Abstract

Vector spaces are one of the most important structures in mathematics. These may be generalized into combinatorial pregeometries by looking at how independence work. In model theory the combinatorial pregeometries arise in a suprising way where it is very important what constants you have in the language. This will in turn make it possible to talk about independence and other vector space properties inside some logical structures and make it possible to prove theorems in both areas.

Pregeometries

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Phd-seminar 11 December

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Pregeometries

Logic

Algebraic Closure

Vectorspaces

$$\mathbb{C}^3 \text{ over } \mathbb{C} \qquad\qquad \bigoplus_{i=1}^{\omega} \mathbb{R} \text{ over } \mathbb{R}$$

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Vectorspaces

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with addition of vectors and scalar multiplication.

Generalized to modules, by replacing the *field* requirement with *ring*.
Ex: $\mathbb{Z}^3$ over $\mathbb{Z}$

Independence:

Described through the span:

$$v \in Span(v_1, ..., v_n) \text{ then } v \text{ is dependent of } v_1, ..., v_n$$

$$v \notin Span(v_1, ..., v_n) \text{ then } v \text{ is independent of } v_1, ..., v_n$$

Generalized to a pregeometry.

Pregeometries

A pregeometry (or matroid) $G = (V, cl)$ consists of a set V and a setfunction $cl : \mathcal{P}(V) \to \mathcal{P}(V)$ satsifying the following for each $X, Y \subseteq V$.

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Theorem

If (V,cl) is a pregeometry then for each $X \subseteq V$, $cl(X) = cl(cl(X))$

Examples

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The examples 2 and 3 are pregeometries, but not geometries.

- V vectors in a vector space and $cl(\emptyset) = \emptyset$. If $X \subseteq V$, choose Y to be the least closed subspace of V such that there is $v_0 \in V$ such that $X \subseteq v_0 + Y$. Then let $cl(X) = v_0 + Y$. *(The affine geometry)*

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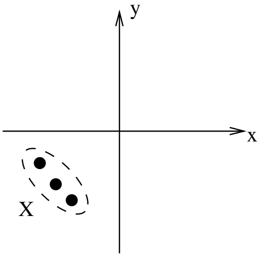

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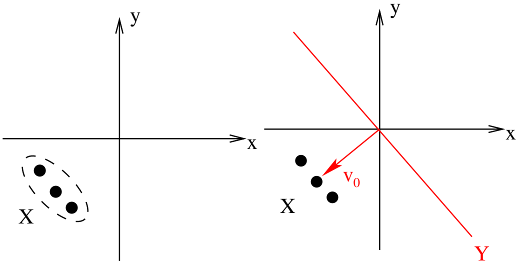

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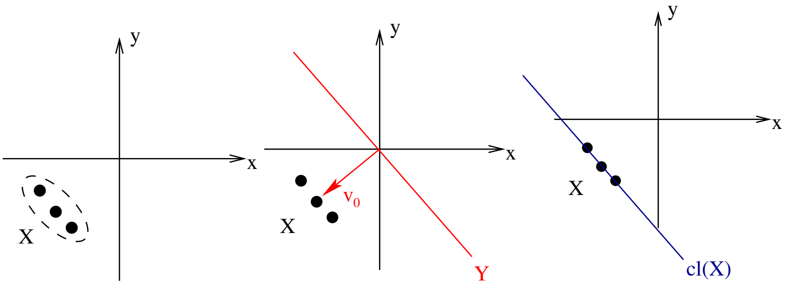

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  (*Projective geometry*)

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($\textit{Projective geometry}$)

- F is an algebaically closed field. For $X \subseteq F$ let $cl(X)$ be the smallest algebraically closed subfield of F which contain X. (so for $\mathbb{C}$, $cl(\{15\})$ would be the algebraic numbers)

Languages

Vocabularies consists of constants, function symbols and relation symbols:

Ex:

Rings: $<0, 1, +, -, \cdot>$

Groups: $<1, (\ \ )^{-1}, \cdot>$

Graphs: $<E>$

Linear orders: $<\leq>$

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Use these to build formulas:

Ex:

$\forall x(0 + x = 1 \cdot x)$

$\forall y(y^{-1} = y \rightarrow y = 1)$

$\exists y \exists z(E(z, x) \wedge E(x, y))$

$\forall x(\neg x \leq x \vee \exists y(y \leq x))$

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The following are not formulas (in the Ring vocabulary):

$\exists x(x + 2 = y)$ $\quad$ $x - \pi = x \cdot x$ $\quad$ $5 \neq x \rightarrow x = 4$

Structures

Add a universe and interpret the vocabulary.

Ex: Linear orders: $<\leq>$

$\mathcal{M} =< \mathbb{N}, \leq >$ where $\leq$ is as we think on $\mathbb{N}$.

$\mathcal{N} =< \mathbb{Z}, \leq >$ where $\leq$ is as we think on $\mathbb{Z}$.

$\mathcal{A} =< \mathbb{C}, \leq >$ where $\leq$ order numbers lexicographic i.e.

$a + bi \leq c + di$ iff $a < c$ or $(a = c$ and $b \leq d)$.

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Add a universe and interpret the vocabulary.
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Notice:
$\mathcal{A} \models \exists x (\pi + i \leq x)$, but this is not a formula in the laguage.

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Ex: Rings: $< 0, 1, +, -, \cdot >$.
$\mathcal{M} =< \mathbb{Z}, 0, 1, +, -, \cdot >$ as we think.
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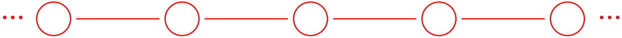
So $\varphi(x)$ describes a unique element in $\mathcal{M}$, while an infinite amount in $\mathcal{N}$.

Ex: Graphs: $< E >$.

Let $\mathcal{M} =< \mathbb{Z}, E >$ where E points out the next and previous number.



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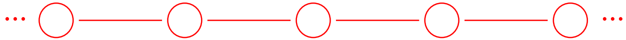

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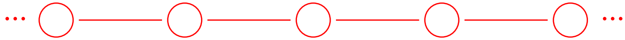

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Theorem

For each formula $\varphi(x,y)$ from the language it is either so that $\mathcal{M} \models \varphi(a,b)$ for an infinite amount of pairs $a, b \in \mathbb{Z}$ or $\mathcal{M} \not\models \varphi(a,b)$ for each pair $a, b \in \mathbb{Z}$

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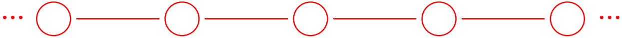

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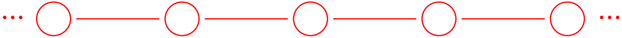

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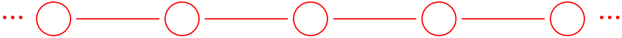

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We can, for each $n \in \mathbb{Z}$, create $\varphi(x)$ s.t. $\varphi(x)$ is only true for n and $-n$.

Adding constants may give us more precise formulas.

Ex: Vectorspaces over $\mathbb{R}$: $< 0, \{f_r\}_{r\in\mathbb{R}}, + >$.
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$v_1 = (1, 0, 0, 0), v_2 = (0, 3, 1, 0)$ to the vocabulary.
Then for each vector $v \in Span(v_1, v_2)$ there is a formula $\varphi_v(x)$ such that only v makes φ_v true. $\varphi_v(x)$ is

$$f_\alpha(v_1) + f_\beta(v_2) = x$$

For appropriate $\alpha, \beta \in \mathbb{R}$.

Study the finitely describeable elements. Define for a structure $\mathcal{M}$ and $X \subseteq \mathcal{M}$:

$$acl(X) = \{a \in \mathcal{M} : \text{there is } \varphi(x) \text{ formula possibly with constants from } X \text{ such that } \mathcal{M} \models \varphi(a) \text{ and only finitely many elements in } \mathcal{M} \text{ make } \varphi(x) \text{ true}\}$$

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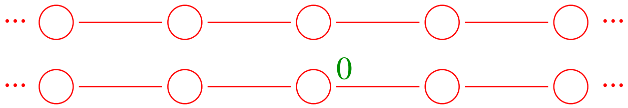
$$\text{in } \mathcal{M} \text{ make } \varphi(x) \text{ true}\}$$

Ex: $\mathcal{M} =< \mathbb{Z}, \leq >$ then $acl(\emptyset) = \emptyset$, but $acl(X) = \mathbb{Z}$ for each $\emptyset \neq X \subseteq \mathbb{Z}$.



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Ex: $\mathcal{M} = < \mathbb{Z}, 0, 1, +, -, \cdot >$ then for each $X \subseteq \mathbb{Z}$, $acl(X) = \mathbb{Z}$, since $1 + 1 + ... + 1 = x$ or $0 - 1 - 1 - ... - 1 = x$.

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Ex: $\mathcal{M} =< \mathbb{C}, 0, 1, +, -, \cdot >$ then $acl(\emptyset) = \mathbb{A}$ and for each X,$acl(X) =$ "The least algebraically closed subfield of $\mathbb{C}$ containing X".

Ex: $\mathcal{M} =< \mathbb{Q}, \leq >$ then $acl(X) = X$ for each $X \subseteq \mathbb{Q}$.

Ex: $\mathcal{M} =< \mathbb{Z}, 0, 1, +, -, \cdot >$ then for each $X \subseteq \mathbb{Z}$, $acl(X) = \mathbb{Z}$, since $1 + 1 + ... + 1 = x$ or $0 - 1 - 1 - ... - 1 = x$.

Ex: $\mathcal{M} =< \mathbb{Q}, 0, 1, +, -, \cdot >$ then for each $X \subseteq, \mathbb{Q}$ $acl(X) = \mathbb{Q}$, since $x + x + x = 1 + 1$ describes $\frac{2}{3}$ etc.

Ex: $\mathcal{M} =< \mathbb{C}, 0, 1, +, -, \cdot >$ then $acl(\emptyset) = \mathbb{A}$ and for each X,$acl(X) =$ "The least algebraically closed subfield of $\mathbb{C}$ containing X".

Ex: $\mathcal{M} =< \mathbb{R}^4, 0, \{f_r\}_{r \in \mathbb{R}}, + >$ as a vectorspace. Then $acl(X) = Span(X)$ for each $X \subseteq \mathbb{R}^4$.

Remember from before (V,cl) is a pregeometry if for each $X, Y \subseteq V$:

Reflexivity $X \subseteq cl(X)$.

Monotonicity $Y \subseteq cl(X) \Rightarrow cl(Y) \subseteq cl(X)$

Finite character $cl(X) = \bigcup\{cl(X_0) : X_0 \subseteq X \text{ and } |X_0| < \infty\}$

Exchange For each $a, b \in V$,
$a \in cl(X \cup \{b\}) - cl(X) \Rightarrow b \in cl(X \cup \{a\})$

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Theorem

For any structure $\mathcal{M}$ with universe M,(M, acl) satisfies the Reflexivity, Monotonicity and finite character property.

A theory T is called strongly minimal if for each $\mathcal{M} \models T$ and formula $\varphi(x)$ (possibly using extra constants from $\mathcal{M}$) we have:

$\varphi(x)$ is satisfied by a finite amount of elements

or $\neg\varphi(x)$ is satisfied by a finite amount of elements

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Theorem

If T is strongly minimal then for each $\mathcal{M} \models T$, $(\mathcal{M}, acl)$ forms a pregeometry.

Theorem (Zilber)

An infinite, locally finite, homogeneous geometry is one of the following

- *Trivial*
- *Affine*
- *Projective*

Thank you for coming!

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