

# Extension of continuous functions in digital spaces with the Khalimsky topology

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## Abstract

The digital space  $\mathbb{Z}^n$  equipped with Efim Khalimsky's topology is a connected space. We study continuous functions  $\mathbb{Z}^n \supset A \rightarrow \mathbb{Z}$ , from a subset of Khalimsky  $n$ -space to the Khalimsky line. We give necessary and sufficient condition for such a function to be extendable to a continuous function  $\mathbb{Z}^n \rightarrow \mathbb{Z}$ . We classify the subsets  $A$  of the digital plane such that every continuous function  $A \rightarrow \mathbb{Z}$  can be extended to a continuous function on the whole plane.

## 1 Introduction

The Tietze extension theorem states that if  $X$  is a normal topological space and  $A$  is a closed subset of  $X$ , then any continuous map from  $A$  into a closed interval  $[a, b]$  can be extended to a continuous function on all of  $X$  into  $[a, b]$ . Real valued, continuous functions on a digital space are not so interesting, but if we replace the real interval with its digital counterpart, a Khalimsky interval, the same question is relevant. However, the Tietze extension theorem is not true in this setup; in fact closedness of the domain is neither sufficient nor necessary. It turns out that the answer is instead related to the connectedness of the domain.

The structure of the article is as follows. First we give some preliminary results. The proofs are straight-forward. These results, (Lemma 2, Prop. 3, Thm. 4 and Prop. 6) are due to Kiselman [5] or small generalizations of his results.

Then we seek similar results in higher dimension. Here, the crucial step is to move the questions to the well known one-dimensional case. We give a sufficient and necessary condition for a function to be extendable. Then we use this result to give a simple criterion on a subset of the digital plane that is equivalent with saying that every continuous function defined on the

subset is extendable to the whole plane. In order to make this article self-contained, we begin by summarizing some necessary definitions and results. A more systematic treatment can be found in [3, 4, 5].

Let us by  $N(x)$  denote the intersection of all neighbourhoods of a point  $x$  in any topological space. Spaces such that  $N(x)$  is always a neighbourhood were introduced and studied by Aleksandroff [1]. We will call these spaces smallest-neighbourhood spaces. It is equivalent to require that the intersection of an arbitrary family of open sets is open, or that a union of closed sets is closed. Finite spaces constitute an important special case. General properties of finite spaces are studied in [6].

A topological space is said to be connected if the only sets which are both open and closed are the empty set and the whole space. If  $X$  and  $Y$  are topological spaces and  $f: X \rightarrow Y$  a continuous mapping, then the image  $f(A)$  of a connected subset  $A$  of  $X$  is connected. This result may be used to define a topology on a space, and to ensure that the space will be connected.

**Proposition 1** *Let  $f: X \rightarrow Y$  be a surjective mapping from a connected topological space onto a set  $Y$ . Suppose  $Y$  is equipped with any topology such that  $f$  is continuous. Then  $Y$  is connected.*

This result is particularly interesting when  $Y$  is equipped with the strongest topology such that  $f$  is continuous. We shall use this result with  $X = \mathbb{R}$  and  $Y = \mathbb{Z}$  to define a connected topology on the digital line  $\mathbb{Z}$ .

## 2 Topologies on the Digital Line

We will now use Proposition 1 to define connected topologies on the digital line. Let  $X = \mathbb{R}$  and  $Y = \mathbb{Z}$ , and let  $f$  be a surjective mapping from  $\mathbb{R}$  to  $\mathbb{Z}$ . Of course there are many surjective mappings  $\mathbb{R} \rightarrow \mathbb{Z}$ . It is natural to think of  $\mathbb{Z}$  as an approximation of the real line, and therefore to consider mappings expressing this idea. Define  $f$  to be the integer closest to  $x$ , unless  $x$  is a half-integer. When  $x = n + \frac{1}{2}$  we have a choice for each  $n$ ; either  $f(x) = n$  or  $f(x) = n + 1$ . If we always choose the first alternative for every  $n$ , then the topology defined in Proposition 1 is called the *right topology* on  $\mathbb{Z}$ ; the second alternative gives the *left topology* on  $\mathbb{Z}$ ; cf. Bourbaki [2] (Exerc. 2).

If we instead decide that the best approximant of a half-integer is always an even integer, the resulting topology is *Khalimsky's topology*. The inverse image of an even number  $n$  is the closed interval  $[n - \frac{1}{2}, n + \frac{1}{2}]$  so that  $\{n\}$  is closed, whereas the inverse image of an odd number  $m$  is open, so that  $\{m\}$  is open.

By construction, the *Khalimsky line* is connected, and it is easy to see that if we remove any point the resulting subspace is disconnected. Using the terminology of [4], where this space is studied in a more general setting, this is to say that every point is a *cut point*. We note that the Khalimsky

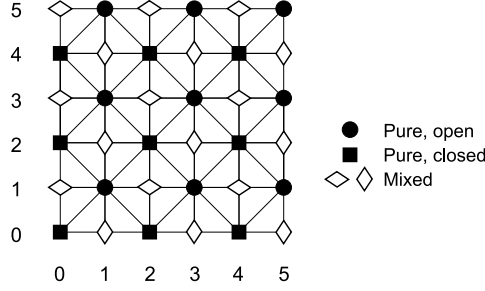


Figure 1: The Khalimsky plane.

line is a smallest-neighbourhood space. All odd points are open:  $N(2k + 1) = \{2k + 1\}$ , and all even points have a smallest neighbourhood  $N(2k) = \{2k - 1, 2k, 2k + 1\}$ . A *Khalimsky interval* is an interval  $[a, b] \cap \mathbb{Z}$  with the topology induced from  $\mathbb{Z}$ .

### 3 Khalimsky $n$ -space

The *Khalimsky plane* is the Cartesian product of two Khalimsky lines, and in general, *Khalimsky  $n$ -space* is  $\mathbb{Z}^n$  with the product topology. A topological space is said to be  $T_{1/2}$  if each singleton set is either open or closed. Clearly a Khalimsky interval is  $T_{1/2}$ . However, the product of two  $T_{1/2}$  spaces need not be  $T_{1/2}$ . In fact Khalimsky  $n$ -space is not  $T_{1/2}$  if  $n \geq 2$ . This is because mixed points (defined below) are neither open nor closed.

Let us examine the structure of the Khalimsky plane a bit more carefully. A point  $x = (x_1, x_2) \in \mathbb{Z}^2$  is open if both coordinates are odd, and closed if both coordinates are even. These points are called *pure*. Points with one odd and one even coordinate are neither closed nor open and are called *mixed*. This definition extends in a natural way to higher dimensions. A point is pure if all its coordinates have the same parity, and mixed otherwise. A part of the Khalimsky plane is shown in Figure 1. A line between two points  $x$  and  $y$  indicate that  $\{x, y\}$  is connected, or in other words that  $x$  and  $y$  are *adjacent*.

We note, for later use, that a diagonal consisting of pure points considered as a subspace of the plane, is homeomorphic to the Khalimsky line, whereas a diagonal consisting of mixed points receives the discrete topology.

Another way to describe the Khalimsky plane is through a subbasis. Let

$$A_2 = \{x \in \mathbb{Z}^2 : \|x\|_\infty \leq 1\} = \{(0, 0), \pm(0, 1), \pm(1, 0), \pm(1, 1), \pm(-1, 1)\}$$

be the smallest neighbourhood of the closed point  $(0, 0)$ . Consider the family of all translates  $A + c$  with  $c_1, c_2 \in 2\mathbb{Z}$ . This family is a subbasis for the topology. The family of all intersections of these sets is a basis, and the

unions of these intersections is the Khalimsky topology in  $\mathbb{Z}^2$ . In general, the topology on Khalimsky  $n$ -space can be constructed in the same way from the sets  $A_n = \{x \in \mathbb{Z}^n : \|x\|_\infty \leq 1\}$ .

## 4 Continuous functions

Let us agree that  $\mathbb{Z}^n$  is equipped with the Khalimsky topology from now on, unless otherwise stated. This makes it meaningful, for example, to talk about continuous functions  $\mathbb{Z} \rightarrow \mathbb{Z}$ . What properties then, does such a function have? First of all, it is necessarily Lipschitz with Lipschitz constant 1. We say that the functions is *Lip-1*. To see this, suppose that somewhere  $|f(n+1) - f(n)| \geq 2$ , then  $f(\{n, n+1\})$  is not connected, in spite of the fact that  $\{n, n+1\}$  is connected. This is impossible if  $f$  is continuous.

However, Lip-1 is not sufficient. Suppose that  $x$  is even and that  $f(x)$  is odd. Then  $U = f(\{x\})$  is open. This implies that  $V = f^{-1}(U)$  is open, and in particular that the smallest neighbourhood  $N(\{x\}) = \{x, x \pm 1\}$  is contained in  $V$ , or in other words that  $f(x \pm 1) = f(x)$ . A similar argument applies when  $f(x)$  is even and  $x$  is odd. Let us first define a binary relation on  $\mathbb{Z}$ . We say that  $a \sim b$  if  $a - b$  is even, i.e., if  $a$  and  $b$  have the same parity.

**Lemma 2** *A function  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  is continuous if and only if*

1.  *$f$  is Lip-1*
2. *For all even  $x$ ,  $f(x) \not\sim x$  implies  $f(x \pm 1) = f(x)$*

*Proof.* That these conditions are necessary is already clear. For the converse, let  $A = \{y-1, y, y+1\}$  where  $y$  is even be any sub-base element. We must show that  $f^{-1}(A)$  is open. If  $x \in f^{-1}(A)$  is odd, then  $\{x\}$  is a neighbourhood of  $x$ . If  $x$  is even, then we have two cases. First, if  $f(x)$  is odd, then condition 2 implies  $f(x \pm 1) = f(x)$  so that  $\{x-1, x, x+1\} \subset f^{-1}(A)$  is a neighbourhood of  $x$ . Second, if  $f(x)$  is even, then  $f(x) = y$ , and the Lip-1 condition implies  $|f(x \pm 1) - y| \leq 1$  so that again  $\{x-1, x, x+1\} \subset f^{-1}(A)$  is a neighbourhood of  $x$ . Thus  $f$  is continuous.  $\square$

*Remark.* In condition 2, we can replace the check for even numbers by the same check for odd numbers, if we prefer. This follows from condition 1.

From this lemma it follows that a continuous function  $\mathbb{Z}^2 \rightarrow \mathbb{Z}$  is Lip-1 if we equip  $\mathbb{Z}^2$  with the  $l^\infty$  metric. For example, if  $f(0,0) = 0$ , then  $f(1,0)$  can be only 0 or  $\pm 1$ . It follows that  $f(1,1) \in \{-2, -1, \dots, 2\}$ , and by checking the parity conditions, one easily excludes the cases  $f(1,1) = \pm 2$ . This result holds in any dimension:

**Proposition 3** *A continuous function  $f: \mathbb{Z}^n \rightarrow \mathbb{Z}$  is Lip-1 with respect to the  $l^\infty$  metric.*

*Proof.* We use induction over the dimension. Suppose therefore that the statement holds in  $\mathbb{Z}^{n-1}$ . Let  $f: \mathbb{Z}^n \rightarrow \mathbb{Z}$  be continuous,  $x' \in \mathbb{Z}^{n-1}$ ,  $x_n \in \mathbb{Z}$  and  $x = (x', x_n) \in \mathbb{Z}^n$ . Assume that  $f(x) = 0$ . We consider the cases  $x_n$  odd and  $x_n$  even. If  $x_n$  is odd, then  $f(x + (0, \dots, 0, 1)) = 0$ , and by the induction hypothesis  $f(x + (1, \dots, 1, 1)) \leq 1$ . On the other hand, it is always true, by the induction hypothesis, that  $f(x + (1, \dots, 1, 0)) \leq 1$ . If  $x_n$  is even and  $f(x + (1, \dots, 1, 0)) = 1$ , then also  $f(x + (1, \dots, 1, 1)) = 1$ . This shows that  $f$  can increase at most 1 if we take a step in every coordinate direction, and by a trivial modification of the argument, also if we step only in some directions. By a similar argument, we can get a lower bound, and hence  $f$  is Lip-1.  $\square$

*Remark.* We will prove a stronger version of this Proposition later (See Proposition 8). However, we need this preliminary result to get there.

Let us say that a function  $f: \mathbb{Z}^n \rightarrow \mathbb{Z}$  is *continuous in each variable separately* or *separately continuous* if for each  $x \in \mathbb{Z}^n$  the  $n$  maps:

$$\mathbb{Z} \rightarrow \mathbb{Z}, x_i \mapsto f(x); x_j \text{ is constant if } i \neq j$$

are continuous. Kiselman has found the following easy, but quite remarkable, theorem. It is stated in two dimensions and proved in a different way in [5].

**Theorem 4**  *$f: \mathbb{Z}^n \rightarrow \mathbb{Z}$  is continuous if and only if  $f$  is separately continuous.*

*Proof.* The only if part is a general topological property. For the other direction, it suffices to check that the inverse image of a subbasis element,  $A = \{y - 1, y, y + 1\}$ , where  $y$  is even, is open. Suppose that  $x \in f^{-1}(A)$ . We show that  $N(x) \subset f^{-1}(A)$ . It is easy to see that

$$N(x) = \left\{ z \in \mathbb{Z}^n; \begin{array}{ll} |x_i - z_i| \leq 1 & \text{if } x_i \text{ is even} \\ z_i = x_i & \text{if } x_i \text{ is odd} \end{array} \right\}$$

Let  $z \in N(x)$ , and  $I = \{i_0, \dots, i_k\}$  be the indices for which  $|x_i - z_i| = 1$ . Let  $x_0, \dots, x_k$  be the sequence of points in  $\mathbb{Z}^n$  such that  $x_0 = x$ ,  $x_k = z$  and

$$x^{j+1} = x^j + (0, 0, \dots, 0, \pm 1, 0, \dots, 0)$$

for  $j = \{0, \dots, k-1\}$  so that  $x^{j+1}$  is one step closer to  $z$  than  $x^j$  in the  $i_j$ :th coordinate direction. Now, if  $f(x)$  is odd, then by separate continuity and Lemma 2 it follows that  $f(x^{j+1}) = f(x^j)$ . In particular  $f(z) = f(x)$  and hence  $z \in f^{-1}(A)$ . If  $f(x)$  is even, it may happen that  $f(x^{j+1}) = f(x^j) \pm 1$  for some index  $j$ . But then  $f(x^{j+1})$  is odd, and must be constant on the remaining elements of the sequence. Therefore  $f(z) \in A$ , and so  $z \in f^{-1}(A)$   $\square$

## 5 Extensions of continuous functions

After the preliminary results of the previous sections, we now turn to the question of extending continuous functions. To be precise, consider a function  $f: A \rightarrow \mathbb{Z}$ , defined on a subset  $A \subset \mathbb{Z}^n$ . We will examine when there is a continuous function  $g: \mathbb{Z}^n \rightarrow \mathbb{Z}$ , such that  $g|_A = f$ .

**Definition 5** Let  $A \subset \mathbb{Z}$ . A **gap** of  $A$  is an ordered pair of integers  $(p, q) \in \mathbb{Z} \times \mathbb{Z}$  such that  $q \geq p + 2$  and  $[p, q] \cap A = \{p, q\}$ .

*Example.* The set  $\{n \in \mathbb{Z} : |n| > 1\}$  has precisely one gap, namely  $(-2, 2)$ .

**Proposition 6** Let  $A \subset \mathbb{Z}$  and  $f: A \rightarrow \mathbb{Z}$  be continuous. Then  $f$  has a continuous extension if and only if for every gap  $(p, q)$  of  $A$  one of the following conditions holds:

1.  $|f(q) - f(p)| < q - p$
2.  $|f(q) - f(p)| = q - p$  and  $p \sim f(p)$ .

*Proof.* There are two possibilities for a point  $x$  not in  $A$ . Either  $x$  is in a gap of  $A$ , or it is not. In the latter case, one of  $x > a$  or  $x < a$  holds for every  $a \in A$ . Let  $(p, q)$  be any gap. We try to extend  $f$  to a function  $g$  that is defined also on the gap. It is clear that the function can jump at most one step at the time. If  $p \not\sim f(p)$ , then it must remain constant the first step  $g(p+1) = f(p)$ , so  $p \sim f(p)$  is clearly necessary when  $|f(q) - f(p)| = q - p$ . It is also sufficient since the conditions implies  $q \sim f(q)$ .

If  $|f(q) - f(p)| < q - p$  it does not matter whether  $p \sim f(p)$ ; the function can always be extended. If  $|f(q) - f(p)| < q - p - 1$  then let  $p_2 = q - 1 - |f(q) - f(p)|$  and define  $g(i) = f(p)$  for  $i = p + 1, p + 2, \dots, p_2$ .

Thus we consider the pair  $(p_2, q)$  where  $|g(q) - g(p_2)| = q - p_2 - 1$ . If  $g(p_2) \not\sim p_2$  then define  $g(p_2 + 1) = g(p_2)$  so that  $(p_2 + 1) \sim g(p_2 + 1)$  and we are in the situation described in condition 2. Similarly for the case  $f(q) \not\sim q$ .

Finally, if  $|f(q) - f(p)| > q - p$  the function is not globally Lip-1 and thus, cannot be extended.

If there is a largest element  $a$  in  $A$ , then  $f$  can always be extended for all  $x > a$  by  $g(x) = f(a)$ , and similarly if there is a smallest element in  $A$ . Since every possibility for an  $x \notin A$  is now covered, we are done.  $\square$

*Remark.* The extension in a gap, if it exists, is unique if  $|f(q) - f(p)| \geq q - p - 1$  and non-unique otherwise.

In the Khalimsky plane, the situation is somewhat more complicated. For example, as we have already noted, a mixed diagonal receives the discrete topology, which makes any function from a mixed diagonal continuous. Of course, most of them are not Lip-1 and thus, cannot be extended.

The mixed diagonal is obviously totally disconnected, but connectedness of the set  $A$  is not sufficient for a continuous function defined on  $A$  to be extendable. If  $A$  has the shape of a horseshoe, a function may fail to be globally Lip-1, even though it is continuous on  $A$ :

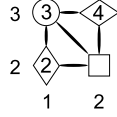


Figure 2: A continuous function on a connected subset of  $\mathbb{Z}^2$ , that is not (globally) Lip-1.

On the other hand, that a function is globally Lip-1 is not sufficient, as already the one dimensional case shows, see Proposition 6. In fact, even connectedness of  $A$  and that  $f$  is globally Lip-1 is not sufficient:

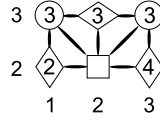


Figure 3: A continuous function defined on a connected subset of  $\mathbb{Z}^2$ , that is globally Lip-1 but still not extendable.

The following definition is basic in the discussion that will follow.

**Definition 7** Let  $A \subset \mathbb{Z}^n$  and  $f: A \rightarrow \mathbb{Z}$  be continuous. Let  $x$  and  $y$  be two distinct points in  $A$ . If one of the following conditions are fulfilled for some  $i = 1, 2, \dots, n$ ,

1.  $|f(x) - f(y)| < |x_i - y_i|$  or
2.  $|f(x) - f(y)| = |x_i - y_i|$  and  $x_i \sim f(x)$ ,

then we say that the function is **strongly Lip-1 with respect to (the points)  $x$  and  $y$** . If the function is strongly Lip-1 with respect to every pair of distinct points in  $A$  then we simply say that  $f$  is **strongly Lip-1**.

*Remark.* If condition 2 is fulfilled for some coordinate direction  $i$  of the points  $x$  and  $y$ , then it follows that also  $y_i \sim f(y)$ , making the relation symmetric.

Intuitively the statement  $f$  is strongly Lip-1 w.r.t.  $x$  and  $y$  can be thought of as that there is enough distance between  $x$  and  $y$  in one coordinate direction for the function to change continuously from  $f(x)$  and  $f(y)$  in this direction.

With this definition at hand, it is now possible to reformulate Proposition 6. It simply reads that a continuous function  $f: A \rightarrow \mathbb{Z}$  is continuously extendable if and only if it is strongly Lip-1. Our goal is to show that this is true in general.

**Proposition 8** *If  $f: \mathbb{Z}^n \rightarrow \mathbb{Z}$  is continuous, then  $f$  is strongly Lip-1.*

*Proof.* Suppose that  $f$  is not strongly Lip-1. Then there are distinct points  $x$  and  $y$  in  $\mathbb{Z}^n$  such that  $f$  is not strongly Lip-1 with respect to  $x$  and  $y$ . Define  $d$  by  $d = |f(x) - f(y)|$ . Since  $x \neq y$ , it is clear that  $d > 0$ . Let  $J$  be an enumeration of the (finite) set of indices for which  $|x_i - y_i| = d$  but  $x_i \not\sim f(x)$ . Let  $k = |J|$ . Define  $x^0 = x$  and for each  $i_j \in J$ ,  $j = 1, 2, \dots, k$ , let  $x^j \in \mathbb{Z}^n$  be a point one step closer to  $y$  in the  $i_j$ :th coordinate direction,

$$x^{j+1} = x^j + (0, 0, \dots, 0, \pm 1, 0, \dots, 0)$$

where the coordinate with  $\pm 1$  is determined by  $i_j$  and the sign by the direction toward  $y$ . If  $J$  happens to be empty, only  $x^0$  is defined, of course. Now we note that for all  $j = 1, 2, \dots, k$ ,  $f(x^{j+1}) = f(x^j)$ . This is because  $f(x^j) \not\sim x_{i_j}^j$  by construction and since  $f$  is necessarily separately continuous. Thus  $f(x^k) = f(x)$ . Also, for all  $i = 1, 2, \dots, n$  it is true that  $|x_i^k - y_i| < d = |f(x^k) - f(y)|$ . This contradicts the fact the  $f$  must be Lip-1 for the  $l^\infty$  metric. (Proposition 3) Therefore  $f$  is strongly Lip-1.  $\square$

To prove the converse, we begin with the following lemmas.

**Lemma 9** *Let  $x$  and  $y$  in  $\mathbb{Z}^n$  be two distinct points and  $f: \{x, y\} \rightarrow \mathbb{Z}$  a function that is strongly Lip-1. Then it is possible, for any point  $p \in \mathbb{Z}^n$ , to extend the function to  $F: \{x, y, p\} \rightarrow \mathbb{Z}$  so that  $F$  is still strongly Lip-1.*

*Proof.* Let  $i$  be the index of a coordinate for which one of the conditions in the definition of strongly Lip-1 functions are fulfilled. Then there is a continuous function  $g: \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $g(x_i) = f(x)$  and  $g(y_i) = f(y)$  by Proposition 6. Define  $h: \mathbb{Z}^n \rightarrow \mathbb{Z}$  by  $h(z) = g(z_i)$ . Obviously  $h$  satisfies the strongly Lip-1 condition in the  $i$ :th coordinate direction for any pair of points, and therefore  $h$  is strongly Lip-1. By construction  $h(x) = g(x_i) = f(x)$  and similarly  $h(y) = f(y)$ . The restriction of  $h$  to  $\{x, y, p\}$  is the desired function.  $\square$

**Lemma 10** *Suppose  $A \subset \mathbb{Z}^n$ , and that  $f: A \rightarrow \mathbb{Z}$  is strongly Lip-1. Then  $f$  can be extended to all of  $\mathbb{Z}^n$  so that the extended function is still strongly Lip-1.*

*Proof.* If  $A$  is the empty set or  $A$  is all of  $\mathbb{Z}^n$  the lemma is trivially true, so we need not consider these cases further. First we show that for any



point where  $f$  is not defined we can define it so that the new function still is strongly Lip-1.

To this end, let  $p$  be any point in  $\mathbb{Z}^n$ , not in  $A$ . For every  $x \in A$  it is possible to extend  $f$  to  $f^x$  defined on  $A \cup \{p\}$  so that the new function is strongly Lip-1 w.r.t  $x$  and  $p$ , for example by letting  $f^x(p) = f(x)$ . It is also clear that there is a minimal (say  $a^x$ ) and a maximal (say  $b^x$ ) value that  $f^x(p)$  can attain if it still is to be strongly Lip-1 w.r.t.  $x$ . It is obvious that  $f^x(p)$  may also attain every value in between  $a^x$  and  $b^x$ . Thus the set of possible values is in fact an interval  $[a^x, b^x] \cap \mathbb{Z}$ . Now define

$$R = \bigcap_{x \in A} [a^x, b^x] \cap \mathbb{Z}$$

If  $R = \emptyset$ , then there is an  $x$  and a  $y$  such that  $b^x < a^y$ . This means that it is impossible to extend  $f$  at  $p$  so that it is strongly Lip-1 with respect to both  $x$  and  $y$ . But this cannot happen according to Lemma 9. Therefore  $R$  cannot be empty. Define  $\tilde{f}(p)$  to be, say, the smallest integer in  $R$  and  $\tilde{f}(x) = f(x)$  if  $x \in A$ . Then  $\tilde{f}: A \cup p \rightarrow \mathbb{Z}$  is still strongly Lip-1.

Now we are in a position to use this result to define the extended function by recursion. If the complement of  $A$  consists of finitely many points, this is easy – just extend the function finitely many times using the result above. Otherwise, let  $(x_j)_{j \in \mathbb{Z}_+}$  be an enumeration of the points in  $\mathbb{Z}^n \setminus A$ . Define  $f_0 = f$  and for  $n = 1, 2, \dots$  let

$$f_{n+1}: A \cup \bigcup_{j=1}^{n+1} \{x_j\} \rightarrow \mathbb{Z}$$

be the extension of  $f_n$  by the point  $x_{n+1}$  as described above.

Finally, define  $g: \mathbb{Z}^n \rightarrow \mathbb{Z}$  by

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ f_n(x) & \text{if } x = x_n \end{cases}$$

Then  $g$  is defined on all of  $\mathbb{Z}^n$ , its restriction to  $A$  is  $f$  and it is strongly Lip-1, so it is the required extension.  $\square$

**Proposition 11** *Suppose  $A \subset \mathbb{Z}^n$ , and that  $f: A \rightarrow \mathbb{Z}$  is strongly Lip-1. Then  $f$  is continuous.*

*Proof* Since we can always extend  $f$  to all of  $\mathbb{Z}^n$  by Lemma 10 and the restriction of a continuous function is continuous, it is sufficient to consider the case  $A = \mathbb{Z}^n$ . But it is clear from the definition of strongly Lip-1 functions, and in view of Lemma 2 that such a function is separately continuous, thus continuous by Theorem 4.  $\square$

We now turn to the main theorem of this section.

**Theorem 12 (Continuous Extensions)** *Let  $A \subset \mathbb{Z}^n$ , and let  $f: A \rightarrow \mathbb{Z}$  be any function. Then  $f$  can be extended to a continuous function on all of  $\mathbb{Z}^n$  if and only if  $f$  is strongly Lip-1.*

*Proof.* That it is necessary that the functions is strongly Lip-1 follows from Proposition 8. For the converse, first use Lemma 10 to find a strongly Lip-1 extension to all of  $\mathbb{Z}^n$  and then Proposition 11 to conclude that this extension is in fact continuous.  $\square$

## 6 Graph-connected sets

We will now use Theorem 12 to prove that a special class of connected sets in the digital plane, that we will call the *graph connected sets*, are precisely the sets on which every continuous function can be extended to a continuous function on the whole plane. This is a digital analogue of the classical Tietze extension theorem.

Let  $I$  be a Khalimsky interval. We call a set  $G$  a *Khalimsky graph* if it is the graph of a continuous function  $\varphi: I \rightarrow \mathbb{Z}$ , i.e.

$$G = \{(x, \varphi(x)) \in \mathbb{Z}^2; x \in I\} \quad (1)$$

or

$$G = \{(\varphi(y), y) \in \mathbb{Z}^2; y \in I\} \quad (2)$$

If  $a, b \in \mathbb{Z}^2$ , we say that  $G$  is a *graph connecting  $a$  and  $b$*  if also  $\varphi(a_1) = a_2$  and  $\varphi(b_1) = b_2$  (if it is a graph of type (1)) or  $\varphi(a_2) = a_1$  and  $\varphi(b_2) = b_1$  (if it is a graph of type (2)).

**Proposition 13** *Let  $a$  and  $b$  in  $\mathbb{Z}^2$  be distinct points. Then there is a graph connecting  $a$  and  $b$  if and only if  $a$  and  $b$  do not lie on the same mixed diagonal.*

*Proof.* Suppose first that  $a$  and  $b$  lie on the same mixed diagonal, and that  $a_1 < b_1$  and  $a_2 < b_2$ . Since  $a$  is mixed, the graph cannot take a diagonal step in  $a$ ; it must either step right or up. But if it steps up, then at some point later, it must step right, and conversely. This is not allowed in a graph. Next, suppose that  $a$  and  $b$  do not lie on the same mixed diagonal. If  $a$  is pure, then we can start by taking diagonal steps toward  $b$  until we have one coordinate equal the corresponding coordinate of  $b$ . Then we step vertically or horizontally until we reach  $b$ . The trip constitutes a graph. If  $a$  is mixed, then by assumption  $b$  is not on the same diagonal as  $a$ , and we can take one horizontal or vertical step toward  $b$ . Then we stand in a pure point, and the construction above can be used.  $\square$

*Remark.* If  $a$  and  $b$  lie on the same pure diagonal, the graph connecting them is unique between  $a$  and  $b$ . It must consist of the diagonal points.

In order to define the graph-connected sets, we first need to handle the mixed diagonals, since there is no graph connecting them. To this end, let  $\text{sgn}: \mathbb{R} \rightarrow \{-1, 0, 1\}$  be the sign function defined by  $\text{sgn}(x) = x/|x|$  if  $x \neq 0$  and  $\text{sgn}(0) = 0$ .

**Definition 14** Suppose  $a$  and  $b$  are distinct points in  $\mathbb{Z}^2$  that lie on the same mixed diagonal. Then the the following set of points

$$\begin{aligned} \mathcal{C}(a, b) = \{ & (a_1, a_2 + \text{sgn}(b_2 - a_2)), (a_1 + \text{sgn}(b_1 - a_1), a_2), \\ & (b_1, b_2 + \text{sgn}(a_2 - b_2)), (b_1 + \text{sgn}(a_1 - b_1), b_2) \} \end{aligned}$$

is called the the **set of connection points of  $a$  and  $b$** .

*Remark.*  $\mathcal{C}(a, b)$  consists of two points if  $\|a - b\|_\infty = 1$  and of four points otherwise.

*Example.* If  $a = (0, 1)$  and  $b = (2, 3)$ , then

$$\mathcal{C}(a, b) = \{(0, 2), (1, 1), (2, 2), (1, 3)\}$$

**Definition 15** A set  $A \in \mathbb{Z}^2$  is called **graph-connected** if for each pair  $a, b$  of distinct points one of the following holds:

1.  $A$  contains a Khalimsky graph connecting  $a$  and  $b$ , or
2.  $a$  and  $b$  do lie on the same mixed diagonal and  $\mathcal{C}(a, b) \subset A$

*Remark.* A graph-connected set is obviously connected. The examples below will show that connected sets are not in general graph connected.

*Example.* Let  $f_i: \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $i = 1, 2, 3, 4$ , be continuous, and suppose that for every  $n \in \mathbb{Z}$ :  $f_1(n) \leq f_2(n)$  and  $f_3(n) \leq f_4(n)$ . Then the set

$$\{(x_1, x_2) \in \mathbb{Z}^2; f_1(x_1) \leq x_2 \leq f_2(x_1) \text{ and } f_3(x_2) \leq x_1 \leq f_4(x_2)\}$$

is graph connected

*Example.* The set of pure points in  $\mathbb{Z}^2$  is graph connected. The set of mixed points is not. (In fact, this subset has the discrete topology, and consists of isolated points.)

*Example.* The set  $A = \{x \in \mathbb{Z}^2; \|x\|_\infty = 1\}$  is not graph connected. There is, for example, no graph connecting the points  $(-1, 0)$  and  $(1, 0)$ . However

the set  $A + (1, 0)$  is graph connected. (The diagonal elements are now on a mixed diagonal.) Thus the translate of a graph-connected set need not be graph-connected, and vice versa. Both these sets are connected, however, so this example shows that the graph connected sets form a proper subset of the connected sets.

**Theorem 16** *Let  $A$  be a subset of  $\mathbb{Z}^2$ . Suppose that every continuous function  $f: A \rightarrow \mathbb{Z}$  can be extended to a continuous function defined on  $\mathbb{Z}^2$ . Then  $A$  is graph-connected.*

*Proof.* We show that if  $A$  is not graph connected, then there is a continuous function  $f: A \rightarrow \mathbb{Z}$  that is not strongly Lip-1, and thus cannot be extended by Theorem 12. There are basically two cases to consider:

Case 1: There are two points  $a$  and  $b$  in  $A$  that are not connected by a graph. Suppose, for definiteness, that  $a_1 \leq b_1$ ,  $a_2 \leq b_2$  and that  $b_1 - a_1 \geq b_2 - a_2$ . It follows that any graph between  $a$  and  $b$  can be described as the image  $(x, \phi(x))$  of the interval  $[a_1, b_1] \cap \mathbb{Z}$ . Let us think of the graph as a travel from the point  $a$  to the point  $b$ . If we are standing in a pure point, then we are free to move in three directions: diagonally up/right, down/right or horizontally right. If we, on the other hand, are standing in a mixed point, we may only go right. Since we are supposed to reach  $b$ , there is an other restriction; we are not allowed to cross the pure diagonals  $\{(b_1 - n, b_2 \pm n); n = 0, 1, 2, \dots\}$  (if  $b$  is pure) or  $\{(b_1 - n - 1, b_2 \pm n); n = 0, 1, 2, \dots\}$  (if  $b$  is mixed). In fact, if we reach one of these *back diagonals*, the only way to get to  $b$  via a graph is to follow the diagonal toward  $b$  (and then take a step right if  $b$  is mixed).

Now, start in  $a$  and try to travel by a graph inside  $A$  to  $b$ . By assumption, this is not possible; we will reach a point  $c$  where it is no longer possible to continue. There are three possibilities. In each case we construct the non-extendable function  $f$ .

Case 1.1:  $c$  is a pure point and  $A \cap M = \emptyset$  where

$$M = \{(c_1 + 1, c_2), (c_1 + 1, c_2 \pm 1)\}$$

If  $c$  is closed, define  $g: \mathbb{Z}^2 \setminus M \rightarrow \mathbb{Z}$  by:

$$g(x) = \begin{cases} 0 & \text{if } x = c \\ 1 & \text{if } x = (c_1, c_2 \pm 1) \\ x_1 - c_1 + 2 & \text{otherwise} \end{cases}$$

(If  $c$  is open, define instead  $g$  by adding 1 everywhere to the function above). It is easy to check that  $g$  is continuous, and so is its restriction to  $A$  called  $f$ . But since  $|f(b) - f(c)| = b_1 - c_1 + 2 = \|b - c\|_\infty + 2$ , it is not Lip-1, and thus not extendable.

Case 1.2:  $c$  is mixed, and  $(c_1 + 1, c_2)$  does not belong to  $A$ . If  $c_1$  is odd, define  $g: \mathbb{Z}^2 \setminus \{(c_1 + 1, c_2)\} \rightarrow \mathbb{Z}$  by:

$$g(x) = \begin{cases} 0 & \text{if } x = c \\ x_1 - c_1 + 1 & \text{otherwise} \end{cases}$$

(If  $c_1$  is even, add again 1 to the above function.)  $g$  is continuous so its restriction  $f$  to  $A$  is continuous. This time  $|f(b) - f(c)| = b_1 - c_1 + 1 = \|b - c\|_\infty + 1$ , and hence  $f$  is not extendable.

Case 1.3: We have reached a back diagonal, and (depending on which diagonal) the point  $(c_1 + 1, c_2 + 1)$  or  $(c_1 + 1, c_2 - 1)$  is not in  $A$ . Let us consider the first case. Then either  $b = (c_1 + n, c_2 + n)$ ,  $n \geq 2$  ( $b$  is pure) or  $b = (c_1 + n + 1, c_2 + n)$ ,  $n \geq 2$  ( $b$  is mixed). In any case, and if  $c$  is closed we define  $g: \mathbb{Z}^2 \setminus \{(c_1 + 1, c_2 + 1)\} \rightarrow \mathbb{Z}$  by:

$$g(x) = \begin{cases} 0 & \text{if } x_1 \leq c_1 \text{ and } x_2 \leq c_2 \\ 1 & \text{if } x_1 = c_1 + 1 \text{ and } x_2 \leq c_2 \\ 1 & \text{if } x_2 = c_2 + 1 \text{ and } x_1 \leq c_1 \\ 2 + \min(x_1 - c_1, x_2 - c_2) & \text{if } x_1 > c_1 \text{ and } x_2 > c_2 \\ 2 & \text{otherwise} \end{cases}$$

As usual, we should add 1 to this function to make it continuous if  $b$  is open. Let  $f$  be the restriction of  $g$  to  $A$ . If  $b$  is mixed, then for some integer  $n \geq 2$

$$|f(c) - f(b)| = f(b) = 2 + \min(b_1 - c_1, b_2 - c_2) = 2 + n = \|c - b\|_\infty + 1$$

and hence it is not Lip-1. If on the other hand  $b$  is pure, then  $f(b) = \|c - b\|_\infty + 2$  and also in this case fails to be Lip-1.

Case 2: For  $a$  and  $b$  on the same mixed diagonal, a connection point is missing. For simplicity, we make the same assumptions on the location of  $a$  and  $b$  as we did in Case 1. Let us also say that it is the point  $(a_1 + 1, a_2)$  that is missing in  $A$ . If  $a_1$  odd, define  $g: \mathbb{Z}^2 \setminus \{(a_1 + 1, a_2)\} \rightarrow \mathbb{Z}$  by:

$$g(x) = \begin{cases} 0 & \text{if } x = a \\ x_2 - a_2 + 1 & \text{otherwise} \end{cases}$$

If  $a_1$  is instead even, we should of course add 1 to the function. As before,  $g$  is continuous, and so is its restriction  $f$  to  $A$ . Also  $|f(a) - f(b)| = \|a - b\|_\infty + 1$ , so that once again  $f$  fails to be extendable. This completes the proof.  $\square$

**Theorem 17** *Let  $A$  be a graph-connected set in  $\mathbb{Z}^2$ , and let  $f: A \rightarrow \mathbb{Z}$  be a continuous function. Then  $f$  can be extended to a continuous function  $g$  on all of  $\mathbb{Z}^2$ . Furthermore,  $g$  can be chosen so that it has the same range as  $f$ .*

*Proof.* First we show that  $f$  is strongly Lip-1. Let  $a$  and  $b$  be distinct in  $A$ , and suppose first that  $a$  and  $b$  are not on the same mixed diagonal. Assume that  $|a_1 - b_1| \geq |a_2 - b_2|$  and that  $a_1 < b_1$ . Let  $I = [a_1, b_1] \cap \mathbb{Z}$  and  $\varphi: I \rightarrow \mathbb{Z}$  be a continuous function such that  $a = (a_1, \varphi(a_1))$  and  $b = (b_1, \varphi(b_1))$ , and that the graph  $\{(x, \varphi(x)) \in \mathbb{Z}^2; x \in I\}$  is contained in  $A$ . The existence of  $\varphi$  follows from the graph connectedness of  $A$  and Proposition 13. But then  $\xi: I \rightarrow \mathbb{Z}$ ,  $x \mapsto f(x, \varphi(x))$  is continuous. By Proposition 8 it is strongly

Lip-1, and since  $\xi(a_1) = f(a)$  and  $\xi(b_1) = f(b)$  and  $\|a - b\|_\infty = b_1 - a_1$  it follows that  $f$  is strongly Lip-1 with respect to  $a$  and  $b$ .

Next, suppose that  $a$  and  $b$  are on the same mixed diagonal. Assume for definiteness that  $a_1 < b_1$  and  $a_1 < b_1$ . Then the connection points  $a + (1, 0)$  and  $a + (0, 1)$  are included in  $A$ . Now,  $a$  is a mixed point by assumption, and therefore  $f$  must attain the value  $f(a)$  also on one of these points; call this point  $c$ . From the previous case, it follows that  $f$  is strongly Lip-1 with respect to  $c$  and  $b$ . But  $c$  is one step closer to  $b$  than  $a$  in one coordinate direction, and since  $f(c) = f(a)$ , we conclude that  $f$  is strongly Lip-1 also with respect to  $a$  and  $b$ .

We have shown that  $f$  is strongly Lip-1, and by Theorem 12 it is extendable to all of  $\mathbb{Z}^2$ . We now prove the assertion about the range. In the extension process it is clear that we can always extend the function at the point  $x$  so that

$$f(x) \in [\min_{p \in A} f(p), \max_{p \in A} f(p)] \cap \mathbb{Z}$$

(Where the interval may be infinite, and assuming that we do this at every point so we do not change the min and max in the extension.) But since  $A$  is graph-connected, and  $f$  must be Lip-1 along the graphs, the range is already this interval, and therefore the extension preserves the range.  $\square$

This theorem has an immediate corollary, that might be of some independent interest:

**Corollary 18** *Let  $A \subset \mathbb{Z}^2$  and  $f: A \rightarrow \mathbb{Z}$  be a function. Suppose that  $G \subset \mathbb{Z}^2$  is graph-connected and that  $A \subset G$ . Then  $f$  can be extended to a continuous function on all of  $\mathbb{Z}^2$  if and only if it can be extended to a continuous function on  $G$*

If  $f$  is defined on relatively large set, and we know from start that  $f$  is continuous there, it might be much easier to check that  $f$  can be extended to a perhaps not so much larger, graph connected set, than to check the strongly Lip-1 condition of Theorem 12.

## 7 Conclusion

We have showed that there is a combined Lipschitz and parity condition on a function that is equivalent with continuity on  $\mathbb{Z}^n$  and in general stronger than continuity on subsets of  $\mathbb{Z}^n$ . Since this condition is global in its nature, while continuity is a local property, it is useful for checking extensibility of functions given on subsets. Using this condition, we have then characterized all subsets of the digital plane with the property that every continuous function defined there can be extended to the whole plane. The criterion turned out to be a special type of connectedness; given two points in the

subset  $A$  of the digital plane, there must be a path in  $A$  between the points which is short enough to ensure that the function values at the points cannot differ too much. In particular there cannot be any holes in  $A$  containing pure points; it is possible to construct a continuous function on the complement of a pure point that cannot be extended.

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## References

- [1] P. Alexandroff, Diskrete Räume, Mat. Sb. 2 (44), (1937), 501–519.
- [2] N. Bourbaki, Topologie générale. 3rd edition. Hermann, Paris (1961)
- [3] E. Khalimsky, Topological structures in computer science. Journal of Applied Math. and Simulation Vol. 1 1, (1987), 25–40.
- [4] E. Khalimsky; R. Kopperman and P. R. Meyer, Computer graphics and connected topologies on finite ordered sets. Topol. Appl. 36 (1990), 1–17.
- [5] C. O. Kiselman, Digital geometry and mathematical morphology, Lecture Notes, Uppsala University, Department of Mathematics, (2002). Available at [www.math.uu.se/~kiselman](http://www.math.uu.se/~kiselman)
- [6] R. E. Stong, Finite topological spaces Trans. Amer. Math. Soc. 123, (1966), 325–340.