UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Theory of Partial Differential Equations, 5 hp
Fall 2010

Project Suggestions

All participants in the course *Theory of Partial Differential Equations* are expected to make an oral presentation from the theory or applications of partial differential equations. The oral presentations are expected to last for around 30 minutes.

The purpose of these presentations is to give participants in the course an overview of interesting material which is not covered in the course curriculum. The projects below are of two kinds. Some have an orientational purpose. I am thinking here specifically of the ones concerning the KdV equation - which initiated of one of the great mathematical adventures of the past decades - namely that of the modern theory of integrable dynamical systems. Others intend to survey general techniques which are needed in many branches of Analysis. For example, a few projects concern methods for obtaining asymptotics such as Laplace's method, the method of stationary phase and the method of steepest descent. Applications to PDE:s discussed during the course are given: to diffusive regularizations of shock waves, to the long-time behavior of linear dispersive waves and to the long-time behavior of diffusion processes. The projects concerning fixed point theorems are also of this kind. Applications to nonlinear PDE:s such as the stationary Navier-Stokes equation are given.

Note: The topics listed below are just suggestions. Feel free to choose different ones.
When you have decided for a project, please send me an E-mail with your project title. In case more than one participant chooses the same topic, the one that E-mailed me first will have priority. Contact me if you want copies of the references suggested below.
Final remark: The presentations are to be of orientational character. Focus on briefly explaining the results and overall ideas. Do *not* dwell too deep into proofs. Thirty minutes passes quickly!

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The first two projects could be considered by two participants working together:

• **Direct and inverse scattering theory for the Schrödinger operator.**
Given a potential explain how to compute the scattering data - eigenvalues, norming constants and the reflection coefficient. This is the forward scattering problem. Next explain how the potential can be reconstructed from the scattering data by solving the Gelfand-Levitan-Marchenko equation. This is the inverse scattering problem.
References:
· Drazin [1], Chapter 3, pages 39-63.
· Lax [2], Chapter II, pages 84-94.
· Palais [3], pages 34-40.

• **The KdV equation.**
Explain how the KdV equation can be solved using the above mentioned scattering theory for the Schrödinger operator.
References:
· Drazin [1], Chapter 4, pages 64-88.
· Lax [2], Chapter III, pages 95-101.
· Palais [3], pages 31-41.

The following three projects discuss asymptotic methods:

• **Laplace's methods for asymptotic expansions of integrals.**
Application: Weakly diffusive regularization of shock waves.
Review the goal and results concerning Laplace's method relevant for above given application. Discuss how the Cole-Hopf transformation linearizes Burger's equation and how Laplace's method can be applied to compute the zero-diffusion limit.
References:
· Miller [4], pages 61-87.

• **The method of stationary phase for asymptotic expansions of osc. integrals.**
Application: Long-time asymptotic behavior of linear dispersive waves.
Review the goal and results concerning the method of stationary phase relevant for above given application. Discuss how it applies to compute the long-time behavior of solutions of linear dispersive wave equations.
References:
· Miller [4], pages 149-171.

• **The method of steepest descent for asymptotic expansions of integrals.**
Application: Long-time asymptotic behavior of diffusion processes.
Review the basic idea concerning the method of steepest descent. Show how it applies to compute the long-time behavior of diffusion processes.
References:
· Miller [4], pages 95-116.

The following two projects concern fixed point methods:

• **The contraction mapping principle.**
Application: Existence of a unique solution to the mean curvature equation.
Discuss the contraction mapping principle and how it implies the Picard and inverse function theorems. Give also an idea of how the latter applies to show existence of a unique solution of the mean curvature equation.
References:
· MwOwen [5], pages 237-243.

• **The Leray-Schauder fixed point theorem.**
Application: Existence of solutions to the stationary Navier-Stokes equation.
Discuss the Leray-Schauder fixed point theorem. Sketch how it applies to prove the existence of a solution to the stationary Navier-Stokes equation.
References:
· MwOwen [5], pages 277-282.

I finally suggest two projects related to numerical analysis:

• **Numerical methods.**
Give an outline of the finite difference and finite element methods.
References:
· Pinchover [7], pages 309-334.

• **Existence of a unique weak solution of the heat equation.**
Sketch a proof of the existence of a unique weak solution of the heat equation using Galerkin's method.
References:
· Evans [6], pages 349-358.

References

[1] P.G. Drazin and R.S Johnson, Solitons: an introduction, Cambridge University Press, 1989.

[2] Peter Lax, Outline of a theory of the KdV equation, Lecture Notes in Mathematics, 1996, Vol. 1640, Springer-Verlag, 1996.

[3] R.S. Palais, The symmetries of solitons, Bull. Amer. Math. Soc. 34, 1997.

[4] Peter Miller, Applied Asymptotic Analysis, Graduate studies in Mathematics, Vol. 75, AMS Publications, Providence, 2006.

[5] Robert McOwen, Partial Differential Equations - Methods and applications, Pearson Education, 2003.

[6] Lawrence Evans, Partial Differential Equations, Graduate studies in Mathematics, Vol. 19, AMS Publications, Providence, 1998.

[7] Yehuda Pinchover and Jacob Rubinstein, An Introduction to Partial Differential Equations, Cambridge University Press, 2005.