Topics in partial differential equations: proposals

All participants in the courses *Theory of differential equations* and *Partial differential equations D*, especially those taking the latter course, are invited to present orally some chapter from the theory or applications of partial differential equations.

The themes proposed below either give some motivation from technology or nature why we should know more about differential equations or else give glimpses of developments beyond the core curriculum.

1. Solitons

Review Chapter 1 and the beginning of Chapter 2 of the book by Alan C. Newell, *Solitons in mathematics and physics.* Philadelphia, PA: Society for Industrial and Applied Mathematics, 1985.

2. Solitons

Study the beginning of M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering,* Cambridge University Press, 1992, and review the sections *Physical derivation of the Kadomtsev–Petviashvili equation* (pp. 8–12); *Travelling wave solution of the Korteweg–de Vries equation* (pp. 13–17); *The discovery of the soliton* (pp. 17–19).

3. Solitons and the Korteweg–de Vries equation

A wave which solves the KdV-equation $u_t + (1+u)u_x + u_{xxx} = 0$ is kept together, while a wave which solves its linearization $u_t + u_x + u_{xxx} = 0$ is spread out and vanishes. Explain how the nonlinearity works to conserve the wave. Reference: P. G. Drazin, *Solitons,* pp. 1–17. Cambridge University Press, 1983.

Note: Proposals 1, 2, and 3 could be considered by two students working together.

4. Minimal surfaces

Review the introduction and the beginning of Chapter 1 of Enrico Giusti, *Minimal surfaces and functions of bounded variation,* Birkhäuser, 1984.

5. Minimal surfaces

Review the first part and the beginning of the second part of the article by Johannes C. C. Nitsche, *Minimal surfaces and partial differential equations.* In: *Studies in partial differential equations* (Ed. Walter Littman), pp. 69–142. Math. Assoc. of America, 1982.

Note: Proposals 4 and 5 could be considered by two students working together.

6. Equations from fluid mechanics

Study the beginning of W. F. Ames, *Nonlinear partial differential equations in engineering,* Academic Press, 1965, and review all of Chapter 1, especially *Equations from diffusion theory, Equations from fluid mechanics, Equations from solid mechanics.* Probably some other work has to be consulted to find careful motivations for all these equations.

7. The Navier–Stokes equation and related equations

Study the beginning of the book by Emmanuele DiBenedetto, *Partial differential equations,* Birkhäuser, 1995, and review Chapter 0, sections 4–8. Problably some other work should be studied to obtain good explanations of these equations.

Note: Proposals 6 and 7 could be considered by two students working together.

8. Equations from combustion theory

Study the beginning of the book by Jerrold Bebernes and David Eberly, *Mathematical Problems from Combustion Theory,* Springer-Verlag, 1989, and review Chapter 1 and section 2.1.

9. Laws of fluid mechanics

Review pages 194–202 or more in S. N. Antontsev, J. I. Diaz, and S. Shmarev, *Energy Methods for Free Boundary Problems.* Boston, Basel, Berlin: Birkhäuser, 2002.

10. Free boundary value problems

Study the beginning of J. I. Díaz, *Nonlinear partial differential equations and free boundary problems, vol. I: Elliptic equations,* Pitman, 1985, and review the introduction and the beginning of chapter 1.

Note: Cf. # 6, #9.

11. Wavelet transforms

Review the essay by Ingrid Daubechies, *Wavelet transforms and orthonormal wavelet bases* in the volume *Different perspectives on wavelets* (Ed. Ingrid Daubechies), Proc. Symp. Appl. Math., vol. 47, Amer. Math. Soc., 1993.

12. The Laplace transformation

Review the methods in several variables in Jean Hladnik, *La transformation de Laplace à plusieurs variables,* Masson, 1969; especially *Historique* and the beginning of Chapter V.

13. Shock waves, I

A solution of the equation $u_t + F(u)u_x = 0$ with inital values $u(x, 0) = g(x)$, where g has compact support and is not identically zero, must have a finite life span for certain choices of the function F. Which are the conditions on F? What does the singularity look like? Give examples from traffic flow etc. Reference: G. B. Whitham, *Lectures on wave propagation,* Springer-Verlag, 1979 [Geobiblioteket].

14. Shock waves, II

A solution $u \in C^1$ to the equation $u_t + uu_x = 0$ (often called Burgers' equvation) with initial values $u(x, 0) = g(x)$, where g has compact support and is not identically zero, must have a finite life span. Describe when the solution must cease to exist. However, by writing the equation as

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

one can accept solutions which are only in L^2: the defining equation becomes

$$-\iint_{t>0}\left(u\frac{\partial \varphi}{\partial t} + \frac{1}{2}u^2\frac{\partial \varphi}{\partial x}\right)dxdt = \int g(x)\varphi(0, x)dx, \qquad \varphi \in C_0^\infty(\mathbf{R}^2).$$

Show that if u is such a solution of class C^1 on both sides of a curve having the equation $x = \gamma(t)$, then $\gamma'(t) = \frac{1}{2}(u_+ + u_-)$, i.e., the shock moves with a velocity which is the mean values of the velocities on both sides. Show also that one can get these weak solutions by solving a diffusion equation. Reference: Lars Hörmander, *Nonlinear hyperbolic differential equations,* pp. 20–24, Lund University, 1988 [private].

15. Nonlinear wave equations
Review Chapter 1, the main parts of the first section, possibly also some parts of the second section, of the book by Fritz John, *Nonlinear wave equations, formation of singularities,* Univ. Lecture Series, American Math. Soc., 1990.

Note: Proposals 13, 14, and 15 could probably be combined in some way.

16. Soft solutions
Review the paper by M. H. Protter, *Difference methods of soft solutions.* In: *Nonlinear partial differential equations* (Ed. W. F. Ames), pp. 161–170, Academic Press, 1967.

17. Similarity analysis
Review the paper by Arthur G. Hansen, *Generalized similarity analysis of partial differential equations.* In: *Nonlinear partial differential equations* (Ed. W. F. Ames), pp. 1–17, Academic Press, 1967.

18. Nonlinear diffusion
Review the article by D. G. Aronson, *The asymptotic speed of propagation of a simple epidemic.* In: *Nonlinear diffusion* (Ed. W. E. Fitzgibbon (III) and H. F. Walker), pp. 1–23, Pitman, 1977.

19. Ill-posed problems
Review Chapter 1 and Chapter II, Sections 1 and 2 of the book by M. M. Lavrentiev, *Some improperly posed problems of mathematical physics.* Springer, 1967.

20. Ill-posed problems
Review pp. 1–25 of the book by Andrey N. Tikhonov and Vasiley Y. Arsenin, *Solution of ill-posed problems.* John Wiley & Sons, 1977.

Note: Proposals 19 and 20 could be considered by two students working together.

21. Ill-posed probelms
Review sections 1.1 and 1.4, Radon problem as an example of an ill-posed problem, in M. M. Lavrent'ev, S. M. Zerkal, and O. E. Trofimov, *Computer Modelling in Tomography and Ill-Posed Problems.* Utrecht, Boston, Köln, Tokyo: VSP, 2001.

22. Ill-posed problems
Review section 2, pp. 557–568, in M. M. Lavrent'ev & L. Ja. Savel'ev, *Teorija operatorov i nekorrektnye zadači.* Novosibirsk: Izdatel'stvo Instituta Matematiki, 1999.

23. Representation formulas for the solutions to $\triangle u - u = 0$
Review the article by Luis A. Caffarelli and Walter Littman, *Representation formulas for the solutions to* $\triangle u - u = 0$ *in* $\mathbf{R}^n$. In: *Studies in partial differential equations* (Ed. Walter Littman), pp. 249–263. Math. Assoc. of America, 1982.

24. Parametrices for elliptic equations

Review Chapter I, §1, of the book by François Treves, *Introduction to pseudodifferential and Fourier integral operators,* Plenum Press, 1980.

25. Leray–Schauder fixed point theorem with applications to elliptic equations

Review sections 10.1–10.3 of the book by David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order,* Springer-Verlag, 1977.

26. Tikhonov's example

There can be nonuniqueness in the Cauchy problem for the heat equation. Review the construction of Tikhonov[1] solutions in section 7.1 in Fritz John, *Partial differential equations,* Springer-Verlag, 1991.

27. Differential forms

The equation $ddu = 0$. Solvability of the equation $du = f$. The operations grad, curl (= rot) and div as special cases of d. How Maxwell's field equations look expressed with the help of differential forms. Other examples from physics. Reference: Harley Flanders, *Differential forms with applications to the physical sciences.* Academic Press, 1963 [beu 53].

28. Pfaffian differential equvations

Transport of differential forms: if $\varphi\colon Y \to X$ is a smooth mapping, a differential form on X can be pulled back to Y; the pull-back of u is written φ^*u. It is determined by the requirements that $\varphi^*u = u \circ \varphi$ if u is a function, $\varphi^*(du) = d(\varphi^*u)$ and $\varphi^*(u \wedge v) = (\varphi^*u) \wedge (\varphi^*v)$. A *Pfaffian differential equation* is of the form $\varphi^*u = 0$, where $u = \sum u_j dx_j$ is of degree one on X, and one tries to find φ from a space one dimenion lower. The problems are rather different when Y is of dimension 1 and of dimension > 1, respectively (curves and surfaces, respectively). An integrating factor is a function g such that $gu = dh$ for some functions h. Hence

$$(g \circ \varphi)\varphi^*u = \varphi^*(gu) = \varphi^*(dh) = d(h \circ \varphi) = dC = 0$$

if φ maps Y into the level surfaces of h, which defines a solution where $g \circ \varphi \neq 0$. Reference: Ian N. Sneddon, *Elements of partial differential equations,* McGraw-Hill, 1957 [disappered; private] (or perhaps some other book).

29. Geometric interpretation of the Hamilton–Jacobi equation

In mechanics one studies the function

$$S(y,t) = \int L(\tau, x(\tau), \dot{x}(\tau))d\tau$$

taken along curves x from 0 to $y = x(t)$ such that the integral is stationary. There is a connection between such curves and the level surfaces of S. They can be interpreted as light rays and wave fronts, respectively. Describe this interpretation and show that the surfaces are equidistant in a certain sense. Reference: Hanno Rund, *The Hamilton–Jacobi theory in the calculus of variation,* Van Nostrand, 1966 [beu 49].

---

[1] Andreĭ Nikolaevič Tikhonov, born 1906, the coathor of the book mentioned in # 19. John writes his name Tychonoff.

30. Fundamental solutions; singular support

A function E is called a *fundamental solution at the point* a to the differential operator $P = P(x, D) = \sum c_\alpha(x) D^\alpha$ if $PE = \delta_a$. Show that if one knows a fundamental solution to P one can solve the equation $Pu = f$ for many f. The singular support of a function is the smallest closed set outside which the function is smooth (C^∞). Show that if $\operatorname{sing\,supp} E = \{a\}$, then all solutions to $Pu = f$ are smooth wherever f is. Reference: François Treves, *Basic Linear Partial Differential Equations*. Academic Press, 1975 (pages 17–21 for constant coefficients).

31. Poisson's formula for a disk and a half-plane

With the help of Green's function for the Laplace operator one can give an explicit representation of a harmonic function in a disk or a half-plane as an integral over the boundary (and correspondingly in higher dimensions). Reference: R. Courant och D. Hilbert, *Methods of mathematical physics*, Chapter IV. Interscience Publishers, 1962 (or some other book)

32. Green's function and compactness

If G is Green's function for a domain Ω, we define an integral operator T by putting

$$(Tf)(x) = \int_\Omega G(x, y) f(y) dy.$$

Show that this defines a continuous operator $T\colon L^2(\Omega) \to L^2(\Omega)$ if Ω is bounded. Show that T is even compact if the dimension is at most 3. Reference: Günter Hellwig, *Differentialoperatoren der mathematischen Physik*. Berlin, Göttingen, Heidelberg: Springer, 1964 [beu 47].

33. Compact operators and spectrum

Prove that a compact operator has a discrete spectrum and that every nonzero eigenvalue is of finite multiplicity. Reference: Günter Hellwig, *Differentialoperatoren der mathematischen Physik*. Berlin. Göttingen, Heidelberg: Springer, 1964 [beu 47].

34. Variational characterization of eigenvalues. Rayleigh–Ritz' method

For a compact symmetric operator the eigenvalues can be obtained using a variational principle: the first eigenvalue is

$$\lambda_1 = \sup_x \big(\langle Ax, x\rangle;\ \langle x, x\rangle = 1\big),$$

while other eigenvalues can be obtained by taking the supremum over suitable subspaces. Reference: S. H. Gould, *Variational methods for eigenvalue problems*. University of Toronto Press, 1966 (pages 70–79) [beu 49].

35. Dirichlet's principle

The solution to Dirichlet's problem

$$\triangle u = 0 \text{ in } \Omega, \quad u = f \text{ on } \partial\Omega$$

can be obtained by minimizing the functional

$$\inf_{u \in D} \int_\Omega \|\operatorname{grad} u(x)\|^2 dx,$$

where D is the set of all $u \in C^1(\Omega)$ with boundary value equal to f. Prove that any extremal function to this functional is harmonic. Reference: R. Courant, *Dirichlet's principle, conformal mapping, and minimal surfaces,* Springer 1977 [Ångström].

36. The Lax–Milgram theorem

The Hilbert space method of solving differential equations is based on the idea of representing equations as conditions on linear functionals. These can then be written as inner products by an element of the Hilbert space (the Riesz representation theorem). Lax and Milgram showed that one can also do this for more general bilinear forms than the inner product. Show this and how one can use it. Reference: David Gilbarg and Neil S. Trudinger, *Elliptic differential equations of second order,* Springer, 1977.

37. Fundamental solutions to the heat equation

Derive them. Reference: François Treves, *Basic linear partial differential equations,* Academic Press, 1975.

38. Maximum principles for parabolic equations

Formulate and prove some maximum principle. Reference: Murray H. Protter och Hans F. Weinberger, *Maximum principles in differential equations,* Springer 1984 [Ångström].

39. Why is the world three-dimensional?

Review the article by Tom Morley, A simple proof that the world is three-dimensional, *SIAM Review* **27** (1985), 69–71.

40. Fundamental solution to the wave equation

Derive them. Reference: François Treves, *Basic linear partial differential equations,* Academic Press, 1975.

41. Huygens' principle

What is Huygens' principle? Give a physical and mathematical description of it. Reference: Richard Courant and David Hilbert, *Methods of mathematical physics,* Interscience Publishers, 1962.

42. The Schrödinger equationen and its relation to the Korteweg–de Vries equation

Discuss the relation. Reference: Peter D. Lax, Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* **21** (1968), 467–490.

Note: Cf. proposals 1, 2, 3.

Christer Kiselman, Uppsala University, Department of Mathematics, P. O. Box 480, SE-751 06 Uppsala, Sweden
Telephone: 018-4713216 (office); 018-300708 (home); 0708-870708 (cellular)
Electronic mail: `kiselman@math.uu.se` Fax: 018-4713201