Sturm-Liouville problems

1. Describe the eigenvalues and eigenfunctions of the Sturm-Liouville problem

\[ \begin{align*}
  y'' + \lambda y &= 0, & 0 < x < 1, \\
  y'(0) &= 0, & y(1) + y'(1) = 0.
\end{align*} \]

Obtain an asymptotic approximation of large eigenvalues.

2. Determine the eigenvalues and eigenfunctions of the periodic Sturm-Liouville problem

\[ \begin{align*}
  y'' + \lambda y &= 0, & 0 < x < L, \\
  y(0) &= y(L), & y'(0) = y'(L).
\end{align*} \]

3. Transverse vibrations of a drumhead \( D = \{(x, y) : x^2 + y^2 < a^2\} \) are governed by the two-dimensional wave equation. Writing the Laplacian in polar coordinates, the problem becomes:

\[ \begin{align*}
  c^{-2}u_{tt} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta \theta}, & \text{for } 0 < r < a, \\
  u &= 0, & \text{when } r = a, \\
  u, u_t &= \text{given functions when } t = 0.
\end{align*} \]

(a) Separating the variables as \( u(r, \theta, t) = T(t)R(r)\Theta(\theta) \), show that

\[ \begin{align*}
  T'' + \lambda c^2 T &= 0, \\
  \Theta'' + \gamma \Theta &= 0, \\
  R'' + \frac{1}{r}R' + \left(\lambda - \frac{\gamma}{r^2}\right)R &= 0,
\end{align*} \]

for some constants \( \lambda, \gamma \).

(b) Since \( \Theta \) must satisfy periodic boundary conditions, use the result of Problem 2 to deduce that \( \gamma = n^2 \), \( n = 0, 1, 2, ... \), so that

\[ R'' + \frac{1}{r}R' + \left(\lambda - \frac{n^2}{r^2}\right)R = 0, \quad \text{where } n = 0, 1, 2, .... \]

(c) Let \( \rho = \sqrt{\lambda r} \) and put \( w(\rho) = R(r) \). Show that \( w \) satisfies the Bessel equation of order \( n \):

\[ \rho^2 w'' + \rho w' + (\rho^2 - n^2)w = 0. \]

Bounded solutions \( w = w_n \) are given by \( w_n(\rho) = J_n(\rho) \), the Bessel functions of order \( n \).

(d) The function \( J_n \) has an infinite number of positive zeros \( \alpha_{n,m} \) such that

\[ 0 < \alpha_{n,1} < \alpha_{n,2} < \alpha_{n,3} < \cdots. \]

Show that the boundary condition at \( r = a \) implies that the eigenvalues \( \lambda = \lambda_{n,m} \) are given by

\[ \lambda_{n,m} = \left(\frac{\alpha_{n,m}}{a}\right)^2. \]

It follows that the natural frequencies that the drum can produce are given by \( c\alpha_{n,m}/a \).
Answers or hints:

1. The eigenvalues and eigenfunctions are $\lambda_n = \omega_n^2$ respectively $y_n(x) = \cos \omega_n x$, $n = 0, 1, 2, ...$, where $\omega_n$ are the positive solutions of the equation $\tan x = 1/x$. By drawing the graphs of these functions, one sees that $\omega_n \approx n \pi$, so that $\lambda_n \approx n^2 \pi^2$, for large values of $n$.

2. The eigenvalues and eigenfunctions are:

   $\lambda_0 = 0$, $y_0(x) = 1$,

   $\lambda_n = \left(\frac{2n\pi}{L}\right)^2$, $y_n(x) = A_n \cos \frac{2n\pi x}{L} + B_n \sin \frac{2n\pi x}{L}$, for $n = 1, 2, 3, ...$.