname{Supp} V$. Then* $\operatorname{Supp} V \subset \lambda \cup \lambda + \Delta$

This is the standard fact about weight modules.

Let us denote by mod-α (mod-$\mathcal{V}_q$) the category of all finitely generated $\mathfrak{A}$-modules ($\mathcal{V}_q$-modules). Let F : mod-$\mathfrak{A} \to$ mod-$\mathcal{V}_q$ be the standard forgetful functor (i.e. $F(V)$ is obtained from V by restriction on $\mathcal{V}_q$).

Proposition 2. *1. $F(V)$ is graded $\mathcal{V}_q$-module for any weight $\mathfrak{A}$-module V.*

2. Let W be a graded $\mathcal{V}_q$-module then there exists a weight $\mathfrak{A}$-module V such that $F(V) \simeq W$.

To prove the first statement fix $\lambda \in \operatorname{Supp} V$. For $x \in \mathbb{Z}^2$ set $(F(V))_x = V_{\lambda+\alpha_x}$. It follows from the definition of $\mathcal{V}_q$ this defines a $\mathbb{Z}^2$-grading on $F(V)$.

For the second statement take some $\lambda \in \mathfrak{H}^*$. For any $x \in \mathbb{Z}^2$ and any $v \in W^x$ set $d_i v = (\lambda + \alpha_x)(d_i)v$. It is straightforward that this defines an $\mathfrak{A}$-module structure on W (we denote this new module by V), and that $W^x = V_{\lambda+\alpha_x}$. Thus the proposition follows.

Proposition 2 reduces the problem of investigation of graded $\mathcal{V}_q$-modules to the problem of investigation of weight modules over $\mathfrak{A}$. But it is not quite clear if F preserves simplicity or not. In the next section we construct a family of simple weight $\mathfrak{A}$-modules and show that corresponding restriction on $\mathcal{V}_q$ will be simple again.

4 Modules $V(a, b)$

As it follows from the classification of simple Harish-Chandra modules over classical Virasoro Lie algebra ([6]) the so-called intermediate series of modules plays important role in the representation theory of Virasoro algebra. This series consists of modules $V(a, b)$,

$a, b \in \mathbb{C}$ with the base $v(i)$, $i \in \mathbb{Z}$, without central charge and the action of canonical generators defined by

$$e(j)v(i) = (i + aj + b)v(i + j), \quad i, j \in \mathbb{Z}.$$

In this section we will construct some analogue of such modules both for algebras $\mathfrak{A}$ and $\mathcal{V}_q$.

First of all we investigate the case of algebra $\mathfrak{A}$. Consider a $\mathbb{C}$-vectorspace V with the base $v(x)$, $x \in \mathbb{Z}^2$ and fix $a, b \in \mathbb{C}$, $\lambda \in \mathfrak{H}^*$. For $x \in P$, $y \in \mathbb{Z}^2$ set

$$e(x)v(y) = (q^{(y_1+a)x_2} - q^{(y_2+b)x_1})v(x + y)$$

and

$$d_i v(y) = (\lambda + \alpha_y)(d_i)v(y), \quad i = 1, 2.$$

Lemma 3. *The formulas above defines on V the structure of a weight $\mathfrak{A}$-module.*

Proof is straightforward calculation.

The module defined above will be denoted by $V(a, b)$. An $\mathfrak{A}$-module W will be called trivial provided $\mathcal{V}_q W = 0$. Let $S \subset \mathbb{C}$ be the set of all complex u such that $q^u = 1$. We note that $\mathbb{Z} \cap T = \{0\}$. Set $T = \mathbb{Z} + S$.

Proposition 3. *1. $V(a, b)$ is simple if and only if $a \notin T$ or $b \notin T$.*

2. If $a, b \in T$ then $V(a, b)$ is a direct sum of two simple modules and one of those modules is trivial.

We consider the case $a \notin T$. It is sufficient to show that $U(\mathfrak{A})v = V(a, b)$ for any non-zero $v \in V(a, b)$. Since $V(a, b)$ is a weight module we can assume that v is a weight element (i.e. $v \in V(a, b)_\mu$ for some $\mu \in \operatorname{Supp} V(a, b)$). Thus we can assume that $v = v(y)$ for some $y \in \mathbb{Z}^2$. Set $N = U(\mathfrak{A})v$. We need only to show that $v(z) \in N$ for all $z \in \mathbb{Z}^2$.

Consider an element $w = e(z - y)v$. If $w \neq 0$ we've done. Suppose that $w = 0$. Thus $q^{(y_1+a)(z_2-y_2)} = q^{(y_2+b)(z_1-y_1)}$. Since $a \notin T$ it follows that $w_1 = e(z - y + (0, 1))v \neq 0$. Finally, we obtain that

$$e((0, -1))v(z + (0, 1)) = q^{-(z_1+1+a)}v(z) \neq 0.$$

This proves the first statement.

To prove the second one consider the unique $a', b' \in Z$ such that $a - a' \in S$, $b - b' \in S$. One can obtain by direct calculation that $V(a, b)_{\lambda - \alpha_{(a',b')}}$ is the trivial direct summand of $V(a, b)$. Set $V(a, b) = V(a, b)_{\lambda - \alpha_{(a',b')}} \oplus V'(a, b)$ Now one can show that for any $\mu \in \operatorname{Supp} V'(a, b)$ either $e((1, 0))V'(a, b)_\mu \neq 0$ or $e((0, 1))V'(a, b)_\mu \neq 0$. Now the second part of proposition follows at the same way as its first part.

Theorem 1. *1. If $a \notin T$ or $b \notin T$ then the module $F(V(a, b))$ is simple.*

2. If $a, b \in T$ then the module $F(V'(a, b))$ is simple.

By proposition 3 it is sufficient to show that each non-trivial submodule of $F(V(a,b))$ $(F(V'(a,b)))$ contains an element of the form $v(x)$ for some $x \in \mathbb{Z}^2$ in both cases. We will consider only the first case. One can do the same for the second one.

Let $M = F(V(a,b))$. Clearly, M is $\mathbb{Z}^2$-graded and all graded components are one-dimensional. Thus U_0 is diagonalizable on M. It is sufficient to show that for any $x, y \in \mathbb{Z}^2$, $x \neq y$, $x_i \neq -y_i$, $i = 1, 2$ there exists $u \in U_0$, $uv(x) = c_1 v(x)$, $uv(y) = c_2 v(y)$ such that $c_1 \neq c_2$.

To prove the last statement consider two elements in U_0: $A_1 = e((0,1))e((0,-1))$ and $A_2 = e((1,0))e((-1,0))$. Let $x \neq y$, $x, y \in \mathbb{Z}^2$, $x_i \neq -y_i$, $i = 1, 2$. Since $x \neq y$ it follows that there exists $j \in \{1, 2\}$ such that $x_j \neq y_j$. Then it follows from direct calculation that $A_j v(x) = c_1 v(x)$, $A_j v(y) = c_2 v(y)$ and $c_1 \neq c_2$. This completes the proof.

Proposition 4. *1. For $a \notin T$ or $b \notin T$ and $a' \notin T$ or $b' \notin T$ modules $F(V(a,b))$ and $F(V(a',b'))$ are isomorphic if and only if $a - a' \in T$ and $b - b' \in T$.*

2. For $a, b, a', b' \in T$ modules $F(V'(a,b))$ and $F(V'(a',b'))$ are isomorphic.

The second part follows immediately from the definition of $V(a,b)$ as good as the "if" statement for the first part. So we need only to prove the "only if" statement. But in the case $a - a' \notin T$ $(b - b' \notin T)$ the modules $F(V(a,b))$ and $F(V(a',b'))$ can be easily distinguished by the eigenvalues of the elements A_2 (A_1) defined in the proof of theorem 1.

5 Support of a simple weight $\mathfrak{A}$-module

In this section we give some sort of description for the support of simple weight module over $\mathfrak{A}$. This results can be viewed as a generalization of the results from [1, 7] for simple finite-dimensional Lie algebras and rank 2 Virasoro algebras. For the proof we will use the technique from [7].

Denote $\Delta' = \Delta \cup \{0\}$. An $\mathfrak{A}$-module V will be called dense if $\operatorname{Supp} V = \lambda + \Delta'$ for some $\lambda \in \operatorname{Supp} V$ (and thus for any such λ). An $\mathfrak{A}$-module V will be called pined if $\operatorname{Supp} V = \lambda + \Delta$ for some $\lambda \in \mathfrak{H}^*$. An $\mathfrak{A}$-module V will be called cut if there exist a non-trivial order $\leq$ on abelian group Δ' such that $\operatorname{Supp} V \subset \lambda + \{\beta \in \Delta' \mid b \leq 0\}$.

Theorem 2. *Let V be simple weight $\mathfrak{A}$-module. Then one of the following holds:*

1. *V is dense.*

2. *V is pined.*

3. *V is cut.*

For a graded $\mathcal{V}_q$-module V let $\operatorname{Supp} V$ to be the set of all $x \in \mathbb{Z}^2$ such that $V^x \neq 0$. We keep the same terminology as for $\mathfrak{A}$-modules.

Corollary 1. *Let V be a simple graded $\mathcal{V}_q$-module. Then one of the following holds:*

1. *V is dense.*

2. V is pined.

3. V is cut.

Follows from theorem 2 and proposition 2.

[Proof of theorem 2] The proof follows the general way of the proof [7, Theorem 2]. So we only sketch it by outlying main steps.

Of course it is sufficient to prove that any $\mathfrak{A}$-module V such that $|(\lambda+\Delta')\setminus \operatorname{Supp} V| > 1$ for some $\lambda \in \operatorname{Supp} V$ is cut. Fix two different $\lambda_1, \lambda_2 \in (\lambda + \Delta') \setminus \operatorname{Supp} V$. Then it is easy to see that for any $\mu \in \operatorname{Supp} V$ and any $k, l \in \mathbb{N}$ holds $U_{k(\lambda_1-\mu)+l(\lambda_2-\mu)} V_\mu = 0$. Using the irreducibility of V and the arguments from [7, Lemma 5] it follows that $k(\lambda_1-\mu)+l(\lambda_2-\mu) \notin \operatorname{Supp} V$ for all $k, l \in \mathbb{N}$. Finally, since the last holds for any $\mu \in \operatorname{Supp} V$ it is not difficult to show that $\operatorname{Supp} V$ lies in a some real halfplane of $\mathfrak{H}^*$ (see the proof of the main theorem in [7]). The last observation completes the proof.

After such description it is natural to search for examples of dense, pined and cut modules. Fortunately, they can be obtained by standard constructions:

Examples 1. *1. For $a \notin T$ or $b \notin T$ the module $V(a, b)$ is simple dense $\mathfrak{A}$-module.*

2. For $a, b \in T$ the module $V'(a, b)$ is simple pined $\mathfrak{A}$-module.

3. The first example of cat module is trivial in all senses: trivial module is cut. To give more interesting example of cut module we construct some analogue of Verma modules ([8]). Let $\Delta = \Delta_+ \cup \Delta_-$ be some decomposition of Δ into disjoint union of sets Δ_+ and Δ_- such that for any $\beta_1, \beta_2 \in \Delta_+$ holds $\beta_1 + \beta_2 \in \Delta_+$ and $-\beta_1 \in \Delta_-$. Consider the corresponding decomposition of $\mathfrak{G}$: $\mathfrak{G} = \mathfrak{G}_- \oplus \mathfrak{G}_+$, where $\mathfrak{G}_\pm = \langle e(x) \,|\, x \in \Delta_\pm \rangle$. Define on $\mathbb{C}$ a structure of trivial $\mathfrak{G}_+$-module. Let

$$M = M(\mathfrak{G}_-) = U \bigotimes_{U(\mathfrak{G}_+)} \mathbb{C}.$$

It follows directly from definition that any simple subquotient of M is cut. Moreover, it is not difficult to verify that for "almost all" partitions of Δ the corresponding module M is a direct sum of trivial $\mathfrak{A}$-module and some simple $\mathfrak{A}$-module.

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Mechanics and Mathematics department
Kyiv Taras Shevchenko University
64, Volodymyrska st.
252033 Kyiv
Ukraine
e-mail: mazorchu@uni-alg.kiev.ua